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The Existence of a Periodic Solution of a Boundary Value Problem for a Linear System

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ABSTRACT : *This work investigates the existence of a T periodic solution of the problem*

$$\begin{cases} X'(t) = F(t,x) \\ X(0) = X(T) \end{cases}$$

Where T is a positive constant, $X(t)$ is an n- vector of t, F(t, x) is a T periodic in t and continuous in X. Here, $X'(t)$ means the derivative of $X(t)$ with respect to t. The linear and non-linear cases are covered and examples are given to illustrate the concept of a limit cycle. Furthermore, Bendixson theorem and Floquet theorem were treated.

Keywords - *Periodic solution, linear and non linear, limit cycle*

I. Introduction

In this paper, we study the existence of a periodic solution of a boundary value problem for a linear system. Before we investigate the existence of such periodic solution we give definitions of some basic concepts with examples and theorem that will be important in the sequel.

Definition I

A first order linear system simply referred to as a linear system of the form

$$X' = A(t)X \qquad 1.1$$

Where A(t) is an nxn matrix function of t, continuous for (+, ε R), a≤ t ≤ b.

Consider the homogenous linear periodic system, where A is a periodic matrix, that is

$$A(t+T) = A(t) \; for \; all \; t\varepsilon R$$

For some fixed T, the period of A. the

question now is whether such a system has solutions of period T, or perhaps of mT, m=1,2,….. This is not always the case. Even in the scalar case, as shown below, the system $U' = sin^2 xU$ has no non-trival 2π periodic solution since

$$U' = sin^2 xU \qquad 1.2$$

$$u(\theta) = u(2\pi)$$

Gives

$$u(x) = \frac{Ae^{\frac{x}{2}}}{e^{\frac{1}{4}sin2x}}$$

Applying the 2π periodic condition we get

$$u(o) = A$$

$$u(2\pi) = Ae^{2\pi}$$

$$A = Ae^{2\pi}$$

$$\Rightarrow \qquad A = 0$$

So, u(x) =0 is the solution of 1.2

Definition 1.2

Let $X_1(t),\ldots,X_n(t)$ be n solutions of (1.1)

If $X^1,\ldots, X^n$ are linearly independent, then X is a fundamental matrix. Also, if X(t) is a fundamental matrix solution of (1.1), then so is X(t)C for non- singular constant matrix C.

Considering the homogenous equation (1.1), since the coefficient matrix A is now a constant matrix, it is defined for all t and hence the solution procedure is to substitute

$$X = e^{\lambda t}U$$

In (1.1), where U is a constant vector. This will be a solution if

$$AU = \lambda U.$$

Hence, λ is an eigenvalue of A, where U is the corresponding eigenvector. There are n-eigenvalue $\lambda_1,\dots,\lambda_n$ which are zeros of

$$\det[A - \lambda E] = 0 \qquad (1.3)$$

and we let $U_1,\dots, U_n$ be the corresponding eigenvectors. Thus we have obtained the solutions to (1.1)

$$X_1 = e^{\lambda_1 t}U_1, \dots, X_n = e^{\lambda_1 t}U_n$$

Where $U_1,\dots,U_n$ are linearly independent eigenvectors, these n solutions are linearly independent and can be used to form a fundamental matrix of (1.1)

LEMMA

Let Ψ, Φ be two fundamental matrices for the system

$$X' = A(t)X$$

Then there exists a non-singular nxn constant matrix c such that

$$\Phi = \Psi C$$

Proof:

Fix t_0 and set $c = (\Psi(t_o))^{-1}\Phi(t_o)$. It suffices to show that $(\Psi(t))^{-1}\Phi(t)$ is independent of t. using equation (1.1) for both Φ *and* Ψ we have immediately

$$\frac{d}{dt}[(\Psi(t))^{-1}\Phi(t)] = [(\Psi(t)^{-1}]'\Phi(t) + (\Psi(t))^{-1}\Phi'(t)$$

But $[\Psi(t)^{-1}]' = \Psi'(t)(\Psi(t))^{-2} = -(\Psi(t)^{-1}\Psi'(t)(\Psi(t))^{-1}$

Substituting, we have

$$\frac{d}{dt}[(\Psi(t))^{-1}\Phi(t)] = -(\Psi(t)^{-1}\Psi'(t)(\Psi(t))^{-1}\Phi(t) + (\Psi(t))^{-1}\Phi'(t)$$

$$= -(\Psi(t)^{-1}A(t)\Psi(t)\Psi(t)^{-1}\Phi(t) + (\Psi(t))^{-1}A(t)\Phi(t)$$

$$= -(\Psi(t)^{-1}A(t)\Phi(t) + (\Psi(t))^{-1}A(t)\Phi(t)$$

$$= 0$$

This completes the proof.

Given a fundamental matrix for a periodic system, the existence of a periodic solution is checked by the following theorem

THEOREM 1.4 (FLOQUET THEOREM)

Let Ψ be a fundamental matrix for a periodic system. Then there exists a matrix P of period T and a constant matrix R such that

$$\Psi(t) = P(t)e^{tR}$$

Proof:

Set $\Phi(t) = (\Psi + T)$. Φ is non singular for all t and satisfies

$$\Phi'(t) = \Psi(t+T)$$

$$= A(t+T)\Psi(t+T)$$

$$= A(t)\Phi(t)(by\, the\, T - periodic\, of\, A)$$

So, Φ is another fundamental matrix. By lemma (1.6), we have $\Phi(t) = \Psi(t)C_1$, for some constant non-singular matrix C_1.

Determine $C_1 = e^{TR}$ and P by $P = \Psi e^{-tR}$. To show that P is periodic with period T, we have that

$$p(t+T) = \Psi(t+T)e^{-(t+T)R}$$

$$= \Phi(t)e^{-(t+T)R}$$

$$= \Psi(t)C_1 e^{-(t+T)R}$$

$$= \Psi(t)C_1 e^{TR}e^{-TR}e^{tR}$$

$$= \Psi(t)e^{-tR}$$

$$= p(t)$$

And this completes the proof.

II. Second Order Autonomous Differential Equations

The second order equation is of general type

$$\ddot{x} = f(x, \ddot{x}\,)$$

$$i.e\ \ddot{X} \ = \ F(x,t)$$

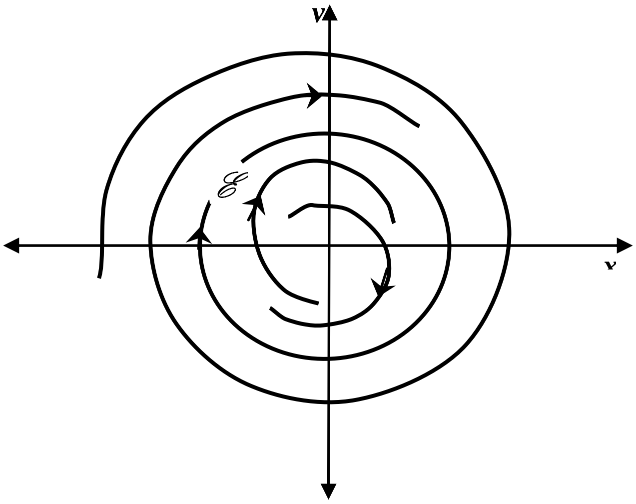
And the associated autonomous equation

$$\dot{X} \ = \ F(X) \ \ where\, X = \binom{x}{y}$$

In what follows, we shall be concerned with autonomous equations.

DEFINITION

LIMIT CYCLE: A limit cycle is an isolated periodic solution represented in the phase plane by an isolated path. The neighboring path are, by definition, not closed but spiral into or away from the limit ɣ as shown below.



Suppose that the phase diagram for a differential equation contains a single, unstable equilibrium point and a limit cycle surrounding it as in the case of the Van-der-pol equation

$$\ddot{x} = e(x^2 - 1)\ddot{x} + x = 0, \qquad e > 0$$

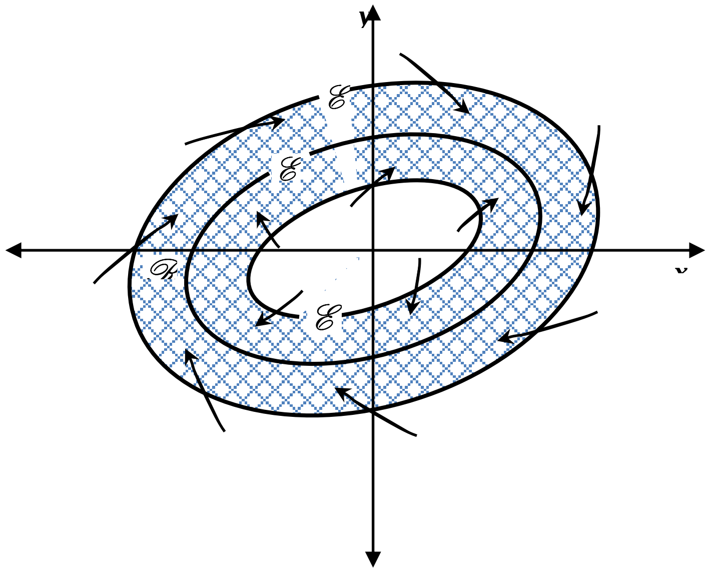
In such cases, the limit cycle is the principal features of the system and it is desirable to be able to decide with certainty whether it is there or not.

We now state, without proof, a theorem on which the result of this chapter is based.

THEOREM 2.2 (THE POINCARE-BENDIXSON THEOREM)

Let R be a closed bounded region consisting of non-singular points of a 2x2 system $\ddot{X} = F(x)$ such that some positive half-path Ʊ of the system lies entirely within R. then either Ʊ is itself a closed path or it approaches a closed path or it terminates at the equilibrium point.

The theorem implies, in particular, that if R contains no equilibrium points and some Ʊ remain in R, then R must contain a periodic solution. The theorem can be used in the following way. Suppose that γ_1 and γ_2 with γ_2 inside γ_1, such that all paths, crossing γ_1 point toward its interior and all paths crossing γ_2 points outwards from it. Then no path which enters the annular region between them can even escape again. The annulus is therefore a region R for the theorem (see fig 2.2). If further, we can arrange that R has no equilibrium points in it. Then the theorem predicts at least one closed path $\bar{\gamma}$ somewhere in R.



The practical difficulty is in finding for a given system, a suitable γ_1 and γ_2 to substantiate a brief in the existence of a limit cycle. An example can be taken to show that a given system has a periodic solution.

EXAMPLE 2.3

Show that the system

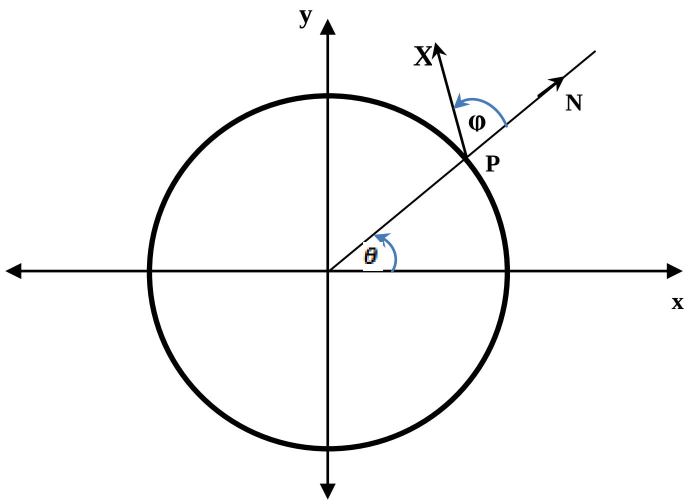
$$\dot{x} = x - y - \left(x^2 + \frac{2}{3}y^2\right)x$$

$$\dot{y} = x + y - \left(x^2 + \frac{1}{2}y^2\right)y$$

Has a periodic solution

SOLUTION:

We shall try to find two circles centered on the origin with the required properties



In fig 2.3, $N = (x, y)$is a normal pointing outward at P from the circle of radius r and $X = (x_1, y_1)$ is in direction of the path through P. Also

$$\cos \Phi = \frac{N.X}{|N||X|}$$

And therefore $N.X$ is positive or negative according to whether X is pointing outwards or inwards. We have

$$N.X = xx_1 + yy_1$$

But $x_1 = \dot{x}\ and\ y_1 = \dot{y}$

$$\Rightarrow N.X = x\dot{x} + y\dot{y}$$

$$= x\left[x - y - \left(x^2 + \frac{3}{2}y^2\right)x\right] + y\left[x + y - \left(x^2 + \frac{1}{2}y^2\right)y\right]$$

$$= x\left[x - y - x^3 - \frac{3}{2}xy^2\right] + y\left[x + y - x^2y - \frac{1}{2}y^3\right]$$

$$= \left[x^2 - xy - x^4 - \frac{3}{2}x^2y^2 + xy + y^2 - x^2y^2 - \frac{1}{2}y^4\right]$$

$$= x^2 + y^2 - x^4 - \frac{5}{2}x^2y^2 - \frac{1}{2}y^4$$

Changing to polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

Substituting, we have

$$N.X = r^2 - r^4cos^4\theta - \frac{5}{2}r^2cos^2\theta r^2sin^2\theta - \frac{1}{2}r^4sin^4\theta$$

$$= r^2 - r^4cos^4\theta - \frac{5}{2}r^4cos^4\theta sin^2\theta - \frac{1}{2}r^4sin^4\theta$$

$$= r^2 - r^4\left[cos^4\theta + \frac{5}{2}cos^2\theta sin^2\theta + \frac{1}{2}sin^4\theta\right]$$

$but\ sin^2\theta = 1 - cos^2\theta$

$$= r^2 - r^4\left[cos^4\theta + \frac{5}{2}cos^2\theta(1 - cos^2\theta) + \frac{1}{2}(1 - cos^2\theta)^2\right]$$

$$= r^2 - r^4\left[cos^4\theta + \frac{5}{2}(cos^2\theta - cos^4\theta) + \frac{1}{2}(1 - 2cos^2\theta + cos^4\theta)\right]$$

$$= r^2 - r^4\left[cos^4\theta + \frac{5}{2}cos^2\theta - \frac{5}{2}cos^4\theta + \frac{1}{2} - cos^2\theta + \frac{1}{2}cos^4\theta\right]$$

$$= r^2 - r^4\left[cos^4\theta - \frac{5}{2}cos^4\theta + \frac{1}{2}cos^4\theta + \frac{5}{2}cos^4\theta - cos^2\theta + \frac{1}{2}\right]$$

$$= r^2 - r^4\left[-cos^4\theta + \frac{3}{2}cos^2\theta + \frac{1}{2}\right]$$

$but\quad cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$cos^4\theta = (\frac{\cos 2\theta + 1}{2})^2$

Substituting, we have

$$N.X = r^2 - r^4\left[-(\frac{\cos 2\theta + 1}{2})^2 + \frac{3}{2}\left[\frac{\cos 2\theta + 1}{2}\right] + \frac{1}{2}\right]$$

$$= r^2 - r^4\left[-\left[\frac{cos^2 2\theta + 2\cos 2\theta + 1}{4}\right] + \frac{3}{4}(cos2\theta + 1) + \frac{1}{2}\right]$$

$$= r^2 - r^4\left[-\frac{1}{4}cos^2 2\theta - \frac{1}{2}\cos 2\theta - \frac{1}{4} + \frac{3}{4}\cos 2\theta + \frac{3}{4} + \frac{1}{2}\right]$$

$$= r^2 - r^4\left[-\frac{1}{4}cos^2 2\theta + \frac{1}{4}\cos 2\theta + 1\right]$$

When, for example, $r = \frac{1}{2}$, this is positive for all θ and so all paths are directed outwards on the circle and when $r = 2$, it is negative, with all paths directed inwards. Therefore, somewhere between $r = \frac{1}{2}$ and $r = 2$ there is at least a periodic solution.

III. The Existence Of A Periodic Solution Of A General System

The Bendixson principle can be employed to obtain theorems covering broad types of differential equation, of which the following is an example.

Theorem 3.1

The differential equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \qquad (3.1)$$

(The Liénard equation), or the equivalent system

$\dot{x} = y, \quad \dot{y} = -f(x, y)y - g(x)$

Or

$\dot{X} = F(X)$

Where

$X = \binom{x}{y}, \quad \dot{X} + \binom{\dot{x}}{\dot{y}}$

F and g are continuous, has at least one periodic solution under the following conditions:

(i) $\exists\, a > 0 \ni f(x, y) > 0 \text{ when } \sqrt{x^2 + y^2} > a$

(ii) $f(0,0) < 0$ $(\text{hence } f(x, y) < 0 \text{ in a neighbourhood of the origin})$

(iii) $g(0) = 0, \ g(x) > 0 \text{when } x > 0 \text{ and } g(x) < 0 \text{when } x < 0.$

(iv) $G(x) = \int_0^{\infty} g(u)du \to \infty \text{ as } x \to \infty$

Proof:

By $g(0) = 0, g(x) < 0 \text{when } x < 0,$ it implies that there is a single equilibrium point at the origin.

Consider the function

$$\varepsilon(x, y) = \frac{1}{2}y^2 + G(x) \qquad (3.2)$$

i.e $\dot{x} = y, \ \dot{y} = -g(x)$

$\int \dot{y}dx = -\int g(x)dx = -G(x) + C$

$but \int \dot{y}dx = \int \frac{dy}{dt}dx = \int \frac{dy}{dt}\frac{dx}{dt}dt$

$\Rightarrow \int ydy = -G(x) + C$

$\frac{y^2}{2} = -G(X) + C$

And $C = \varepsilon(x, y)$

$\varepsilon(x,y) = \frac{1}{2}y^2 + G(x)$

$\varepsilon(x, y)$ represents the energy of the system, G(0)=0, G(x)>0 when x$\neq 0$ and G is monotonic increasing (by (iv); and is continuous). Therefore

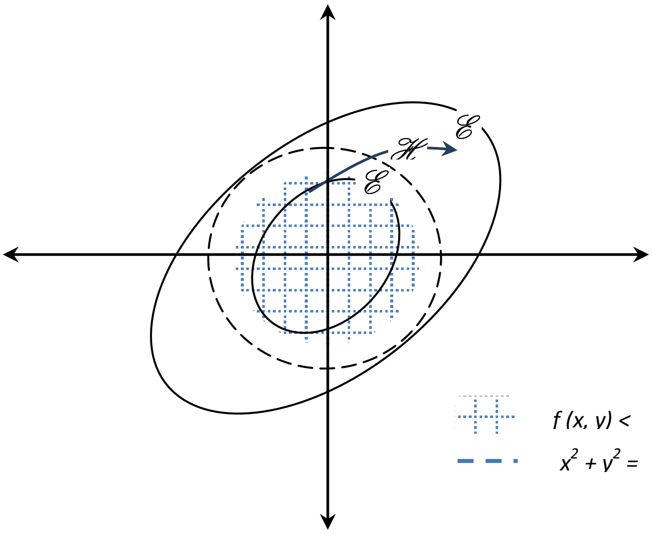
$\varepsilon(0,0) = 0 \text{ and } \varepsilon(x, y) > 0$ for x$\neq 0$ or y$\neq 0$ (ε is positive definite).

Also ε is continuous and increases monotonically.

Let

$$\varepsilon(x, y) = C \qquad (3.3)$$

consists of simple closed curves encircling the origin. As C tends to zero, they approach the origin and as C to infinity, they become infinitely “remote” (the principal consequence of (iv)).



We can choose C, C =C_1 small enough for the corresponding contourγ_1, to be entirely within the neighborhood of the origin where, by (ii), f(x, y)< 0. Examining a half path $\bar{\mathcal{O}}$ starting at a point on γ_1 we consider $\dot{\varepsilon}(x,y)$ on $\bar{\mathcal{O}}$

$\dot{\varepsilon}(x,y) = g(x)\dot{x} + y\dot{y}$

$= g(x)y + y\{-f(x,y)y - g(x)\}$

$= g(x)y - f(x,y)y^2 - g(x)y$

$$= -f(x,y)y^2 \qquad (3.4)$$

$= -y^2 f(x,y)$

This is positive except at y=0, on γ_1 choose $\bar{\mathcal{O}}$ to start at a point other than y=0 on γ_1. Then it leaves γ_1 in the outward direction and it can never reappear inside γ_1 since to do so, it must cross same interior contours in the outward direction, which are impossible since by (3.4); $\dot{\varepsilon} \geq 0$ on all contour near γ_1 as well as onγ_1. Considering a contour γ_2 for large C, C=C_2, say, γ_2 can be chosen by (iv) to lie entirely outside the circle$x^2 + y^2 = a^2$, so that by (i), f(x, y)> 0 onγ_2. By (3.4), with f(x, y)> 0, all paths crossing γ_2 cross inwardly or are tangential (at y=0) and by a similar argument to the above, no positive half-path once inside γ_2, can escape.

Therefore $\bar{\mathcal{O}}$ remains in the region bounded by γ_1 and γ_2 and by theorem (2.1), here is a periodic solution in this region.

Example 3.2

Show that the equation

$\dot{x} + \varepsilon(x^2 + \dot{x}^2 - 1)\dot{x} + x^3 = 0$

Has a limit cycle and locate it, between two curves $\varepsilon(x,y)$=0 constant.

Solution

From the theorem,

$f(x,y) = x^2 + y^2 - 1$

$g(x) = x^3$

$G(x) = \int_0^x g(u)du = \int_0^x u^3 du$

$= \frac{U^4}{4}\Big|_0^x = \frac{x^4}{4}$

$G(x) = \frac{1}{4}x^4$

Therefore (3.2) gives the contours of

$\varepsilon\colon \frac{1}{4}x^4 + \frac{1}{2}y^2 = C$

The periodic solution located by the theorem lies between two such contours, one inside, the other outside, of the curve f(x, y)=0 or $x^2 + y^2 = 1$ and is most closely fixed by finding the smallest contour lying outside this circle and the largest lying inside. We require respectively minimum/maximum of $x^2 + y^2$ subject to $\frac{1}{4}x^4 + \frac{1}{2}y^2 = C, C$ being then chosen so that the minimum/maximum is equal to 1.

Minimum/Maximum $x^2 + y^2$

Subject to $\frac{1}{4}x^4 + \frac{1}{2}y^2 = C$

Calculating by mean of a Lagrange multiplier, we have

H(x, y, λ) =I(x, y) + λ (I(x, y)-C)

H(x, y, λ) = $x^2 + y^2 + \lambda(\frac{1}{4}x^4 + \frac{1}{2}y^2 - C)$

$$\frac{\partial H}{\partial x} = 2x + \lambda x^3 = 0 \qquad (1)$$

$$\frac{\partial H}{\partial y} = 2y + \lambda y = 0 \qquad (2)$$

$$\frac{\partial H}{\partial \lambda} = \frac{1}{4}x^4 + \frac{1}{2}y^2 - C = 0 \qquad (3)$$

Form (2)

$2y + \lambda y = 0$

$y(2+\lambda) = 0 \Rightarrow y = 0 \; or \; \lambda = -2$

Substituting for λ=-2 in equation (1)

$2x + \lambda x^3 = 0$

$2x - 2x^3 = 0 \Rightarrow 2x(1 - x^2) = 0$

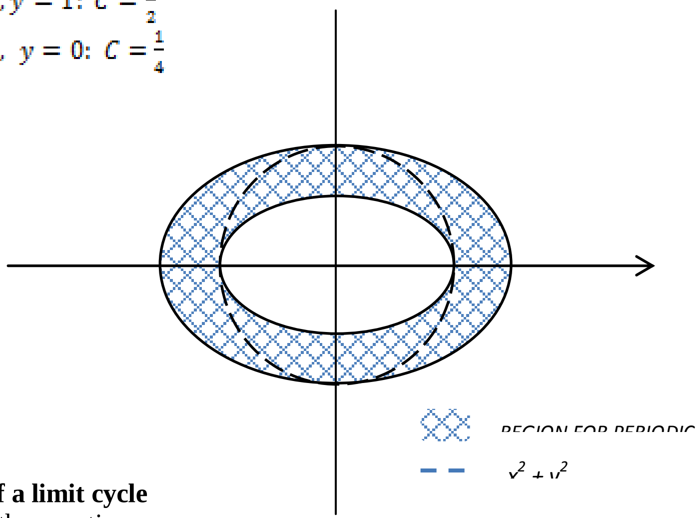
$x = 0 \quad or \; x = \pm 1$

Since C is chosen so that the minimum/maximum is equal to ! it implies that if x=0, y+1 and if x=1, y=0 substituting in (3) we have

(a) $x = 0, y = 1: \; C = \frac{1}{2}$

(b) $x = 1, \; y = 0: \; C = \frac{1}{4}$

(See fig 3.1B)



An existence result of a limit cycle

We consider the equation

$$\dot{x} + f(x)\dot{x} + g(x) = 0 \qquad (3.5)$$

Where, broadly speaking, f(x) is positive when $|x|$ is large, and negative when $|x|$ is small and where g is such that in the absence of damping term, F(x)$\dot{x}$, we expect periodic solutions for small x. Example is the Van-der Pol's equation.

$\ddot{x} + e(x^2 - 1)\dot{x} + x = 0, \;\; e > 0$

It will provide for us the simplest form of equation having this pattern of positive and negative damping which gives rise to a limit cycle.

Effective, the theorem demonstrates a pattern of expanding and contracting spirals about a limit cycle. Paths far from the origin spend part of their time in regions of energy input and part in regions of energy loss.

The proof is carried out using the equivalent system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x) \qquad (3.6)$$

(the Liénard plane) where

$F(x) := \int_0^x f(u)du$

The use of this plane enables the shape of the paths to be simplified (under the conditions of theorem (3.1) y=0 only on x=0) without losing symmetry, and , additionally, allows the burden of the conditions to rest on F rather than on f, f being thereby less restricted. We now state without proof a proof a theorem on which the result of this section is based.

Theorem 3.3

The equation

$\ddot{x} = f(x)\dot{x} + g(x) = 0$

Has a unique periodic solution if f and g are continuous, and

(i) F(x) is an odd function

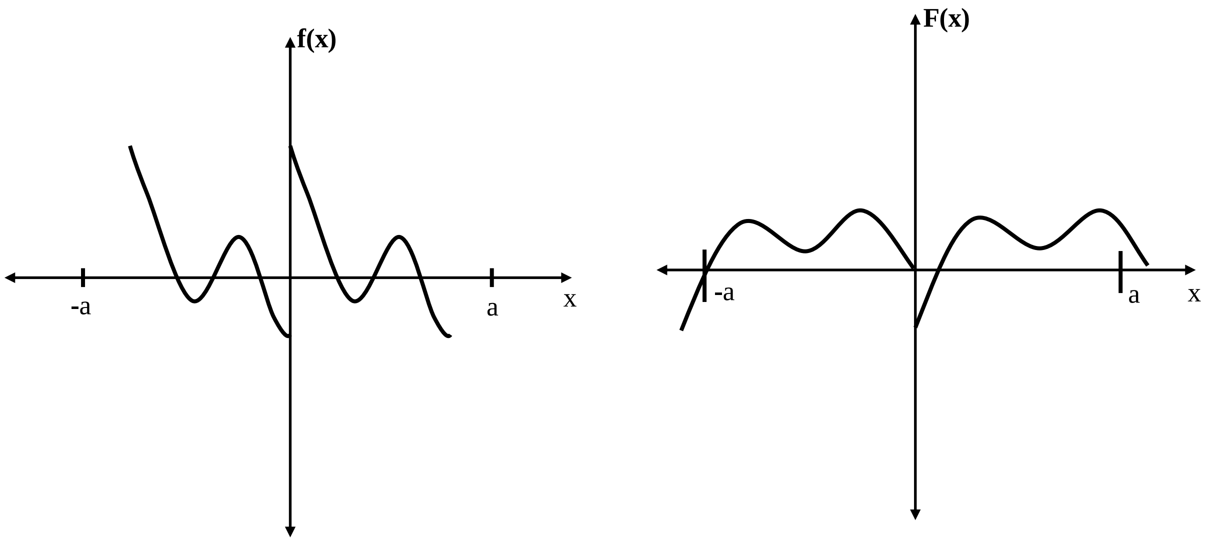
(ii) F(x) is zero only at x=0, x=a, x=-a, for some a> 0

(iii) F(x) $\rightarrow \infty$ as x$\rightarrow \infty$ monotonically for ax$> a$

(iv) g(x) is an odd function and g(x)> 0 for x> 0

(These conditions imply that f(x) is even, f(0)< 0 and f(x)> 0 for x$> a$)

The general shape of f(x) and F(x) is shown in (fig 3.2) below



The general pattern of the paths can be obtained from the following considerations.

(a) If x(t), y(t) is a solution, so is –x(t), -y(t) (since F and g are odd); therefore the phase diagram is symmetrical about the origin (but not necessary the individual phase paths).

(b) The slope of a path is given by

$$\frac{dy}{dx} = \frac{-g(x)}{y - F(x)}$$

So the paths are horizontal only on x=0 (from (iv)) and are vertical only on the curve y=F(x). above y=F(x), $\dot{x} > 0$ and below, $\dot{x} < 0$

(c) $\dot{y} < 0$ for $x > 0$, and $y > 0$ for $x < 0$ by (iv)

Illustration of the existence of a limit cycle

The Van-der-pol equation

$\ddot{x} + \varepsilon(x^2 - 1)\ddot{x} + x = 0, \quad \varepsilon > 0$

Has a unique limit cycle

Here, $f(x) = \varepsilon(x^2 - 1)$

$g(x) = x$

$$F(x) = \int_0^x f(u)\,du = \varepsilon(\tfrac{1}{3}x^3 - x)$$

i.e

$$\varepsilon \int_0^x (u^2 - 1)\,du = \varepsilon\left[\frac{U^3}{3} - U\right]\Big|_0^x$$

$F(x) = \varepsilon\left(\frac{1}{3}x^3 - x\right)$

Proof:

(i) F(x) is an odd function since

$$F(-x) = e(x - \frac{1}{3}x^3)$$

$$= -e\left(\frac{1}{3}x^3 - x\right) = -F(x)$$

$\Rightarrow (i)$ is satisfied

(ii)

$$e\left(\frac{1}{3}x^3 - x\right) = 0$$

$$ex\left(\frac{1}{3}x^2 - 1\right) = 0$$

$$x = 0$$

$$x^2 - 3 = 0$$

$$x = \pm\sqrt{3}$$

Here, $a = \sqrt{3}$

⇒ (ii) is satisfied with a=$\sqrt{3}$

(iii) Is satisfied also since for x> a, f(x)→ ∞ as x→ ∞

For example,

X = 3 ⇒ f(x) = 6

X = 4 ⇒ f(x) = $\frac{52}{3}$, …

(iv) Is also satisfied since

g(-x) = -x = -g(x)

Since all the four axioms are satisfied, if follows therefore that a limit cycle exists, its x- extremities must be beyond x=$\pm\sqrt{3}$

IV. Conclusion

We have thus obtained explicitly the method of calculation of fundamental matrices, *floquent multipliers and floquent exponents.* Also, the poincaré-Bendixson theorem was discussed as regards to the existence of periodic solutions and limit cycles. Some application were also given.

References

[1]. Hakl, R; Lomtatidze, A and Sremr, J.O: *on a boundary-value problem for for linear solutions*. Masarky University, Brno (2002).

[2]. Belmekki, M; Nieto,J.J and Ford, N.D*: Existence of periodic solutions for a linear system*, Boundary value problems. Georgian Math. J.6 (2000), No 1, 47-66.

[3]. F. Li and Z. Liang, *Existence of positive periodic solution to nonlinear second order differential equations.* Applied Mathematics Letters, Vol. 18, No. 11, 2005, pp. 1256-1264.

[4]. K.Q. Lan, *Nonzero positive solutions of systems of boundary value problems,* AMS, vol.139, No. 12, 2011, pp 4343-4349.