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Some Studies on Control Theory InvolvingSchrodinger Group

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Abstract: *In this project we tried to cover the possible aspects of mathematical control theory, especially Left invariant optimal control on the very important physical Group \"Schrodinger Lie Group\". Here we tried to restrict our-selves only in the mathematical aspects, as a result we concluded thestability factors of Schrodinger Lie Group at various equilibrium states. However this group is related to many physical phenomenon, which are interesting to physicist .We hope our study will be lead to a small step towards such type of investigation.*

I. Introduction

1 Some Terms Associated With Control Theory

The theme of this chapter is to introduce some basic concepts and results related to Control theory which will be required latter. The rest two sections deals with control theory on a vector space .The particular sec-tion contains some de notions and results for control systems on a vector space. Whereas the last section provides an preface of geometric control theory. A description of control theory on a manifold is given. The chapter ends with some well-known results of control theory on Lie groups. A general exposition is given and some results without proofs are presented. Further details can be found in the cited references.

1.1 Control System

Let V be an n-dimensional vector space, called the state space, and let (xV) be a state vector. A control system on V is defined by

$$\dot{y} = f(y, u(t)), x(t_0) = x_0 \tag{1}$$

Where the control functions u belongs to a class **U**of admissible controls with values in a subset of R^m and f is continuously differentiable.It isprovided that sufficiently smooth control function (u U), is a solution of the system termed as a trajectory, and is determined. Such type of solution can be explained using the transition function. Specifically (t, t_0, x_0, u) denotes the state that results at time t if the system was in state x_0 at time t_0 and the control u was applied.

Definition the state z can bereached fromthe state **x** if and only ifthere is a trajectory of whose initial state is **x** and whose final state is **z**,that is , if there exist u**ϵU** such that $(t_f, 0, x, u) = z$. One can also say that x can be controlled to z. The controllable set at t_1 is the set of initialstates that can be controlled to the origin in time t_1 using an admissible control, that is,

$$C(t_1) = x_0 : \phi(t_1, 0, x_0, u) = 0 \text{For some u.}$$

The controllable set C is the set of states that can be controlled to the origin in any finite time i.e,

$$C = u_{t_1 \geq 0} C(t_1)$$

The system is called controllable at x if z can be controlled to x for all z V .Therefore**σ,** is controllable at the origin if and only if C=V.

If all initial states can be controlled to x for all x $\in$ V , then the system is controllable.

1.2 Linear Control Systems

A linear control system is defined as

$$\dot{x} = Ax + Bu$$

Where$A(n,n) and\ B(n,m)$scalar matrices and the dimension of the state space are is n and the control u $\in$ U, where U is the class of integrable functions of t.

We can de ne the exponential of a matrix by the using the definition ofinfinite series

$$\boldsymbol{exp(A)} = \frac{\{\boldsymbol{A^k}\}}{\boldsymbol{k}}$$

It follows that $x_0 \in C(t_1)$ if and only if there is an admissible control $u \in U$.
The following lemma shows some controllability equivalences for a linear system.

Lemma 1. If σis a linear control system , then

(i) State space(x) is controllable to another State space (z) iff the origin 0 is controllable to $z - exp(At)x$.
(ii) The Control system is controllable iff the origin is controllable to $y\ for\ all\ y\ R^m$.

Proof . (i) Note that x can be controlled to z that implies there exist an admissible control u U such that

$$z = \exp(At)(x + z \exp(At)x$$
$$= exp(At)(exp(-A)$$

the origin can be controlled to $z - exp(At)x$by definition.

Proposition: $C(t1) and\ C$ are both symmetric and convex.

Example: Consider the linear system given by the following state equations

$$x_1 = x_1 + u, x_2 = x_2 + u$$

where $u \in U$ and the matrices $A\ and\ B$ are given by

$$A = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad B = \begin{matrix} 1 \\ 1 \end{matrix}$$

so we have that $x = (x_1, x_2)\ belongs\ to\ C(t_1)$ if

$$x_1 = -\int_0^{t_1} exp(-\tau)u_1 d\tau = x_2$$

$\overline{since|u| \leq 1, then}|x_1| \leq 1 - exp(-t_1)$.

Therefore, $C(t_1)$ is the closed diagonal segment

$C(\mathrm{t}_1) = \mathrm{x}_1{=}\mathrm{x}_2\colon |x_1| \leq 1 - exp(-t_1)$
and C is the open diagonal segment

$C = \mathrm{x}_1{=}\mathrm{x}_2\colon |x_1| \leq 1$

In general, it would be impossible to control both components simultaneously with identical controls. To control both components using the same control, the initial deviation of the component must be equal.

To get controllability there are two necessary conditions on the controllable set, namely it must have full dimension and be bounded.

2.2 Relation between Lie group and its Lie algebra

In the general theory it has been shown that the structure (vector space) of the Lie algebra of a Lie group is isomorphic to the tangent space at the identity element of the Lie group. Consider in $GL\ (n, C)$ a subset ofoperators depends onto a real parameter t and satisfies$A(0) = 1$, where 1 is the identity operator on the given vector space V. Now we can consider the tangent vector at $t = 0$ with the aid of Taylor's expansion of $A(t)$ upto first order term.

$$A(t) = A(0) + N(t) + O(t^2)$$

Where N determines the derivative of $A(t)\ at\ t = 0$.

Hence these linear operators N obtained by this way are the elements of Lie algebra $of\ GL(n, C)$. With the help of the basis elements $k_1;\ k_2; ::: ;\ k_n$ these operators can be represented by $n \times\ n$ matrices say n_{ij} .On consideration of all possible smooth curves through the unit element of the group, we will be having a vector space of the Lie algebras whose dimension will be n^2 consisting of $n \times n$ matrices. Now in order to obtain the Lie bracket of the elements i.e. for$(\ M, N)\ \in\ N(n;\ C)$, assume the group commutator:

$$C(t) = A(t)B(t)A^{-1}(t)B^{-1}(t)$$

Here $\dot{A}(0) = M$ and $\dot{A}(0) = N$represents the tangent vectors of $C(t)$.After a very simple calculation we will be having the following result:

$$[M, N] = MN - NM$$

It can be easily seen they both can be easily achieved by using exponential mapping.

II. Schrodinger Group

The Schrodinger group for one dimensional particle is given by :

$$i\hbar\frac{\partial\psi(y,t))}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(y,t)}{\partial y^2}$$

where the particle has to be described by the wave function which is given by $\psi(y', t')$as it satisfies the above equation.

It ha s already proven that the Schrodinger equation is invariant under conformal coordinate transformations

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, y' = \frac{ay + ut + c}{\gamma t + \delta}, a^2 = \delta\alpha - \beta\gamma \neq 0$$

Hence as a result it forms a group known as Schrodinger group, whose basis elements are given by:

$$A_1 = -i\hbar\frac{\partial}{\partial y}$$

$$A_2 = \frac{A_1^2}{2m}$$

$$A_3 = tA_1 - my$$

$$A_4 = tA_2 - \frac{1}{4}(yA_1 + A_1 y)$$

$$A_5 = t^2 A_2 - \frac{t}{2}(yA_1 + A_1 y) + \frac{m}{2}y^2$$

$$A_6 = 0.$$

Now using the Exponential mapping we can obtain the Schrodinger Lie Algebra from the group. Also it can be very easily seen that the basis elements for this Schrodinger Group also satisfies all the properties of Lie Bracket. Hence they form a algebra. The commutation table for the elements of Schrodinger Lie algebra is given by:

$$\begin{bmatrix} [.,.] & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_1 & 0 & -2A_1 & A_2 & 0 & A_4 & 0 \\ A_2 & 2A_1 & 0 & -2A_3 & A_4 & -A_5 & 0 \\ A_3 & -A_2 & 2A_3 & 0 & A_5 & 0 & 0 \\ A_4 & 0 & -A_4 & -A_5 & 0 & A_6 & 0 \\ A_5 & -A_4 & A_5 & 0 & & 0 & 0 \\ A_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

III. Left-Invariant Control Problem on the Schrodinger Group

In this section we are considering the left invariant control affine system on Schrodinger Lie Group. Here we are showing that any left invariant control problem can be lifted to a Hamiltonian system on the dual of Schrodinger lie algebra. Also here we are deriving the reduced Hamiltonian equations associated with extremal curves obtained by describing Lie-Poisson structure on dual space of Schrodinger Lie algebra.

Theorem: The constants of structure of the Schrodinger Lie algebra aregiven by: where $A_1, A_2, A_3, A_4, A_5, A_6$ is the canonical basis of Schrodinger Lie algebra, i.e.:

Then we have successively:

$$[A_1, A_1] = c_{11}^k A_k$$

$$c_{11}^1 = c_{11}^2 = c_{11}^3 = c_{11}^4 = c_{11}^5 = c_{11}^6 = 0$$

$$[A_1, A_2] = c_{12}^k A_k$$

$$c_{12}^1 = c_{12}^2 = c_{12}^3 = c_{(12)}^4 = c_{12}^5 = c_{12}^6 = 2$$

$$[A_1;\ A_3] = c_{13}^k A_k$$

$$c_{13}^1 = c_{13}^2 = c_{13}^3 = c_{13}^4 = c_{13}^5 = c_{13}^6 = 1$$

$$[A_1, A_4] = c_{14}^k A_k$$

$$c_{14}^1 = c_{14}^2 = c_{14}^3 = c_{14}^4 = c_{14}^5 = c_{14}^6 = 0$$

$$[A_1, A_5] = c_{15}^k A_k$$

$$c_{15}^1 = c_{15}^2 = c_{15}^3 = c_{15}^4 = c_{15}^5 = c_{15}^6 = 1$$

$$[A_1, A_6] = c_{16}^k A_k$$

$$c_{16}^1 = c_{16}^2 = c_{16}^3 = c_{16}^4 = c_{16}^5 = c_{16}^6 = 0$$

$$[A_2, A_2] = c_{22}^k A_k$$

$$c_{22}^1 = c_{22}^2 = c_{22}^3 = c_{22}^4 = c_{22}^5 = c_{22}^6 = 0$$

$$[A_2, A_3] = c_{23}^k A_k$$

$$c_{23}^1 = c_{23}^2 = c_{23}^3 = c_{23}^4 = c_{23}^5 = c_{23}^6 = 2$$

$$[A_2, A_4] = c_{24}^k A_k$$

$$c_{24}^1 = c_{24}^2 = c_{24}^3 = c_{24}^4 = c_{24}^5 = c_{24}^6 = 1$$

$$[A_2, A_5] = c_{25}^k A_k$$

$$c_{25}^1 = c_{25}^2 = c_{25}^3 = c_{25}^4 = c_{25}^5 = c_{25}^6 = 1$$

$$[A_2, A_6] = c_{26}^k A_k$$

$$c_{26}^1 = c_{26}^2 = c_{26}^3 = c_{26}^4 = c_{26}^5 = c_{26}^6 = 0$$

$$[A_3, A_3] = c_{33}^k A_k$$

$$c_{33}^1 = c_{33}^2 = c_{33}^3 = c_{33}^4 = c_{33}^5 = c_{33}^6 = 0$$

$$[A_3, A_4] = c_{34}^k A_k$$

$$c_{34}^1 = c_{34}^2 = c_{34}^3 = c_{34}^4 = c_{34}^5 = c_{34}^6 = 1$$

$$[A_3, A_5] = c_{35}^k A_k$$

$$c_{35}^1 = c_{35}^2 = c_{35}^3 = c_{35}^4 = c_{35}^5 = c_{35}^6 = 0$$

$$[A_3, A_6] = c_{36}^k A_k$$

$$c_{36}^1 = c_{36}^2 = c_{36}^3 = c_{36}^4 = c_{36}^5 = c_{36}^6 = 0$$

$$[A_4, A_4] = c_{44}^k A_k$$

$$c_{44}^1 = c_{44}^2 = c_{44}^3 = c_{44}^4 = c_{44}^5 = c_{44}^6 = 0$$

$$[A_4, A_5] = c_{45}^k A_k$$

$$c_{45}^1 = c_{45}^2 = c_{45}^3 = c_{45}^4 = c_{45}^5 = c_{45}^6 = 1$$

$$[A_4, A_6] = c_{46}^k A_k$$

$$c_{46}^1 = c_{46}^2 = c_{46}^3 = c_{46}^4 = c_{46}^5 = c_{46}^6 = 0$$

$$[A_5, A_5] = c_{55}^k A_k$$

$$c_{55}^1 = c_{55}^2 = c_{55}^3 = c_{55}^4 = c_{55}^5 = c_{55}^6 = 0$$

$$[A_5, A_6] = c_{56}^k A_k$$

$$c_{56}^1 = c_{56}^2 = c_{56}^3 = c_{56}^4 = c_{56}^5 = c_{56}^6 = 0$$

$$[A_6, A_6] = c_{66}^k A_k$$

$$c_{66}^1 = c_{66}^2 = c_{66}^3 = c_{66}^4 = c_{66}^5 = c_{66}^6 = 0$$

As a consequence we obtain:

Theorem: The minus Lie-Poisson structure on dual of Schrodinger Lie algebra is given by the matrix:

$$\pi_- = \begin{pmatrix} 0 & 2A_1 & -A_2 & 0 & -A_6 & 0 \\ -2A_1 & 0 & 2A_3 & -A_4 & A_5 & 0 \\ A_2 & -2A_3 & 0 & -A_5 & 0 & 0 \\ 0 & A_4 & A_5 & 0 & -A_6 & 0 \\ A_4 & -A_5 & 0 & A_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark . It is easy to see that the function C given by

$$C(P_1;\ P_2;\ P_3;\ P_4;\ P_5;\ P_6)\ =\ P_1P_2P_3P_4P_5P_6$$

is a Casimir of our configuration :

$$(\mathrm{h}(3), \pi_-) i.e.$$
$$(\triangle C)^t . \pi_- = 0$$

Theorem : There exist the following two types of controllable drift-free leftinvariant systems on Schrodinger Lie Group, namely

$$\dot{X}\ =\ X.(A_1u_1\ +\ A_2u_2\ +\ A_3u_3\ +\ A_4u_4\ +\ A_5u_5\ +\ A_6u_6)$$
$$and$$
$$\dot{X}\ =\ X.(A_1u_1\ +\ A_2u_2\ +\ A_3u_3\ +\ A_4u_4\ +\ A_5u_5)$$

Proof :
The proof is a consequence of the Table 1 and Chow's theorem.

IV. An optimal control problem on Schrodinger Lie Group

Let $J(u_1, u_2, u_3, u_4, u_5, u_6) = 1/2(\int_0^{T_f} (c_1u_1^2+c_2u_2^2+c_3u_3^2+c_4u_4^2+c_5u_5^2+c_6u_6^2)dt$

$(c_1;\ c_2;\ c_3;\ c_4;\ c_5;\ c_6\ >\ 0)$ be the cost function. Then the problem which we intend to solve is the following: $find\, u_1;\ u_2;\ u_3;\ u_4;\ u_5;\ u_6$ that minimize J and steer the above system from $= 0\, at\, t\ =\ 0\, to\, X\ =\ X_f\ at\ t\ =\ t_f$. We have the following results:

Theorem : The optimal controls of the above problem for our system aregiven by

$$u_1\ = \frac{P_1}{c_1}, u_2\ = \frac{P_2}{c_2}, u_3\ = \frac{P_3}{c_3}, u_4\ = \frac{P_4}{c_4}, u_5\ = \frac{P_5}{c_5}, u_6\ = \frac{P_6}{c_6}$$

Where$p_i^{'}s$ are the solution given by:

$$\dot{P}_1 = \frac{2P_1P_2}{c_2} - \frac{P_2P_3}{c_3} - \frac{P_5^2}{c_5}$$
$$\dot{P}_2 = \frac{-2P_1^2}{c_1} - \frac{2P_3^2}{c_3} - \frac{P_4^2}{c_4} + \frac{P_5^2}{c_5}$$
$$\dot{P}_3 = \frac{P_1P_2}{c_1} - \frac{2P_3P_2}{c_2} - \frac{P_4P_5}{c_4}$$
$$\dot{P}_4 = \frac{P_4P_2}{c_2} + \frac{P_3P_5}{c_3} - \frac{P_6P_5}{c_5}$$
$$\dot{P}_4 = \frac{P_4P_1}{c_1} + \frac{P_2P_5}{c_2} - \frac{P_6P_4}{c_4}$$
$$\dot{P}_6 = 0$$

Proof.

Let us take the extended Hamiltonian H given by:

$$H\ =\ P_1u_1\ +\ P_2u_2\ +\ P_3u_3\ +\ 1\backslash 2\ (c_1u_1^2\ +\ c_2u_2^2\ +\ c_3u_3^2\ +\ c_4u_4^2\ +\ c_5u_5^2\ +\ c_6u_6^2)$$

Then using the maximum principle, we have the conditions:

$$\frac{\partial H}{\partial u_1}=0,\frac{\partial H}{\partial u_2}=0,\frac{\partial H}{\partial u_3}=0,\frac{\partial H}{\partial u_4}=0,\frac{\partial H}{\partial u_5}=0,\frac{\partial H}{\partial u_6}=0$$

which lead us to:

$$P_1=c_1u_1,P_2=c_2u_2,P_3=c_3u_3,P_4=c_4u_4,P_5=c_5u_5,P_6=c_6u_6$$

and so the reduced Hamiltonian (or the optimal Hamiltonian) is given by:

$$H=\frac{1}{2}(\frac{P_1^2}{c_1}+\frac{P_2^2}{c_2}+\frac{P_3^2}{c_3}+\frac{P_4^2}{c_4}+\frac{P_5^2}{c_5}+\frac{P_6^2}{c_6})$$

It follows that the reduced Hamilton equations have the following expressions:

$$\begin{bmatrix}\dot{P}_1\\ \dot{P}_2\\ \dot{P}_3\\ \dot{P}_4\\ \dot{P}_5\\ \dot{P}_6\end{bmatrix}=\begin{bmatrix}0 & 2A_1 & -A_2 & 0 & -A_6 & 0\\ -2A_1 & 0 & 2A_3 & -A_4 & A_5 & 0\\ A_2 & -2A_3 & 0 & -A_5 & 0 & 0\\ 0 & A_4 & A_5 & 0 & -A_6 & 0\\ A_4 & -A_5 & 0 & A_6 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\times\begin{bmatrix}\frac{P_1}{c_1}\\ \frac{P_2}{c_2}\\ \frac{P_3}{c_3}\\ \frac{P_4}{c_4}\\ \frac{P_5}{c_5}\\ \frac{P_6}{c_6}\end{bmatrix}$$

as required. It is easy to see that the reduced Hamilton's equations can be put in the equivalent form:

$$\begin{aligned}\dot{P}_1&=\frac{2P_1P_2}{c_2}-\frac{P_2P_3}{c_3}-\frac{P_5^2}{c_5}\\ \dot{P}_2&=\frac{-2P_1^2}{c_1}+\frac{2P_3^2}{c_3}-\frac{2P_4^2}{c_4}+\frac{2P_5^2}{c_5}\\ \dot{P}_3&=\frac{P_1P_2}{c_1}-\frac{2P_2P_3}{c_2}-\frac{P_5P_4}{c_4}\\ \dot{P}_4&=\frac{P_4P_2}{c_2}+\frac{P_5P_3}{c_3}-\frac{kP_5}{c_5}\\ \dot{P}_5&=\frac{P_1P_4}{c_1}-\frac{P_2P_5}{c_2}-\frac{kP_4}{c_4}\\ P_6&=k\end{aligned}$$

Theorem :. The controls$u_1;\ u_2;\dots.u_n$are given by sinusoidal s, more exactly

$$\begin{aligned}u_1&=\frac{l_1}{c_1}\cos\sqrt{\frac{c_1}{c_2}}\left(\frac{-p_2\dot{p_1}+p_1\dot{p_2}p_1^2+p_2^2}{}\right)t+C_1\\ u_2&=\frac{l_2}{c_2}\sin\sqrt{\frac{c_3}{c_4}}\left(\frac{-p_4\dot{p_3}+p_3\dot{p_4}p_3^2+p_4^2}{}\right)t+C_2\\ u_3&=\frac{l_3}{c_3}\cos\sqrt{\frac{c_5}{c_6}}\left(\frac{-p_6\dot{p_5}+p_5\dot{p_6}p_5^2+p_6^2}{}\right)t+C_3\end{aligned}$$

Proof:
Let us assume that

$$\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} = l_1^2$$
$$\frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} = l_2^2$$
$$\frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} = l_3^2$$

On substituting these values in Reduced Hamiltonian system of equation, the equation becomes:

$$\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} + \frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} + \frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} = l^2$$
$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 + l_6^2 = l^2$$

For the convenience of the proof let us assume

$$p_1 = l_1\sqrt{c_1}\cos\Theta_1, p_2 = l_1\sqrt{c_2}\sin\Theta_1$$
$$p_3 = l_2\sqrt{c_3}\cos\Theta_2, p_4 = l_2\sqrt{c_4}\sin\Theta_2$$
$$p_5 = l_3\sqrt{c_5}\cos\Theta_3, p_1 = l_3\sqrt{c_6}\sin\Theta_3$$

such that we have:

$$u_1 = \frac{p_1}{c_1} = \frac{l_1\cos\Theta_1}{\sqrt{c_1}}, u_2 = \frac{p_2}{c_2} = \frac{l_1\sin\Theta_1}{\sqrt{c_2}}$$
$$u_3 = \frac{p_3}{c_3} = \frac{l_2\cos\Theta_2}{\sqrt{c_3}}, u_4 = \frac{p_4}{c_4} = \frac{l_2\sin\Theta_2}{\sqrt{c_4}}$$
$$u_5 = \frac{p_5}{c_5} = \frac{l_3\cos\Theta_3}{\sqrt{c_5}}, u_6 = \frac{p_6}{c_6} = \frac{l_3\sin\Theta_3}{\sqrt{c_6}}$$

now on simplifying we have:

$$\dot{\Theta}_1 = \sqrt{\frac{c_1}{c_2}}(\frac{p_1\dot{p_2} - \dot{p_1}p_2}{p_1^2 + p_2^2})$$
$$\dot{\Theta}_2 = \sqrt{\frac{c_3}{c_4}}(\frac{p_3\dot{p_4} - \dot{p_3}p_4}{p_3^2 + p_4^2})$$
$$\dot{\Theta}_3 = \sqrt{\frac{c_5}{c_6}}(\frac{p_5\dot{p_6} - \dot{p_5}p_6}{p_5^2 + p_6^2})$$

as a result we obtain the solutions given by :

$$u_1 = \frac{l_1}{c_1}\cos\sqrt{\frac{c_1}{c_2}}\left(\frac{-p_2\dot{p_1} + p_1\dot{p_2}p_1^2 + p_2^2}{}\right)t + C_1$$
$$u_2 = \frac{l_2}{c_2}\sin\sqrt{\frac{c_3}{c_4}}\left(\frac{-p_4\dot{p_3} + p_3\dot{p_4}p_3^2 + p_4^2}{}\right)t + C_2$$
$$u_3 = \frac{l_3}{c_3}\cos\sqrt{\frac{c_5}{c_6}}\left(\frac{-p_6\dot{p_5} + p_5\dot{p_6}p_5^2 + p_6^2}{}\right)t + C_3$$

V. Stability associated with Lie Groups

Stability associated with Schrodinger Lie Algebra

We investigate the stability nature of the dynamical system shown above,the equilibrium states are

$$\mathrm{P}_e^M 1 = (M,0,0,0,0,0), P_e^M 2 = (0,M,0,0,0,0),$$
$$P_e^M 3 = (0,0,M,0,0,0),\ P_e^M 4 = (0,0,0,M,0,0),$$
$$P_e^M 5 = (0,0,0,0,M,0),\ P_e^M 6 = (0,0,0,0,0,M),$$

here, $M \in R(0)$ and the origin $(0, 0, 0,0,0,0)$.

Proposition

The equilibrium state $P_{M_1}^e = (M;\ 0;\ 0;\ 0;\ 0;\ 0)$ has the following behavior:
1. If k(constant) is positive, then state is non linearly stable.
2. If k(constant) is negative, the state is not stable. For all $c_1, c_2, c_3, c_4, c_5, c_6$ the equilibrium state is unstable:

Proof: For calculating the Eigen values of the derived dynamical system we have to obtain the linearization matrix which is actually the Jacobian matrix of the dynamics of the system. That is, we have $\dot{\boldsymbol{P}} = \boldsymbol{F}(\boldsymbol{P})$, so matrix of the linearization is the Jacobian of F is given by :

$$D(F(p)) = \begin{bmatrix} \frac{2P_2}{c_2} & \frac{2p_1}{c_2} - \frac{P_3}{c_3} & \frac{-P_2}{c_3} & \frac{-P_5}{c_5} & \frac{-P_4}{c_5} & 0 \\ \frac{-4P_1}{c_1} & 0 & \frac{4P_3}{c_3} & \frac{-2P_4}{c_4} & \frac{2P_5}{c_5} & 0 \\ \frac{P_2}{c_1} & \frac{P_1}{c_1} - \frac{2P_3}{c_3} & \frac{-2P_2}{c_2} & \frac{-P_5}{c_4} & \frac{-P_4}{c_4} 0 & \\ 0 & \frac{P_4}{c_2} & \frac{P_5}{c_3} & \frac{P_2}{c_2} & \frac{P_3}{c_3} - \frac{k}{c_3} & 0 \\ \frac{P_4}{c_1} & \frac{-P_5}{c_2} & 0 & \frac{P_1}{c_1} & \frac{-P_2}{c_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus The matrix of the linearization of the system at P_e^M 1 is

$$\begin{bmatrix} 0 & \frac{2M}{c_2} & 0 & 0 & 0 & 0 \\ \frac{-4M}{c_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{M}{c_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-k}{c_3} & 0 \\ 0 & 0 & 0 & \frac{M}{c_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence by using the various properties of stability and energy Casmir functions the above results have been proved.
Similarly for another equilibrium states following results have been obtained.

Proposition. The equilibrium $state P_e^{M_2} = (0;\ M; 0; 0; 0; 0)$ has the following behavior:
1. If $c_1 c_3 \leq c_2^2$, then state is not linearly stable.

2. if $c_1 c_3 \geq c_2^2$, the state is non linearly stable.

Proposition. The equilibrium $state\ P_e^{M_3} = (0; 0;\ M; 0; 0; 0), P_e^{M_4} = (0; 0; 0;\ M; 0; 0) and\ P_e^{M_5} = (0;\ 0;\ 0;\ 0;\ M;\ 0)$ are non linearly stable.

VI. Conclusion

In this project we tried to cover the possible aspects of mathematical control theory, especially Left invariant optimal control on the very important physical Group "Schrodinger Lie Group". Here we tried to restrict our-selves only in the mathematical aspects, as a result we concluded the stability factors of Schrodinger Lie Group at various equilibrium states. However this group is related to many physical phenomenon, which are interesting to physicist .We hope our study will be lead to a small step towards such type of investigation.

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