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Cone 2- Metric Spaces and an Extension of Fixed Point Theorems for Contractive Mappings

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Abstract: *The purpose of this paper is to extend and improve fixed point theorem on cone2- metric spaces for contractive mapping. Our results generalize and unify some well know results in the literature of [15].*
Key words: *Cone 2- metric spaces, contractive mappings, fixed point, non- normal cone, contractive mapping*

I. Introduction

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis. The theory itself is a beautiful mixture of analysis, topology and geometry. Over the last years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of non-linear phenomena. In this area, the first important and significant result was proved by Banach [1] in 1922 for a contraction mapping in a complete metric space. The well knows Banach contraction Theorems may be stated as follows:

"Every contraction mapping of a complete metric space in X into itself has a unique fixed point."

Since then, this principle has been extended and generalized in several ways either by using the contractive condition or by imposing some additional conditions on an ambient space. This principle is one of the cornerstones in the development of fixed point theory. From inspiration of this work, several mathematicians heavily studied this field.

Motivated by this work, several authors introduced similar concepts and proved analogous fixed point theorem in 2-metric and 2- Banach space. Gahler ([2], [3], and [4]) investigated the concept of 2- metric space and give the definition as follows:

Definition 1.1: Let X be a non-empty set and let $d: X \times X \times X \to R$ i.e. $d : X^3 \to R$ satisfying the following conditions:

(i) $d(x, y, x) = 0$ only if at least two x, y, z are equal.
(ii) $d(x, y, z) = d(p(x\, y, z))$ for all $x, y, z \in X$ and for all permutation $p(x\, y, z)$ of $x\, y, z$.
(iii) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$. Then d is called a 2-metric on X and (X, d) is called a 2-metric space which will sometimes be denoted simply by X, when there is no confusion. It can be easily seen that d is a non-negative function.

Perhaps Iseki [5-7] obtained for the first time basic results on fixed point of operators in 2- metric space and in 2- Banach space. After the work of Iseki, several authors extended and generalized fixed point theorems in 2- metric and 2- Banach spaces for different types of operators of contractive type.
Recently, In 2007, Haung and Zhang[8] introduced the concept of cone metric space by generalized the concept of metric space, replacing the set of real numbers, by an ordered Banach space and obtained the following fixed point theorems for mapping satisfying different contractive conditions.

Theorem 1.1: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq Kd(x, y)$ for all $x, y \in X$,where $k \in [0,1]$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to a fixed point.

Theorem 1.2 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$,and $x \neq y$. Then T has a unique fixed point in X.

Theorem 1.3 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq K(d(Tx, y) + d(Ty, x))$ for all $x, y \in X$,where $k \in [0,1]$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to a fixed point.

Theorem 1.4 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq K(d(Tx, x) + d(Ty, y))$ for all $x, y \in X$,where $k \in [0,1]$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to a fixed point.

Subsequently, many authors have studied the strong convergence to a fixed point with contractive constant in cone metric space, see for instance ([9],[10],[11],[12][13], [14]). On the other hand, B. Singh, S. Jain, and P. Bhagat [15] introduced cone 2-metric space by replacing real number in 2- metric space by an ordered Banach space and some fixed point theorem for contractive mappings on complete cone2- metric space with assumption of normality on the cone.

The purpose of this paper is to extend and improves the fixed point theorems of B. Singh, S. Jain, and P. Bhagat [15].

II. Preliminary Notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [8].

Definition 2.1 [3]: Let(E, τ) be a topological vector space and $P \subset E$. Then P is called a cone if and only if:

(i) P is closed, non – empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all x, $y \in P$ and non – negative real number a, b;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P, by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in \text{int } P$, int P denotes the interior of P.

The cone is called normal if there exists a number $K > 0$ such that

$$x \leq y \text{ Implies } \|x\| \leq K\|y\| \text{ for all } x, y \in P.$$

The least number satisfying above is called the normal constant of P.The cone P is called regular if every non decreasing sequence in p, which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \cdots \ldots \ldots \ldots \leq . x_n \leq \cdots \ldots \ldots \ldots \leq y$$

For some y in E, there exist $x \in P$ such that $\|x_n - x\| \to 0 (n \to \infty)$ equivalently the cone P is regular if and only if every decreasing sequence in p, which is bounded from below is convergent. It can be easily proved that a regular conies a normal cone.

Remark 2.2[16]: If E is a real Banach space with cone P and if $a \leq \boldsymbol{\lambda} a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

Definition 2.3 [15]: Let X be a non-empty set. Suppose the mapping $d: X \times X \times X \to E$ $i.e. d : X^3 \to E$ satisfying the following conditions:

1. $0 \leq d(x, y, x)$, for all $x, y, z \in X$ and $d(x, y, x) = 0$ if and only if at least two x, y, z are equal.
2. $d(x, y, z) = d(p(x\, y, z))$ for all $x, y, z \in X$ and for all permutation $p(x\, y, z)$ of $x\, y, z$.
3. $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$. Then d is called a con 2-metric on X and (X, d) will be called cone 2-metric space. Cone 2-metric space will be called normal, if the cone P is normal cone.

Example 2.4[15]: Let $E = R^2$, $P = \{(x, y) \in E: x, y \geq 0\} \subset R^2, X = R$ and $d: X \times X \times X \to E$ such that $d(x, y, a) = (l^n, \alpha l)$ where $l = min(|x - y|, |y - z|, |z - x|)$ and l and n are some fixed positive integers. Then(X, d) is a cone 2- metric space.

Definition 2.5[15]: Let (X, d) be a cone 2-metric space with respect to a cone P in a real Banach space$E$. $\{x_n\}_{n \geq 1}$ a sequence in X and $x \in X$. Then,

(i) $\{x_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x, a) \ll c$ for all $n \geq N$ and for all $a \in X$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x, as\ n \to \infty$.

(ii) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a Natural number N such that $d(x_n, x_m x, a) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is called a complete cone 2- metric space if every Cauchy sequence in X is Convergent.

Lemma 2.6[15]: (1) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K,$\{x_n\}_{n\geq 1}$ a sequence in X and $x\in X$.Then $\{x_n\}_{n\geq 1}$ converges to $x \in X$ if and only $d(x_n, x_m x, a) \to 0$as $n \to \infty$ for all $a \in X$.

(2) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K, $\{x_n\}_{n\geq 1}$ a sequence in X and $x\in X$.Then limit of $\{x_n\}_{n\geq 1}$ is unique if it exist.

(3) Let (X, d) be a cone 2-metric space, $\{x_n\}_{n\geq 1}$ be a sequence in X and $x\in X$.If $\{x_n\}_{n\geq 1}$ converges to $x \in X$, then$\{x_n\}_{n\geq 1}$ is a Cauchy sequence

(4) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K. Let $\{x_n\}_{n\geq 1}$ be a sequence in X and $x\in X$.Then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$ as $(n, m \to \infty)$, for all $a \in X$.

(5) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K. Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequence in X, $x_n \to x, y_n \to y (n \to \infty)$. Then

$$d(x_n, y_n, a) \to (x, y, a)(n \to \infty).$$

Definition 2.7[15]: Let (X, d) be a cone 2-metric space. If for every sequence $\{x_n\}_{n\geq 1}$ in X , there is a subsequence ,$\{x_n\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ convergent in X. Then X is called a sequently compact cone 2- metric space.

III. Main Results

In this section we shall prove some fixed point theorems for contractive maps by using normality of the cone.

Theorem 3.1: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx,Ty,a) \leq a_1 d(x,y,a) + a_2 d(Tx, x, a) + a_3 d(Ty, y, a) + a_4 d(Ty, x, a) + a_5 d(Tx, y, a)$$

For all $x, y \in X$ .and $a_i, i = 1,2,3,4,5$ are all non negative constants with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Let $x_0 \in X$ be fixed. Let $x_1 = Tx_0$ and Let $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in N$.

From (2.1) Taking $x = x_n$ and $y = x_{n-1}$, we have

$$\begin{aligned} d(x_{n+1,}x_n, a) &= d(Tx_n, Tx_{n-1}, a) \\ &\leq a_1 d(x_n, x_{n-1}, a) + a_2 d(Tx_n, x_n, a) + a_3 d(Tx_{n-1}, x_{n-1}, a) \\ &\quad + a_4 d(Tx_{n-1}, x_n, a) + a_5 d(Tx_n, x_{n-1}, a) \\ &\leq a_1 d(x_n, x_{n-1}, a) + a_2 d(x_{n+1}, x_n, a) + a_3 d(x_n, x_{n-1}, a) \\ &\quad + a_5 d(x_n, x_{n-1}, a + a_5 d(x_n, x_{n+1}, a \\ &\leq (a_1 + a_3 + a_5) d(x_n, x_{n-1}, a) + (a_2 + a_5) d(x_{n+1}, x_n, a) \end{aligned}$$

This implies

$$1 - (a_2 + a_5) \, d(x_{n+1,}x_n, a) \leq (a_1 + a_3 + a_5) \, d(x_n, x_{n-1}, a)$$

$$\Rightarrow \quad d(x_{n+1,}x_n, a) \leq L \, d(d(x_n, x_{n-1}, a) ,$$

Where L = $\frac{(a_1+a_3+a_5)}{1-(a_2+a_5\ )}$…… ………………………………………………………(2)

Hence

$$\begin{aligned} &\leq L^2\{\mathrm{d}(x_{n-1,}x_{n-2}, a)\} \\ &\dots \\ &\leq \dots\dots\dots\dots \leq L^n F\{d(x_1, x_0, a)\} \quad \dots\dots\dots\dots\dots\dots (3) \end{aligned}$$

Also for $m > k$, we have

$$\begin{aligned} d(x_m \, x_{m-1}, x_k) &\leq L d(x_{m-1,} x_{m-2}, x_k) \\ &\leq L^2 d(x_{m-2,} x_{m-3}, x_k) \\ &\leq \dots\dots\dots\dots \leq \\ &\leq L^{m-k-1} d(x_{k+1}, x_k, k) \\ &= 0 \quad \dots\dots\dots\dots\dots\dots (4) \end{aligned}$$

For $n > m$, with using (3) and (4) we have

$$\begin{aligned} d(x_n, x_m, a) &\leq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, a) + + d(x_{n-1}, x_m, a) \\ &\leq L^{2n-1} d(x_1, x_0, a) + d(x_{n-1}, x_m, x_{n-2}) + d(x_{n-1}, x_{n-2}, a) \\ &\quad + d(x_{n-2}, x_{n-3}, a) + d(x_{n-3}, x_m, a) \\ &\leq (L^{n-1} + L^{n-2} + L^{n-3}) d(x_1, x_0, a) + d(x_{n-3}, x_m, a) \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ &\leq (L^{n-1} + L^{n-2} + \cdots \dots \dots . . + L^{m+1}) d(x_1, x_0, a) + d(x_{m+1}, x_m, a) \\ &\leq (L^{n-1} + L^{n-2} + \cdots \dots \dots . . + L^{m+1} + L^m) d(x_1, x_0, a) \end{aligned}$$

$$= L^m(1 + L + L^2 + \cdots \dots \dots \dots + L^{n-m-1})\, d(x_1, x_0, a)$$
$$\leq \frac{L^m}{1-L}, d(x_1, x_0, a) \text{ as } L < 1 \text{ and } P \text{ is closed}$$

Thus we have $\|d(x_n, x_m, a)\| \leq \frac{L^m}{1-L} \|d(x_1, x_0, a)\|$. This implies $d(x_1, x_0, a) \to 0, (n, m \to \infty)$, for all $a \in X$. Hence $\{x_n\}$ is a Cauchy sequence in (X, d) is a complete cone 2- metric space, there exist $u \in X$ such that $x_n \to u (n \to \infty)$. $i.e.\ lim_{n\to\infty} x_n = u$.
Now for any $a \in X$, then we have

$$d(Tu, u, a) \leq d(Tu, u, Tx_n) + d(Tu, Tx_n, a) + d(Tx_n, u, a)$$
$$\leq a_1 d(u, x_n, a) + a_2 d(Tu, u, a) + a_3 d(Tx_n, x_n, a) + a_4 d(Tx_n, u, a)$$
$$+ a_5 d(Tu, x_n, a) + d(x_{n+1}, u, a)$$
$$\leq a_1 d(u, x_n, a) + a_2 d(Tu, u, a) + a_3 d(x_{n+1}, x_n, a) + a_4 d(x_{n+1}, u, a)$$
$$+ a_5 d(Tu, x_n, a) + d(x_{n+1}, u, a)$$
$$\leq a_1 d(u, x_n, a) + a_2 d(Tu, u, a) + a_3 [d(u, x_n, a) + d(x_{n+1}, u, a)]$$
$$+ a_4 d(x_{n+1}, u, a) + a_5 [d(u, x_n, a) + d(Tu, u, a)] + d(x_{n+1}, u, a)$$
$$(1 - a_2 - a_5)\, d(Tu, u, a) \leq (a_1 + a_3 + a_5)\, d(u, x_n, a) + (a_3 + a_4)\, d(u, x_{n+1}, a) + d(x_{n+1}, u, a)$$
$$\Rightarrow \quad d(Tu, u, a) \leq \frac{(a_1 + a_3 + a_5)}{(1 - a_2 - a_5)} d(u, x_n, a) + \frac{(a_3 + a_4)}{(1 - a_2 - a_5)} d(x_{n+1}, x_n, a) + \frac{1}{(1 - a_3 - a_4)} d(x_{n+1}, u, a) \to 0.$$

On taking limit as $n \to \infty$ and by using 2.6(4), we get
$$d(Tu, u, a) = 0$$

This implies $Tu = u$. So x is a fixed point for T in$X$. Now if v is another fixed point of T in X, then,
$$d(u, v, a) = d(Tu, Tv, a)$$
$$\leq a_1 d(u, v, a) + a_2 d(Tu, u, a) + a_3 d(Tv, v, a) + a_4 d(Tv, u, a) + a_5 d(Tu, v, a)$$
$$= (a_1 + a_4 + a_5) d\ \ d(u, v, a)$$

By using remark 2.2, we obtain that $d(u, v, a) = 0$. Thus $= v$. Therefore the fixed point of T in X is unique.

On taking $a_1 = k$ and $a_2 = a_3 = a_4 = a_5 = 0$ in theorem 3.1, we get the following corollary in the setting of cone 2- metric space.

Corollary 3.2: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,
$$d(Tx, Ty, a) \leq k d(x, y, a)$$
for all $x, y, a \in X$ where $k \in [0,1]$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Corollary 3.3: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ satisfy the contractive condition,
$$d(Tx, Ty, a) \leq \alpha[d(Tx, x, a) + d(Ty, y, a)]$$
$$+ \beta[d(Ty, x, a) + d(Tx, y, a)]$$
for all $x, y, a \in X$ .and$\alpha, \beta \in [0,1]$ are all non negative constants with $\alpha + \beta < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_1 = 0$, $a_2 = a_3 = \alpha$ and $a_4 = a_5 = \beta$ in theorem 3.1, we get the required result easily.

Remark. Corollary 3.2 is the result of corollary 2.2 of [15].

Corollary 3.4: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,
$$d(Tx, Ty, a) \leq a_1 d(x, y, a) + a_2 d(Tx, x, a) + a_3 d(Ty, y, a)$$
$$+ a_4[d(Ty, x, a) + d(Tx, y, a)]$$
for all $x, y \in X$ .and $a_i, i = 1,2,3,4,$ are all non negative constants with $a_1 + a_2 + a_3 + a_4 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_4 = a_5 = a_4$ in theorem 3.1, we get the required result easily.

Corollary 3.5: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,

$d(Tx,Ty,a) \leq \alpha d(x,y,a) + \beta[d(Tx,x,a)+d(Ty,y,a)]$, for all $x,y,a \in X$ .and $\alpha,\beta \in [0,1]$ are all non negative constants with $+2\beta < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_1 = \alpha, a_2 = a_3 = \beta$ and $a_4 = a_5 = 0$ in theorem 3.1, then we get the required result easily.

Theorem 3.6: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx,Ty,a) \leq a_1 d(x,y,a) + a_2 d(x,Tx,a) + a_3 d(y,Ty,a) + a_4[d(x,Ty,a) + d(y,Tx,a)] \quad \text{….(3.6.1)}$$

For all $x,y,a \in X$ .and $a_i, i = 1,2,3,4,$ are all non negative constants with $a_1 + a_2 + a_3 + a_4 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Let $x_0 \in X$ be fixed. Let $x_1 = Tx_0$ and Let $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in N$.
From (2.1) Taking $x = x_n$ and $y = x_{n-1}$, we have

$$\begin{aligned} d(x_{n+1,}x_n,a) &= d(Tx_n,Tx_{n-1},a) \\ &\leq a_1 d(x_n,x_{n-1},a) + a_2 d(x_n,Tx_n,a) + a_3 d(x_{n-1},Tx_{n-1},a) \\ &+a_4[d(x_n,Tx_{n-1},a) + d(x_{n-1},Tx_n,a)] \\ &\leq a_1 d(x_n,x_{n-1},a) + a_2 d(x_n,x_{n+1},a) + a_3 d(x_{n-1},x_n,a) \\ &+ a_4[d(x_n,x_{n-1},a + d(x_n,x_{n+1},a)] \\ &\leq (a_1 + a_3 + a_4)\, d(x_n,x_{n-1},a) + (a_2 + a_4) d(x_{n+1},x_n,a) \end{aligned}$$

This implies

$$1 - (a_2 + a_4)\, d(x_{n+1,}x_n,a) \leq (a_1 + a_3 + a_4)\, d(x_n,x_{n-1},a)$$

$$\Rightarrow \quad d(x_{n+1,}x_n,a) \leq L\, d(d(x_n,x_{n-1},a),$$

Where $L = \frac{(a_1+a_3+a_4)}{1-(a_2+a_4)}$ …… ……………………………………………………………(3.6.2)

Hence

$$\begin{aligned} &\leq L^2\{d(x_{n-1,}x_{n-2},a)\} \\ &\dots \\ &\leq \ldots\ldots\ldots\ldots \leq L^n F\{d(x_1,x_0,a)\} \quad \text{……………………………… (3.6.3)} \end{aligned}$$

Also for $m > k$, we have

$$\begin{aligned} d(x_m\, x_{m-1},x_k) &\leq L d(x_{m-1,}\, x_{m-2},x_k) \\ &\leq L^2 d(x_{m-2,}\, x_{m-3},x_k) \\ &\leq \ldots\ldots\ldots\ldots \leq \\ &\leq L^{m-k-1} d(x_{k+1},x_k,k) \\ &= 0 \quad \text{……………………………………………………………(3.6.4)} \end{aligned}$$

For $n > m$, with using (3) and (4) we have

$$\begin{aligned} d(x_n,x_m,a) &\leq d(x_n,x_m,x_{n-1}) + d(x_n,x_{n-1},a) + d(x_{n-1},x_m,a) \\ &\leq L^{n-1} d(x_1,x_0,a) + d(x_{n-1},x_m,x_{n-2}) + d(x_{n-1},x_{n-2},a) \\ &+ d(x_{n-2},x_{n-3},a) + d(x_{n-3},x_m,a) \\ &\leq (L^{n-1} + L^{n-2} + L^{n-3}) d(x_1,x_0,a) + d(x_{n-3},x_m,a) \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ &\leq (L^{n-1} + L^{n-2} + \cdots \ldots\ldots + L^{m+1}) d(x_1,x_0,a) + d(x_{m+1},x_m,a) \\ &\leq (L^{n-1} + L^{n-2} + \cdots \ldots\ldots + L^{m+1} + L^m) d(x_1,x_0,a) \\ &= L^m(1 + L + L^2 + \cdots \ldots\ldots\ldots + L^{n-m-1})\, d(x_1,x_0,a) \\ &\leq \frac{L^m}{1-L}, d(x_1,x_0,a) \quad \text{as } L < 1 \text{ and } P \text{ is closed} \end{aligned}$$

Thus we have $\|d(x_n,x_m,a)\| \leq \frac{L^m}{1-L} \|d(x_1,x_0,a)\|$. This implies $d(x_1,x_0,a) \to 0, (n,m \to \infty)$, for all $a \in X$. Hence $\{x_n\}$ is a Cauchy sequence in (X,d) is a complete cone 2- metric space, there exist $u \in X$ such that $x_n \to u(n \to \infty)$. $i.e.\ lim_{n\to\infty} x_n = u$.
Now for any $a \in X$, then we have

$$\begin{aligned} d(Tu,u,a) &\leq a_1 d(u,x_n,a) + a_2 d(u,Tu,a) + a_3 d(x_{n,}\,Tx_n,a) + a_4[d(u,Tx_n,a) + d(x_n,Tu,a] \\ &+ d(x_{n+1},u,a) \\ &\leq a_1 d(u,x_n,a) + a_2 d(Tu,u,a) + a_3 d(x_n,x_{n+1},a) + a_4[d(u,x_{n+1},a) + d(Tu,x_n,a)] \end{aligned}$$

$+ d(x_{n+1}, u, a)$
$\leq a_1 d(u, u, a) + a_2 d(Tu, u, a) + a_3 d(u, u, a) + a_4[d(u, u, a) + d(Tu, u, a)]$
$+ d(x_{n+1}, u, a)$
$= (a_2 + a_4)\, d(Tu, u, a)$

On taking limit as $n \to \infty$ and by using 2.6(4), we get

$$d(Tu, u, a) = 0$$

This implies $Tu = u$. So x is a fixed point for T in$X$. Now if v is another fixed point of T in X, then,

$d(u, v, a) = d(Tu, Tv, a)$
$\leq a_1 d(u, v, a) + a_2 d(u, Tu, a) + a_3 d(v, Tv, a) + a_4[d(u, Tv, a) + a_5 d(v, Tu, a)]$
$\leq a_1 d(u, v, a) + a_2 d(u, u, a) + a_3 d(v, v, a) + a_4[d(u, v, a) + a_5 d(v, u, a)]$
$= (a_1 + 2a_4) d\ d(u, v, a)$
$= (a_1 + 2a_4) d\ d(u, v, a)$

By using remark 2.2, we obtain that $d(u, v, a) = 0$. Thus $= v$. Therefore the fixed point of T in X is unique.
Theorem 3.6 yields the following corollaries:

Corollary 3.7: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx, Ty, a) \leq a_1 d(x, y, a) + a_2 d(x, Tx, a)$$

For all $x, y, a \in X$ where $a_1, a_2, \in [0,1]$ are all non negative constants with $a_1 + a_2 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_3 = a_4 = 0$ in theorem 3.6, then we get the required result.

Corollary 3.8: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping T: $X \longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx, Ty, a) \leq a_1 d(x, y, a) + a_2 d(x, Tx, a) + a_3 d(y, Ty, a)$$

For all $x, y, a \in X$ where$a_1, a_2, a_3 \in [0,1]$ are all non negative constants with $a_1 + a_2 + a_3 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_4 = 0$ in theorem 3.6, then we get the required result easily.

Corollary 3.9: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx, Ty, a) \leq \alpha[d(x, Ty, a) + d(y, Tx, a)]$$

for all $x, y, a \in X$ where$\alpha \in [0, \frac{1}{2}]$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_1 = a_2 = a_3 = 0$ and $a_4 = \alpha$ in theorem 3.6, then we get the required result.

Theorem 3.10: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx, Ty, a) \leq a_1 d(x, y, a) + a_2 d(x, Ty, a) + a_3 d(y, Tx, a) + a_4[d(x, Tx, a) + d(y, Ty, a)] \quad \text{.......(3.10.1)}$$

For all $x, y, a \in X$ .and $a_i, i = 1,2,3,4,$ are all non negative constants with $a_1 + a_2 + a_3 + a_4 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Let $x_0 \in X$ be fixed. Let $x_1 = Tx_0$ and Let $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in N$.
From (2.1) Taking $x = x_n$ and $y = x_{n-1}$, we have

$d(x_{n+1,} x_n, a) = d(Tx_n, Tx_{n-1}, a)$
$\leq a_1 d(x_n, x_{n-1}, a) + a_2 d(x_n, Tx_{n-1}, a) + a_3 d(x_{n-1}, Tx_n, a)$
$+ a_4[d(x_n, Tx_n, a) + d(x_{n-1}, Tx_{n-1}, a)]$
$\leq a_1 d(x_n, x_{n-1}, a) + a_2 d(x_n, x_n, a) + a_3 d(x_{n-1}, x_{n+1}, a)$
$+ a_4[d(x_n, x_{n+1}, a + d(x_{n-1}, x_n, a)]$
$\leq (a_1 + a_3 + a_4)\, d(x_n, x_{n-1}, a) + (a_3 + a_4) d(x_{n+1}, x_n, a)$

This implies $1 - (a_3 + a_4)\, d(x_{n+1,} x_n, a) \leq (a_1 + a_3 + a_4)\, d(x_n, x_{n-1}, a)$

⇨ $d(x_{n+1,} x_n, a) \leq L\, d(d(x_n, x_{n-1}, a)$, Where L = $\frac{(a_1+a_3+a_4)}{1-(a_2+a_4\ )}$……. …………………………………………………………(3.10.2)

Hence $\leq L^2\{d(x_{n-1,}x_{n-2},a)\}$

$\dots$

$\leq$…………..$\leq L^n F\{d(\mathrm{x}_1,x_0,a)\}$ ………………………… (3.10.3)

Also for $m > k$, we have

$$d(x_m\, x_{m-1},x_k) \leq Ld(x_{m-1,}\,x_{m-2},x_k)$$
$$\leq L^2 d(x_{m-2,}\,x_{m-3},x_k)$$
$$\leq \ \ldots\ldots\ldots\ldots \leq$$
$$\leq L^{m-k-1}d(x_{k+1},x_k,k)$$
$$= 0 \ \ldots\ldots\ldots\ldots\ldots\ldots \ (4)$$

For $n > m$, with using (3) and (4) we have

$$d(x_n,x_m\,,a) \leq \ d(x_n,x_m,x_{n-1}) + d(x_n,x_{n-1},a) + d(x_{n-1},x_m,a)$$
$$\leq L^{n-1}d(x_1,x_0,a) + \ d(x_{n-1},x_m,x_{n-2}) + d(x_{n-1},x_{n-2},a)$$
$$+ d(x_{n-2},x_{n-3},a) + d(x_{n-3},x_m,a)$$
$$\leq (L^{n-1} + L^{n-2} + L^{n-3})d(x_1,x_0,a) + d(x_{n-3},x_m,a)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\leq (L^{n-1} + L^{n-2} + \cdots \ldots\ldots..+L^{m+1})d(x_1,x_0,a) + d(x_{m+1},x_m,a)$$
$$\leq (L^{n-1} + L^{n-2} + \cdots \ldots\ldots..+L^{m+1} + L^m)d(x_1,x_0,a)$$
$$=$$
$$L^m(1 + L + L^2 + \cdots \ldots\ldots\ldots + L^{n-m-1})\ d(x_1,x_0,a)$$
$$\leq \frac{L^m}{1-L},\ d(x_1,x_0,a) \text{ as } L < 1 \text{ and } P \text{ is closed}$$

Thus we have $\|d(x_n,x_m\,,a)\| \leq \frac{L^m}{1-L}\ \|d(x_1,x_0,a)\|$. This implies $d(x_1,x_0,a) \to 0,(n,m \to \infty)$, for all $a \in X$. Hence $\{x_n\}$ is a Cauchy sequence in (X,d) is a complete cone 2- metric space, there exist $u \in X$ such that $x_n \to u(n \to \infty)$. $i.e.\ lim_{n\to\infty}\, x_n = u$.

Now for any $a \in X$, then we have

$$d(Tu,u,a) \leq a_1 d(\,u,x_n,a) + \ a_2 d(u,Tx_n,a) + a_3 d(x_{n,}Tu,a)$$
$$+a_4[d(u,Tu,a) + d(\,x_n,Tx_n,a\,] + d(x_{n+1},u,a)$$
$$\leq \ a_1 d(\,u,x_n,a) + \ a_2 d(u,x_{n+1},a) + a_3 d(x_n,Tu,a\,)$$
$$+a_4[d(\,u,Tu,a\,) \ + d(\,x_n,x_{n+1},a)\ ] + d(x_{n+1},u,a)$$
$$\leq \ a_1 d(\,u,u,a) + \ a_2 d(u,u,a) + a_3 d(Tu,u,a\,)$$
$$+a_4[d(\,Tu,u,a\,) \ + d(u,u,a)\ ] + d(x_{n+1},u,a)$$
$$= (\,a_3 + a_4)\ d(Tu,u,a)$$

On taking limit as $n \to \infty$ and by using 2.6(4), we get

$$d(Tu,u,a) = 0$$

This implies $Tu = u$. So x is a fixed point for T in$X$. Now if v is another fixed point of T in X, then,

$$d(u,v,a) = d(Tu,Tv,a)$$
$$\leq a_1 d(\,u,v,a) + \ a_2 d(u,Tv,a) + a_3 d(v,Tu,a)$$
$$+a_4[d(u,Tu,a) + a_5 d(\,v,Tv,a)\ ]$$
$$\leq a_1 d(\,u,v,a) + \ a_2 d(u,v,a) + a_3 d(v,u,a) \ + a_4[d(u,u,a) + d(\,v,v,a)\ ]$$
$$= (a_1 + a_2 + a_3)\ \ d(\,u,v,a)$$

By using remark 2.2, we obtain that $d(u,v,a) = \ 0$. Thus $= v$. Therefore the fixed point of T in X is unique.

Corollary 3.7: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping T: X $\longrightarrow X$ **satisfy** the contractive condition,

$$d(Tx,Ty,a) \leq a_1 d(x,y,a) + a_2 d(x\,,Ty,a)$$

For all $x,y,a\ \in X$ where $a_1,a_2, \in [0,1]$ are all non negative constants with $a_1 + a_2 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_3 = a_4 = 0$ in theorem 3.6, then we get the required result.

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$$d(Tx,Ty,a) \leq a_1 d(x,y,a) + a_2 d(x\ ,Ty,a) + a_3 d(y,Tx,a)$$

For all $x, y, a\ \in X$ where $a_1, a_2, a_3 \in [0,1]$ are all non negative constants with $a_1 + a_2 + a_3 < 1$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

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$$d(Tx,Ty,a) \leq\ \alpha[d(x,Tx,a) + d(y,Ty,a)]$$

for all $x, y, a\ \in X$ where $\alpha \in [0,\frac{1}{2}]$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n{}_x\}$ is convergent to the fixed point.

Proof: Putting $a_1 = a_2 = a_3 = 0$ and $a_4 = \alpha$ in theorem 3.6, then we get the required result.

IV. Conclusion

Theorem concerning the existence and uniqueness of the solutions to concept of cone 2- metric space and established and improved some fixed point theorems for contractive mappings of the results of B. Singh, S. Jain, and P. Bhagat [15].

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