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Stability of Quadratic and Cubic Functional Equations in Paranormed Spaces

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Abstract : *In this paper, we prove Hyers-Ulam Stability of a class of Quadratic and Cubic functional equations in Paranormed spaces.*
Keywords: *Hyers-Ulam Stability, Quadratic Functional Equations, Cubic Functional Equations, Paranormed Spaces*

I. Introduction

The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1940, S. M. Ulam [11] asked the question concerning the stability of group homomorphisms. Next year Hyers [12] gave the first positive answer to the question of Ulam for Banach spaces. In 1978 Th.M. Rassias [13] provided a generalization of the Hyers theorem which allows the Cauchy difference to be unbounded. After this result many of mathematicians were attracted and motivated to investigate the stability problems of functional equations. In particular, the Stability probelems of different functional equations have been investigated in various spaces.

Recently C.Park and D.Y.Shin[1] presented Hyers-Ulam Stability of a class of Quadratic, Cubic and Quartic functional equations in paranormed spaces.

The functional equation

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+16f(x) \qquad (1.1)$$

is a quadratic functional equation and every solution of the quadratic functional equation is said to be a quadratic function.

The Functional equation

$$3f(x+3y)+f(x-3y)=15f(x+y)+15f(x-y)+80f(y) \qquad (1.2)$$

is a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function.

In this paper, we investigate the Hyers-Ulam Stability of the Quadratic Equation (1.1) and Cubic equation (1.2) in Paranormed spaces. This paper is organized as follows: In Section 3, we prove the Hyers-Ulam stability of quadratic functional equation (1.1) in paranormed space. In Section 4, we prove the Hyers-Ulam stability of cubic functional equation (1.2) in paranormed space.

II. Preliminaries

Throughout this paper, we assume that (X,P) is a Frechet space and that $(Y,\|\cdot\|)$ is a Banach Space.

Definition 2.1 A Normed Space over *K* is a pair $(V,\|\cdot\|)$, where *V* is a vector space over *K* and $\|\cdot\|:V\to R^+$, such that

(i) $\|x\|=0 \text{ iff } x=0$

(ii) $\|\lambda x\|=|\lambda|\|x\| \text{ for all } \lambda\in K \text{ and } v\in V$

(iii) $\|x+y\|\le\|x\|+\|y\| \text{ for all } x,y\in V$

Definition 2.2[1] Let X be a vector space. A paranorm $P:X\to[0,\infty)$ is a function on X such that

(i) $P(0)=0$

(ii) $P(-x)=P(x)$

(iii) $P(x+y)\le P(x)+P(y)\,(\textit{Triangle Inequality})$

(iv) *If* $\{t_n\}$ *is a sequence of scalar with* $t_n\to t$ *and* $\{x_n\}\subset X$ *with* $P(x_n-x)\to 0$, *then* $P(t_nx_n-tx)\to 0$ holds. Then the pair (X,P) is called a paranormed space.

Definition 2.3

The Paranorm is called total if in addition we have *P*(*x*) = 0 implies *x* = 0

Definition 2.4

The Frechet space is total and complete paranormed space.

III. Hyers - ulam stability of quadratic functional equations

In this section, we deal with the stability problem for the following quadratic functional equation in paranormed spaces.

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+16f(x)$$

Theorem 3.1

Let r,θ be positive real numbers with $r>2$, and let $f:Y\to X$ be a mapping satisfying f(0) = 0 and

$$P\big(f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-16f(x)\big)\le\theta\left(\|x\|^r+\|y\|^r\right) \tag{3.1}$$

for all $x,y\in Y.$ Then there exists a unique quadratic mapping $Q_2:Y\to X$ such that

$$P\big(f(x)-Q_2(x)\big)\le\frac{4}{4^r-16}\theta\|x\|^r \tag{3.2}$$

for all $x\in Y.$

Proof:

Putting *y* = *x* in (3.1), we get

$$P\big(f(4x)-16f(x)\big)\le 2\theta\|x\|^r$$

for all $x\in Y.$ So

$$P\left(f(x)-16f\left(\frac{x}{4}\right)\right)\le 2\theta\left\|\frac{x}{4}\right\|^r$$

$$\le\frac{2}{4^r}\theta\|x\|^r$$

$$P\left(f\left(\frac{x}{4}\right)-16f\left(\frac{x}{4^2}\right)\right)\le\frac{2}{4^r}\theta\left\|\frac{x}{4}\right\|^r$$

$$\le\frac{2}{4^r4^r}\theta\|x\|^r$$

for all $x,y\in Y.$ Hence

$$P\left(16^l f\left(\frac{x}{4^l}\right)-16^m f\left(\frac{x}{4^m}\right)\right)\le\sum_{j=1}^{m-1}P\left(16^j f\left(\frac{x}{4^j}\right)-16^{j+1}f\left(\frac{x}{4^{j+1}}\right)\right)$$

$$\le\frac{2}{4^r}\sum_{j=1}^{m-1}\frac{16^j}{4^{rj}}\theta\|x\|^r \tag{3.3}$$

for all nonnegative integers *m* and *l* with *m* > *l* and for all $x\in Y.$ It follows from (3.3) that the sequence $\left\{16^n f\left(\frac{x}{4^n}\right)\right\}$ is a Cauchy sequence for all $x\in Y.$ Since *x* is complete, the sequence $\left\{16^n f\left(\frac{x}{4^n}\right)\right\}$ converges. So one can define the mapping $Q_2:Y\to X$ by

$$Q_2(x):=\lim_{n\to\infty}16^n f\left(\frac{x}{4^n}\right) \tag{3.4}$$

for all $x\in Y.$

Moreover, letting *l* = 0 and passing the limit $m\to\infty$ in (3.3), we get (3.2).

It follows from (3.1) that

$$P\left(Q_2(3x+y)+Q_2(3x-y)-Q_2(x+y)-Q_2(x-y)-16Q_2(x)\right)$$
$$=P\left(16^n\left(f\left(\frac{3x+y}{4^n}\right)+f\left(\frac{3x-y}{4^n}\right)-f\left(\frac{x+y}{4^n}\right)-f\left(\frac{x-y}{4^n}\right)-16f\left(\frac{x}{4^n}\right)\right)\right)$$
$$\le \lim_{n\to\infty}16^n P\left(f\left(\frac{3x+y}{4^n}\right)+f\left(\frac{3x-y}{4^n}\right)-f\left(\frac{x+y}{4^n}\right)-f\left(\frac{x-y}{4^n}\right)-16f\left(\frac{x}{4^n}\right)\right)$$
$$\le \lim_{n\to\infty}16^n\theta\left(\left\|\frac{x}{4^n}\right\|^r+\left\|\frac{y}{4^n}\right\|^r\right)$$
$$\le \lim_{n\to\infty}\frac{16^n}{4^{nr}}\theta\left(\|x\|^r+\|y\|^r\right)$$
$$=0$$

for all $x, y \in Y.$

Hence

$$Q_2(3x+y)+Q_2(3x-y)=Q_2(x+y)+Q_2(x-y)+16Q_2(x)$$

for all $x, y \in Y.$ and so the mapping $Q_2 : Y \to X$ is quadratic. Now let $T : Y \to X$ be another quadratic mapping satisfying (3.2). Then we have

$$P\left(Q_2(x)-T(x)\right)=P\left(16^n\left(Q_2\left(\frac{x}{4^n}\right)-T\left(\frac{x}{4^n}\right)\right)\right)$$
$$\le 16^n P\left(Q_2\left(\frac{x}{4^n}\right)-T\left(\frac{x}{4^n}\right)\right)$$
$$\le 16^n P\left(Q_2\left(\frac{x}{4^n}\right)-T\left(\frac{x}{4^n}\right)+f\left(\frac{x}{4^n}\right)-f\left(\frac{x}{4^n}\right)\right)$$
$$\le 16^n\left(P\left(Q_2\left(\frac{x}{4^n}\right)-f\left(\frac{x}{4^n}\right)\right)+P\left(T\left(\frac{x}{4^n}\right)-f\left(\frac{x}{4^n}\right)\right)\right)$$
$$\le \frac{16^n.8}{\left(4^r-16\right)4^{nr}}\theta\|x\|^r$$

which tends to zero as $n \to \infty$ for all $x \in Y.$ So we can conclude that $Q_2(x) = T(x)$ for all $x \in Y.$ This proves the uniqueness of Q_2. Thus the mapping $Q_2 : Y \to X$ is a unique quadratic mapping satisfying (3.2).

Theorem 3.2

Let r be a real positive number with $r < 2$, and let $f : X \to Y$ be a mapping satisfying *f*(0) = 0 and

$$\left\|f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-16f(x)\right\| \le P(x)^r+P(y)^r \tag{3.5}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\left\|f(x)-Q_2(x)\right\| \le \frac{4}{16-4^r}P(x)^r \tag{3.6}$$

for all $x \in X$

Proof

Letting *y* = *x* in (3.5), we get

$$\left\|f(4x)-16f(x)\right\| \le 2P(x)^r$$

and so

$$\left\| f(x) - \frac{1}{16} f(4x) \right\| \le \frac{1}{8} P(x)^r$$

for all $x \in X$. Similarly

$$\left\| f(4x) - \frac{1}{16} f(4^2 x) \right\| \le \frac{4^r}{8} P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{16^l} f(4^l x) - \frac{1}{16^m} f(4^m x) \right\| &\le \sum_{j=1}^{m-1} \left\| \frac{1}{16^j} f(4^j x) - \frac{1}{16^{j+1}} f(4^{j+1} x) \right\| \\ &\le \frac{1}{8} \sum_{j=1}^{m-1} \frac{4^{rj}}{16^j} P(x)^r \end{aligned} \tag{3.7}$$

for all nonnegative integers *m* and *l* with *m* > *l* and for all $x \in X$. It follows from (3.6) that the sequence $\left\{ \frac{1}{16^n} f(4^n x) \right\}$ is a Cauchy sequence for all $x \in X$. Since *Y* is complete, the sequence $\left\{ \frac{1}{16^n} f(4^n x) \right\}$ converges. So one can define the mapping $Q_2 : X \to Y$ by

$$Q_2(x) := \lim_{n \to \infty} \frac{1}{16^n} f(4^n x) \tag{3.8}$$

for all $x \in X$.

Moreover, letting *l* = 0 and passing the limit $m \to \infty$ in (3.7), we get (3.6).

It follows from (3.5) that

$$\begin{aligned} &\| Q_2(3x+y) + Q_2(3x-y) - Q_2(x+y) - Q_2(x-y) - 16Q_2(x) \| \\ &\le \lim_{n\to\infty} \left\| \frac{1}{16^n} f(4^n(3x+y)) + \frac{1}{16^n} f(4^n(3x-y)) - \frac{1}{16^n} f(4^n(x+y)) - \frac{1}{16^n} f(4^n(x-y)) - \frac{1}{16^n} 16 f(4^n x) \right\| \\ &\le \lim_{n\to\infty} \frac{1}{16^n} \| f(4^n(3x+y)) + f(4^n(3x-y)) - f(4^n(x+y)) - f(4^n(x-y)) - 16 f(4^n x) \| \\ &\le \lim_{n\to\infty} \frac{4^{nr}}{16^n} \left(P(x)^r + P(y)^r \right) \\ &= 0 \end{aligned}$$

for all $x, y \in X$.

Thus

$$Q_2(3x+y) + Q_2(3x-y) = Q_2(x+y) + Q_2(x-y) + 16Q_2(x)$$

for all $x, y \in X$ and so the mapping $Q_2 : X \to Y$ is quadratic. Now let $T : X \to Y$ be another quadratic mapping satisfying (3.6). Then we have

$$\begin{aligned} \| Q_2(x) - T(x) \| &= \frac{1}{16^n} \| Q_2(4^n x) - T(4^n x) \| \\ &\le \frac{1}{16^n} \left(\| Q_2(4^n x) - f(4^n x) \| + \| T(4^n x) - f(4^n x) \| \right) \\ &\le \frac{1}{16^n} \left(\frac{4.4^{nr}}{(16 - 4^r)} P(x)^r + \frac{4.4^{nr}}{(16 - 4^r)} P(x)^r \right) \\ &\le \frac{8.4^{nr}}{(16 - 4^r) 16^n} P(x)^r \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q_2(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q_2. Thus the mapping $Q_2 : X \to Y$ is a unique quadratic mapping satisfying (3.5).

IV. Hyers – ulam stability of cubic functional equation

In this section we prove the Hyers – Ulam stability of the following cubic functional equation in paranormed spaces.

$$3f(x+3y) + f(3x-y) = 15f(x+y) + 15f(x-y) + 80f(y)$$

Theorem 4.1

Let r, θ be positive real numbers with $r > 3$, and let $f : Y \to X$ be a mapping such that

$$P\big(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y)\big) \le \theta\left(\|x\|^r + \|y\|^r\right) \qquad (4.1)$$

for all $x, y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P\big(f(x) - C(x)\big) \le \frac{1}{3^r}\theta\|x\|^r \qquad (4.2)$$

for all $x \in Y$.

PROOF:

Putting $y = 0$ in (4.1), we get

$$P\big(f(3x) - 27f(x)\big) \le \theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x) - 27f\left(\frac{x}{3}\right)\right) \le \frac{1}{3^r}\theta\|x\|^r$$

for all $x \in Y$. Now

$$P\left(f\left(\frac{x}{3}\right) - 27f\left(\frac{x}{3^2}\right)\right) \le \frac{1}{3^r}\theta\left\|\frac{x}{3}\right\|^r$$

$$\le \frac{1}{3^r 3^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$P\left(27^l f\left(\frac{x}{3^l}\right) - 27^m f\left(\frac{x}{3^m}\right)\right) \le \sum_{j=1}^{m-1} P\left(27^j f\left(\frac{x}{3^j}\right) - 27^{j+1} f\left(\frac{x}{3^{j+1}}\right)\right)$$

$$\le \sum_{j=1}^{m-1} 27^j P\left(f\left(\frac{x}{3^j}\right) - 27f\left(\frac{x}{3^{j+1}}\right)\right)$$

$$\le \sum_{j=1}^{m-1} \frac{27^j}{3^r.3^{rj}}\theta\|x\|^r$$

$$\le \frac{1}{3^r}\sum_{j=1}^{m-1} \frac{27^j}{3^{rj}}\theta\|x\|^r \qquad (4.3)$$

for all nonnegative integers *m* and *l* with *m* > *l* and for all $x \in Y$. It follows from (4.3) that the sequence $\left\{27^n f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence for all $x \in Y$. Since *x* is complete, the sequence $\left\{27^n f\left(\frac{x}{3^n}\right)\right\}$ converges. So one can define the mapping $C : Y \to X$ by

$$C(x) := \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right)$$

for all $x \in Y$.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.3), we get (4.2).
It follows from (4.1) that

$$\begin{aligned}
&P\big(3C(x+3y)+C(3x-y)-15C(x+y)-15C(x-y)-80C(y)\big)\\
&\quad = \lim_{n\to\infty} P\left(27^n\left(3f\left(\frac{x+3y}{3^n}\right)+f\left(\frac{3x-y}{3^n}\right)-15f\left(\frac{x+y}{3^n}\right)-15f\left(\frac{x-y}{3^n}\right)-80f\left(\frac{y}{3^n}\right)\right)\right)\\
&\quad \le \lim_{n\to\infty} 27^n P\left(3f\left(\frac{x+3y}{3^n}\right)+f\left(\frac{3x-y}{3^n}\right)-15f\left(\frac{x+y}{3^n}\right)-15f\left(\frac{x-y}{3^n}\right)-80f\left(\frac{y}{3^n}\right)\right)\\
&\quad \le \lim_{n\to\infty} 27^n \frac{1}{3^{nr}}\theta\left(\|x\|^r+\|y\|^r\right)\\
&\quad = 0
\end{aligned}$$

for all $x, y \in Y$.
Hence

$$3C(3x+y)+C(3x-y)=15C(x+y)+15C(x-y)+80C(y)$$

for all $x, y \in Y$ and so the mapping $C : Y \to X$ is cubic. Now let $T : Y \to X$ be another quadratic mapping satisfying (4.2). Then we have

$$\begin{aligned}
P\big(C(x)-T(x)\big) &= P\left(27^n\left(C\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)\right)\right)\\
&\le 27^n P\left(C\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)\right)\\
&\le 27^n P\left(C\left(\frac{x}{3^n}\right)-T\left(\frac{x}{3^n}\right)+f\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right)\right)\\
&\le 27^n\left(P\left(C\left(\frac{x}{4^n}\right)-f\left(\frac{x}{4^n}\right)\right)+C\left(T\left(\frac{x}{4^n}\right)-f\left(\frac{x}{4^n}\right)\right)\right)\\
&\le \frac{2.27^n}{\left(3^r-27\right)3^{nr}}\theta\|x\|^r
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $C(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of C. Thus the mapping $C : Y \to X$ is a unique cubic mapping satisfying (4.2).

Theorem 4.2

Let r be a real positive number with $r > 3$, and let $f : X \to Y$ be a mapping such that

$$\|3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y)\| \le P(x)^r + P(y)^r \qquad (4.4)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $C : X \to Y$ such that

$$\|f(x)-C(x)\| \le \frac{1}{27-3^r}P(x)^r \qquad (4.5)$$

for all $x \in X$.

Proof

Letting $y = 0$ in (4.4), we get

$$\|f(3x)-27f(x)\| \le P(x)^r$$

for all $x \in X$ and so

$$\left\|f(x)-\frac{1}{27}f(3x)\right\| \le \frac{1}{27}P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{27^l} f\left(3^l x\right)-\frac{1}{27^m} f\left(3^m x\right)\right\| \leq \sum_{j=1}^{m-1}\left\|\frac{1}{27^j} f\left(3^j x\right)-\frac{1}{27^{j+1}} f\left(3^{j+1} x\right)\right\|$$

$$\leq \frac{1}{27}\sum_{j=1}^{m-1}\frac{3^{rj}}{27^j} P(x)^r \qquad (4.6)$$

for all nonnegative integers *m* and *l* with *m* > *l* and for all $x \in X$. It follows from (4.6) that the sequence $\left\{\frac{1}{27^n} f\left(3^n x\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since *Y* is complete, the sequence $\left\{\frac{1}{27^n} f\left(3^n x\right)\right\}$ converges. So one can define the mapping $C: X \rightarrow Y$ by

$$C(x) := \lim_{n\to\infty}\frac{1}{27^n} f\left(3^n x\right)$$

for all $x \in X$.

Moreover, letting *l* = 0 and passing the limit $m \rightarrow \infty$ in (4.6), we get (4.5).

It follows from (4.4) that

$$\left\|3C(x+3y)+C(3x-y)-15C(x+y)-15C(x-y)-80C(y)\right\|$$

$$= \lim_{n\to\infty}\left\|\frac{3}{27^n} f\left(3^n(x+3y)\right)+\frac{1}{27^n} f\left(3^n(3x-y)\right)-\frac{15}{27^n} f\left(3^n(x+y)\right)-\frac{15}{27^n} f\left(3^n(x-y)\right)-\frac{80}{27^n} f\left(3^n y\right)\right\|$$

$$\leq \lim_{n\to\infty}\frac{1}{27^n}\left\| f\left(3^n(x+3y)\right)+ f\left(3^n(3x-y)\right)-15 f\left(3^n(x+y)\right)-15 f\left(3^n(x-y)\right)-80 f\left(3^n y\right)\right\|$$

$$\leq \lim_{n\to\infty}\frac{3^{nr}}{27^n}\left(P(x)^r + P(y)^r\right)$$

$$= 0$$

for all $x, y \in X$.

Thus

$$3C(3x+y)+C(3x-y)=15C(x+y)+15C(x-y)+80C(y)$$

for all $x, y \in X$ and so the mapping $C: X \rightarrow Y$ is cubic. Now let $T: X \rightarrow Y$ be another quadratic mapping satisfying (4.5). Then we have

$$\left\|C(x)-T(x)\right\| = \frac{1}{27^n}\left\|C\left(3^n x\right)-T\left(3^n x\right)\right\|$$

$$\leq \frac{1}{27^n}\left(\left\|C\left(3^n x\right)- f\left(3^n x\right)\right\|+\left\|T\left(3^n x\right)- f\left(3^n x\right)\right\|\right)$$

$$\leq \frac{2.3^{nr}}{\left(27-3^r\right)27^n} P(x)^r$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $C(x) = T(x)$ for all $x \in X$. This proves the uniqueness of C. Thus the mapping $C: X \rightarrow Y$ is a unique cubic mapping satisfying (4.5).

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