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Fuzzy Rings and Anti Fuzzy Rings With Operators

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Abstract: *In this paper, we studied the theory of fuzzy rings, the concept of fuzzy ring with operators, fuzzy ideal and anti-fuzzy ideal with operators, fuzzy homomorphism with operators etc., and their some elementary properties.*
Keywords:*Fuzzy rings, fuzzy ring with operators, fuzzy ideal with operators, anti-fuzzy ideal, and homomorphism.*

I. Introduction

In 1982 W.J.Liu [1] introduced the concept of fuzzy ring. In 1985 Y.C.Ren [2] established the notion of fuzzy ideal and quotient ring. In this paper we studied the theory of fuzzy ring, and the concept of fuzzy ring with operators, fuzzy ideal and anti-fuzzy ideal with operators, homomorphism and their related elementary properties have discussed.

II. Prliminaries

Definition 2.1:[Liu [1]] Let R be a ring, a fuzzy set A of R is called a fuzzy ring of R if

(i) $A(x - y) \geq \min(A(x), A(y))$, for all x, y in R
(ii) $A(x\, y) \geq \min(A(x), A(y))$, for all x, y in R

Definition 2.2: [Liu [1]] Let R be a ring, a fuzzy ring A of R is called a ring with operator (read as M – fuzzy ring) iff for any $t \in [0,1]$, A_t is a ring with operator of R (i.e M – subring of R), when $A_t \neq \emptyset$.
Where $A_t = \{x \in R : A(x) \geq t\}$

Definition 2.3: Let A be M – fuzzy ideal of R, is a M – fuzzy subring of R such that
(iii) $A(y + x - y) \geq A(x)$
(iv) $A(x\, y) \geq A(y)$
(v) $A((x + z)y - x\, y) \geq A(z)$
For all $x, y, z \in R$
Note that A is a M – fuzzy left ideal of R if it satisfies (i), (ii), (iii) and (iv), and A is said to be a M – fuzzy right ideal of R, if it satisfies (i), (ii), (iii) and (v).

Definition 2.4: Let R be a ring, a fuzzy set A of R is called anti fuzzy subring of R, if for all $x, y \in R$
(AF_1) $A(x - y) \leq \max\{A(x), A(y)\}$
(AF_2) $A(x\, y) \leq \max\{A(x), A(y)\}$

Definition 2.5: Let R be a ring, a fuzzy ring A of R is called an anti-fuzzy subring with operator (read as anti M- fuzzy subring) iff for any $t \in [0,1]$, A_t is an anti-ring with operator of R (i.e.anti M-fuzzy subring of R), when $A_t \neq \emptyset$.
Where $A_t = \{x \in R : A(x) \leq t\}$

Definition 2.6: Let A be M-fuzzy anti ideal of R, if A is a anti M-fuzzy subring of R such that the following conditions are satisfied
(AF_3) $A(y + x - y) \leq A(x)$
(AF_4) $A(x\, y) \leq A(y)$
(AF_5) $A((x + z)\, y - x\, y) \leq A(z)$
For all $x, y, z \in R$.
Note that A is an anti M-fuzzy left ideal of R if it satisfies (AF_1), (AF_2), (AF_3) and (AF_4), and A is called an anti M-fuzzy right ideal of R if it satisfies (AF_1), (AF_2), (AF_3) and (AF_5).

Example: Let R = { a_1, a_2, a_3, a_4 }, be a set with two binary operations as follows

+	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_3	a_4
a_2	a_2	a_1	a_4	a_3
a_3	a_3	a_4	a_2	a_1
a_4	a_4	a_3	a_1	a_2

x	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_1	a_1
a_4	a_1	a_1	a_2	a_2

Then (R,+ , .) is a ring. We define a fuzzy subset A : R → $[0,1]$, by A(a_3) = A(a_4) > A(a_2) > A(a_1).

Then A is an anti M-fuzzy right (left) ideal of ring R. Every anti M-fuzzy right (left) ideal of ring R is an anti M-fuzzy subring of R, but converse is not true as shown in the following example.

Example: Let R = { a_1, a_2, a_3, a_4 }, be a set with two binary operations as follows

+	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_3	a_4
a_2	a_2	a_1	a_4	a_3
a_3	a_3	a_4	a_2	a_1
a_4	a_4	a_3	a_1	a_2

x	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_1	a_1	a_1
a_4	a_1	a_1	a_1	a_1

Then (R, +, .) is a ring. We define a fuzzy subset A : R → $[0,1]$, by $A(a_3) = A(a_4) > A(a_2) > A(a_1)$. Then A is anti M-fuzzy subring of R. But A is not an anti M-fuzzy right ideal of R, since

A $((a_1 + a_2)\, a_3 - a_1 a_3) = A(a_3) > A(a_2)$

Theorem 2.1: *Let R be a M-ring and a fuzzy set A on R is anti M-fuzzy ring in R if and only if A^c is a M-fuzzy ring in R.*

Proof : First we suppose that Abe an anti M-fuzzy subring in R. Then we have to show thatA^c is an M-fuzzy subring in R.

Let$x, y \in R$, we have

$$A^c(x-y) = 1 - A(x-y)$$
$$A^c(x-y) \geq 1 - \max\{A(x), A(y)\}$$
$$= \min\{1-A(x), 1-A(y)\}$$
$$= \min\{A^c(x), A^c(y)\}$$

Also,
$$A^c(xy) = 1 - A(xy)$$
$$A^c(xy) \geq 1 - \max\{A(x), A(y)\}$$
$$= \min\{1-A(x), 1-A(y)\}$$
$$= \min\{A^c(x), A^c(y)\}$$

Therefore, A^c is a M-fuzzy subring in R.

Conversely, suppose that A^c is an anti M-fuzzy subring in R, then we have to show that A is a M-fuzzy subring in R. Let x, y, $\in R$, we have

$$A(x-y) = 1 - A^c(x-y)$$
$$A(x-y) \geq 1 - \max\{A^c(x), A^c(y)\}$$
$$= \min\{1-A^c(x), 1-A^c(y)\}$$
$$= \min\{A(x), A(y)\}$$

Also,
$$A(xy) = 1 - A^c(xy)$$
$$A(xy) \geq 1 - \max\{A^c(x), A^c(y)\}$$

$$= \min\left\{1 - A^c(x), 1 - A^c(y)\right\}$$
$$= \min\left\{A(x), A(y)\right\}$$

Thus A is M-fuzzy subring in R.

Theorem 2.2: *Let R be a M- ring and A be a M-fuzzy subring in R .Then A is an anti M-fuzzy ideal in R if and only if A^c is a M-fuzzy ideal in R.*

Proof : First we suppose that A is anti M-fuzzy ideal in R, then we have to show that A^c is a M-fuzzy ideal in R. Let x, y,∈ R, by definition of anti M-fuzzy ideal

$$A(x+y-x) \le A(x) \text{ Since } A_t = \left\{x \in R : A(x) \le t\right\}, \text{ where } t \in [0,1]$$
$$\Rightarrow 1 - A(x+y-x) \ge 1 - A(x)$$
$$\Rightarrow A^c(x+y-x) \ge A^c(x)$$

Also, $A(xy) \le A(y)$

$$\Rightarrow 1 - A(xy) \ge 1 - A(y)$$
$$\Rightarrow A^c(xy) \ge A^c(y)$$

And $A((x+z)y - xy) \le A(z)$

$$\Rightarrow 1 - A((x+z)y - xy) \ge 1 - A(z)$$
$$\Rightarrow A^c((x+z)y - xy) \ge A^c(z)$$

Hence, A^c satisfies all the condition of M-fuzzy ideal.

Conversely, if we suppose that A^c is an M-fuzzy ideal in R, then we have to show that A is anti M-fuzzy ideal in R.

Let $x, y \in R$ and A^c is an M-fuzzy ideal in R, then by the definition of M-fuzzy ideal we have

$$A^c(x+y-x) \ge A^c(x) \because A_t = \left\{x \in R : A(x) \ge t\right\},\ t \in [0,1]$$
$$\Rightarrow A(x+y-x) \le A(x)$$

Also, $A^c(xy) \ge A^c(y)$

$$\Rightarrow A(xy) \le A(y)$$

And $A^c((x+z)y - xy) \ge A^c(z)$

$$\Rightarrow A((x+z)y - xy) \le A(z)$$

Hence, A satisfies all the condition of anti M-fuzzy ideal in R.

Definition 2.7*: Let R be a M-ring and a family $\left\{A_i : i \in I\right\}$ of M-fuzzy ideal of R, then the intersection $\underset{i \in I}{\cap} A_i$ of M-fuzzy ideal of R defined by*

$$\left(\underset{i \in I}{\cap} A_i\right)(x) = \inf\left\{A_i(x) : i \in I\right\}, \quad \text{for all } x \in R$$

Theorem 2.3: *Let R be a M-ring, $\left\{A_i : i \in I\right\}$ be a family of M-fuzzy ideal of R, then $\underset{i \in I}{\cap} A_i$ is an M-fuzzy ideal of R.*

Proof: Let $\left\{A_i : i \in I\right\}$ be a family of M-fuzzy ideal of R, and $x, y \in R$ then we have

$$\left(\underset{i \in I}{\cap} A_i\right)(x-y) = \inf\left\{A_i(x-y) : i \in I\right\}, \quad \text{for all } x, y \in R$$
$$\ge \inf\left\{\min\left(A_i(x), A_i(y)\right), i \in I\right\}$$

$$= \min\left\{\inf\left(A_i(x): i \in I\right)\inf\left(A_i(y): i \in I\right)\right\}$$

$$= \min\left\{\left(\bigcap_{i\in I} A_i\right)(x), \left(\bigcap_{i\in I} A_i\right)(y)\right\}$$

Also, $\left(\bigcap_{i\in I} A_i\right)(xy) = \inf\left\{A_i(xy): i \in I\right\}$

$$\geq \inf\left\{\min\left(A_i(x), A_i(y)\right): i \in I\right\}, \text{ for all } x, y \in R$$

$$= \min\left\{\inf\left(A_i(x): i \in I\right), \inf\left(A_i(y): i \in I\right)\right\}$$

$$= \min\left\{\left(\bigcap_{i\in I} A_i\right)(x), \left(\bigcap_{i\in I} A_i\right)(y)\right\}$$

Hence $\bigcap_{i\in I} A_i$ is a M-fuzzy ring of R.

For any $x, y, z \in R$ we have

$$\left(\bigcap_{i\in I} A_i\right)(y + x - y) = \inf\left\{A_i(y + x - y): i \in I\right\}$$

$$\geq \inf\left\{A_i(x): i \in I\right\}$$

$$= \left(\bigcap_{i\in I} A_i\right)(x)$$

Also, $\left(\bigcap_{i\in I} A_i\right)(xy) = \inf\left\{A_i(xy): i \in I\right\}$

$$\left(\bigcap_{i\in I} A_i\right)(xy) \geq \inf\left\{A_i(y): i \in I\right\}$$

$$= \left(\bigcap_{i\in I} A_i\right)(y)$$

And $\left(\bigcap_{i\in I} A_i\right)\left((x+z)y - xy\right) = \inf\left\{A_i\left((x+z)y - xy\right): i \in I\right\}$

$$\geq \inf\left\{A_i(z): i \in I\right\}$$

$$= \left(\bigcap_{i\in I} A_i\right)(z)$$

Hence $\bigcap_{i\in I} A_i$ is a M-fuzzy ideal of R.

Definition 2.8: *Let R be a M-ring and* $\{A_i : i \in I\}$ *be a family of anti M-fuzzy ideal of R, then the union* $\bigcup_{i\in I} A_i$ *of anti M-fuzzy ideal of R defined by*

$$\left(\bigcup_{i\in I} A_i\right)(x) = \sup\left\{A_i(x): i \in I\right\}, \text{ for all } x \in R$$

Theorem 2.4: *Let R be a M-ring, and* $\{A_i : i \in I\}$ *be a family of anti M-fuzzy ideal of R, then* $\bigcup_{i\in I} A_i$ *is an anti M-fuzzy ideal of R.*

Proof : Let $\{A_i : i \in I\}$ be a family of anti M-fuzzy ideal of R, and $x, y \in R$,then we have

$$\left(\bigcup_{i\in I} A_i\right)(x - y) = \sup\left\{A_i(x - y): i \in I\right\}$$

$$\leq \sup\left\{\max\left(A_i(x), A_i(y)\right): i \in I\right\}$$

$$= \max\left\{\sup\left(A_i(x): i \in I\right), \sup\left(A_i(y): i \in I\right)\right\}$$

$$= \max\left\{\left(\bigcup_{i\in I} A_i\right)(x), \left(\bigcup_{i\in I} A_i\right)(y)\right\}$$

Also $\left(\bigcup_{i\in I} A_i\right)(xy) = \sup\left\{A_i(xy) : i \in I\right\}$

$$\leq \sup\left\{\max\left(A_i(x), A_i(y)\right) : i \in I\right\}$$

$$= \max\left\{\sup\left(A_i(x) : i \in I\right), \sup\left(A_i(y) : i \in I\right)\right\}$$

$$= \max\left\{\left(\bigcup_{i\in I} A_i\right)(x), \left(\bigcup_{i\in I} A_i\right)(y)\right\}, \text{ for all } x, y \in R$$

Hence, $\bigcup_{i\in I} A_i$ is a anti M-fuzzy ring of R.

For any $x, y, z \in R$, we have

$$\left(\bigcup_{i\in I} A_i\right)(y + x - y) = \sup\left\{A_i(y + x - y) : i \in I\right\}$$

$$\leq \sup\left\{A_i(x) : i \in I\right\}$$

$$= \left(\bigcup_{i\in I} A_i\right)(x)$$

And $\left(\bigcup_{i\in I} A_i\right)(xy) = \sup\left\{A_i(xy) : i \in I\right\}$

$$\leq \sup\left\{A_i(y) : i \in I\right\}$$

$$= \left(\bigcup_{i\in I} A_i\right)(y)$$

Similarly, $\left(\bigcup_{i\in I} A_i\right)\left((x+z)y - xy\right) = \sup\left\{A_i\left((x+z)y - xy\right) : i \in I\right\}$

$$\leq \sup\left\{A_i(z) : i \in I\right\}$$

$$= \left(\bigcup_{i\in I} A_i\right)(z)$$

Therefore, $\bigcup_{i\in I} A_i$ is an anti M-fuzzy ideal of R

Definition 2.9: Let R and S be two M-ring and f is a function from R onto S

(i) If B is a M-fuzzy ring in S, then the pre-image of B under f is a M-fuzzy ring in R, defined by $f^{-1}(B)(x) = B\left(f(x)\right)$, for each $x \in R$

(ii) If A is a M-fuzzy ring of R, then the image of A under f is a M-fuzzy ring in S defined by

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x), & if \quad f^{-1}(y) \neq \phi \\ = 0, & otherwise \end{cases}$$

For each $y \in S$

Theorem 2.5: *Let R and S be M-ring and* $f : R \to S$ *,be an M-homomorphism from R onto S and*

(i) If B is an M-fuzzy ring of S, then $f^{-1}(B)$ *is an M-fuzzy ring of R.*

(ii) If A is an M-fuzzy ring of R, then $f(A)$ *is an M-fuzzy ring of S.*

Proof: (i) Let $x_1, x_2 \in R$, then we have

$$f^{-1}(B)(x_1 - x_2) = B\left(f(x_1) - f(x_2)\right)$$

$$\geq \min\left\{B\left(f(x_1)\right), B\left(f(x_2)\right)\right\}$$

$$= \min\left\{f^{-1}(B)(x_1), f^{-1}(B)(x_2)\right\}$$

And $f^{-1}(B)(x_1.x_2) = B(f(x_1).f(x_2))$

$$\geq \min\{B(f(x_1)), B(f(x_2))\}$$

$$= \min\{f^{-1}(B)(x_1), f^{-1}(B)(x_2)\}$$

Hence, $f^{-1}(B)$ is a M-fuzzy ring of R.

(ii) Let $y_1, y_2 \in S$ then we have

$$\{x : x \in f^{-1}(y_1 - y_2)\} \supseteq \{x_1 - x_2 : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$

Hence $f(A)(y_1 - y_2) = \sup\{A(x) : x \in f^{-1}(y_1 - y_2)\}$

$$\geq \sup\{A(x_1 - x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$

$$\geq \sup\{\min(A(x_1), A(x_2)) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$

$$= \min\{\sup(A(x_1)) : x_1 \in f^{-1}(y_1), \sup(A(x_2)) : x_2 \in f^{-1}(y_2)\}$$

$$= \min\{f(A)(y_1), f(A)(y_2)\}$$

Since $\{x : x \in f^{-1}(y_1.y_2)\} \supseteq \{x_1.x_2 : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$$f(A(y_1.y_2)) = \sup\{A(x) : x \in f^{-1}(y_1.y_2)\}$$

$$\geq \sup\{A(x_1.x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$

$$\geq \sup\{\min(A(x_1), A(x_2)) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$

$$= \min\{\sup(A(x_1) : x_1 \in f^{-1}(y_1)), \sup(A(x_2) : x_2 \in f^{-1}(y_2))\}$$

$$= \min\{f(A(y_1)), f(A(y_2))\}$$

Hence $f(A)$ is a M-fuzzy ring of S

Theorem 2.6 :*Let R and S be M-ring and* $f : R \to S$ *be an M-homomorphism from R onto S and*

I. *If B is an M-fuzzy ideal of S, then* $f^{-1}(B)$ *is an M-fuzzy ideal of R*

II. *If A is an M-fuzzy ideal of R, then* $f(A)$ *is an M-fuzzy ideal of S*

Proof: (i) Let B be an M-fuzzy ideal of S, then $f^{-1}(B)$ is an M-fuzzy ring of R from theorem 2.5(i)

Let $x_1, x_2, x_3 \in R$ we have

$$f^{-1}(B)(x_1 + x_2 - x_1) = B(f(x_1) + f(x_2) - f(x_1))$$

$$\geq B(f(x_2))$$

$$= f^{-1}(B)(x_2)$$

Also, $f^{-1}(B(x_1.x_2)) = B(f(x_1.x_2))$

$$\geq B(f(x_2))$$

$$= f^{-1}(B)(x_2)$$

And $f^{-1}(B)((x_1 + x_2)x_3 - x_1x_3) = B((f(x_1) + f(x_2))f(x_3) - f(x_1)f(x_3))$

$$\geq B(f(x_2))$$

$$= f^{-1}(B)(x_2)$$

Hence, $f^{-1}(B)$ is an M-fuzzy ideal of R.

(II) Let A be an M-fuzzy ideal of R, then $f(A)$ is an M-fuzzy ring of S from theorem 2.5(ii).Now for any $y_1, y_2, y_3 \in S$ we have

$$\begin{aligned} f(A)(y_1+y_2-y_1) &= \sup\{A(x) : x \in f^{-1}(y_1+y_2-y_1)\} \\ &\geq \sup\{A(x_1+x_2-x_1) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{A(x_2) : x_2 \in f^{-1}(y_2)\} \\ &= f(A)(y_2) \end{aligned}$$

Also

$$\begin{aligned} f(A)(y_1.y_2) &= \sup\{A(x) : x \in f^{-1}(y_1.y_2)\} \\ &\geq \sup\{A(x_1.x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{A(x_2) : x_2 \in f^{-1}(y_2)\} \\ &= f(A)(y_2) \end{aligned}$$

And $$f(A)((y_1+y_2)y_3-y_1y_3) = \sup\{A(x) : x \in f^{-1}((y_1+y_2)y_3-y_1y_3)\}$$

$$f(A)((y_1+y_2)y_3-y_1y_3) \geq \sup\{A((x_1+x_2)x_3-x_1x_3) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2), x_3 \in f^{-1}(y_3)\}$$

$$\begin{aligned} &\geq \sup\{A(x_2) : x_2 \in f^{-1}(y_2)\} \\ &= f(A)(y_2) \end{aligned}$$

Hence $f(A)$ is an M-fuzzy ideal of S.

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