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Some Fixed Point Theorems of Contractive Mappings in Complete G-Metric Space

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Abstract : *In this paper, we prove some fixed point theorems in complete G-Metric Space for self mapping satisfying various contractive conditions. We also discuss that these mapping are G- continuous on such a fixed point.*

Keywords: ***G****-Metric Spaces, Fixed Point, G-convergent***.**

I. Introduction

Some generalizations of the notion of a metric space have been proposed by some authors. Gahler [1, 2] coined the term of 2-metric spaces. This is extended to D-metric space by Dhage (1992) [3, 4]. Dhage proved many fixed point theorems in D-metric space. Recently, Mustafa and Sims [7] showed that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, in 2006 they introduced a new notion of generalized metric space called G-metric space [5]. In fact, Mustafa et al. studied many fixed point results for a self mapping in G-metric spaces under certain conditions; see [5, 6, 7, 8, 9 and 10].

Now, we give preliminaries and basic definitions which are used throughout the paper.

II. Definitions and Preliminaries

Definition 2.1 [5] Let X be a non empty set, and let $G: X \times X \times X \to [0,\infty)$ be a function satisfying the following axioms

$(G1)$ $G(x,y,z) = 0$ if $x = y = z$,
$(G2)$ $G(x,x,y) > 0$ for all $x,y \in X$, with $x \neq y$.
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$, with $y \neq z$.
$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables)
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$ (rectangular inequality)

Then the function G is called a generalized metric, or more specially a G-metric on X, and the pair (X,G) is called a G-metric space.

Example: Let (X,d) be a usual metric space. Then (X,G_s) and (X,G_m) are G-metric spaces, where

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z) \text{ for all } x,y,z \in X$$

and

$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\} \text{ for all } x,y,z \in X.$$

Definition 2.2 [5] Let (X,G) and (X',G') be G-metric spaces and let $f:(X,G) \to (X',G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\varepsilon > 0$ there exist $\delta > 0$ such that $x,y \in X, G(a,x,y) < \delta$ implies that $G'(fa,fx,fy) < \varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 2.3 [5] Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore; we say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$; that is ,for any $\varepsilon > 0$, there exist N$\in N$ such that $G(x,x_n,x_m) < \varepsilon$ for all $n.m \geq$N.We call $x$is the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.4 [5] Let (X,G) and (X',G') be G metric spaces, then a function $f: X \to X$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous, that is, whenever $\{x_n\}$ is G-convergent to x, $\{fx_n\}$ is G-convergent to $f(x)$.

Proposition 2.5 [5] Let (X,G) be a G-metric space. Then the following statements are equivalent

(a) $\{x_n\}$ is G-convergent to x.
(b) $G(x_n,x_n,x) \to 0$ as $n \to \infty$.
(c) $G(x_n,x,x) \to 0$ as $n \to \infty$.
(d) $G(x_n,x_m,x) \to 0$ as $n \to \infty$.

Proposition 2.6 [5] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy sequence if given $\varepsilon > 0$, there is N$\in N$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq$N; that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2.7 [5] In a G-metric space (X,G), the following two statements are equivalent.

(1) The sequence $\{x_n\}$ is G-cauchy.
(2) For every $\varepsilon > 0$, there exist $N \in N$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$.

Definition 2.8 [5] A G-metric space (X, G) is said to be G-complete (or a complete G-metric pace) if every G-cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.9 [5] Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 2.10 [5] A G-metric space $(X, G)$is called a symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \text{ for all } x, y \in X.$$

Proposition 2.11 [5] Every G-metric space (X, G) defines a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Note that, if (X, G) is a symmetric space G-metric space, then

$$d_G(x, y) = 2\, G(x, y, y) \text{ for all } x, y \in X.$$

However, if (X, G) is not asymmetric space, then it holds by the G-metric properties that

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \text{ for all } x, y \in X.$$

In general, these inequalities cannot be improved.

Proposition 2.12 [5] A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.

Proposition 2.13 [5] Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that

(1) If $G(x, y, z) = 0$ then $x = y = z$.
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.
(3) $G(x, y, y) \leq 2\, G(y, x, x)$.
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.
(5) $G(x, y, z) \leq \frac{2}{3}\{G(x, a, a) + G(y, a, a) + G(z, a, a)\}$.

III. MAIN RESULTS

Theorem 3.1 Let (X, G) be a complete G-metric space and let $T: X \to X$ be a mapping which satisfies the following condition, for all $x, y, z \in X$,

$$G(Tx, Ty, Tz) \leq k\, max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), G(x, Ty, Ty), G(y, Tz, Tz), G(z, Tx, Tx), G(x, Ty, Ty), G(y, Tz, Tz), G(z, Tx, Tx)\}, \quad (3.1)$$

Where $k \in [0, \frac{1}{2})$, then T has a unique fixed point (say u) and T is G-continuous at u.

Proof: Suppose that T satisfies condition (1), let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_n = T^n x_0$, that is

$$x_1 = T^1 x_0 = Tx_0,$$
$$x_2 = T^2 x_0 = T(Tx_0) = Tx_1,$$
$$x_3 = T^3 x_0 = T(T^2 x_0) = Tx_2,$$

$$x_n = Tx_{n-1}, \quad x_{n+1} = Tx_n .$$

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

Then by (3.1), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n), G(x_{n-1}, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1}), G(x_{n-1}, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n)\}.$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}.$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}). \quad (3.2)$$

So, by $G(5)$ we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}.$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\},$$

$G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n) + k\, G(x_n, x_{n+1}, x_{n+1}),$

$(1-k)\, G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n)$

This implies

$G(x_n, x_{n+1}, x_{n+1}) \leq \left(\frac{k}{k-1}\right) G(x_{n-1}, x_n, x_n),$

Let $q = \left(\frac{k}{k-1}\right), q < 1$ as $0 \leq k < \frac{1}{2}$, we obtain

$G(x_n, x_{n+1}, x_{n+1}) \leq q\, G(x_{n-1}, x_n, x_n),$ (3.3)

Repeated application of inequality (3.3), we obtain

$G(x_n, x_{n+1}, x_{n+1}) \leq q^n\, G(x_0, x_1, x_1)$ (3.4)

Then, for all $m, n \in N, m > n$, we have by repeated use of rectangular inequality $(G5)$,

$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + - - - - - - + G(x_{m-1}, x_m, x_m)$

By (3.3), we get,

$G(x_n, x_m, x_m) \leq q^n G(x_0, x_1, x_1) + q^{n+1} G(x_0, x_1, x_1) + - - - - - - - + q^{m-1} G(x_0, x_1, x_1)$

$G(x_n, x_m, x_m) \leq (q^n + q^{n+1} + - - - - - - - + q^{m-1})\, G(x_0, x_1, x_1).$

$G(x_n, x_m, x_m) \leq q^n (1 + q + q^2 + \cdots)\, G(x_0, x_1, x_1)$

$G(x_n, x_m, x_m) \leq \frac{q^n}{1-q}\, G(x_0, x_1, x_1)$ (3.5)

Then $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.

For $n, m, l \in N$, by rectangular inequality of G-metric space implies that

$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l)$

Taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$.

So $\{x_n\}$ is G-cauchy sequence. By completeness of (X, G), there exist $u \in X$ such that $\{x_n\}$ is G-converges to u.

Suppose that $Tu \neq u$, then

$$G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$$

by (3.1), we have

$G(x_n, Tu, Tu) \leq k \max\{G(x_{n-1}, u, u), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(u, Tu, Tu), G(u, Tu.Tu),$
$G(x_{n-1}, Tu, Tu), G(u, Tu, Tu), G(u, Tx_{n-1}, Tx_{n-1}),$
$G(x_{n-1}, Tu, Tu), G(u, Tx_{n-1}, Tx_{n-1}), G(u, Tu, Tu)\}.$

$G(x_n, Tu, Tu) \leq k \max\{G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u, Tu, Tu), G(u, Tu.Tu),$
$G(x_{n-1}, Tu, Tu), G(u, Tu, Tu), G(u, x_n, x_n),$
$G(x_{n-1}, Tu, Tu), G(u, x_n, x_n), G(u, Tu, Tu)\}.$

$G(x_n, Tu, Tu) \leq k \max\{G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u, Tu, Tu), G(x_{n-1}, Tu, Tu), G(u, x_n, x_n)\}.$ (3.6)

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variables, we have,

$G(u, Tu, Tu) \leq k \max\{G(u, u, u), G(u, u, u), G(u, Tu, Tu), G(u, Tu.Tu), G(u, u, u)\},$

$G(u, Tu, Tu) \leq k \max\{0,0, G(u, Tu, Tu), G(u, Tu.Tu), 0\},$

$G(u, Tu, Tu) \leq k\, G(u, Tu, Tu).$

This is a contradiction since $0 \leq k < \frac{1}{2}$

So $Tu = u.$ (3.7)

That is u is a fixe point of T.

To prove uniqueness of the fixed point, suppose that v is another fixed point,

That is $Tv = v$ (3.8)

$G(u, v, v) = G(Tu, Tv, Tv),$

Then from (3.1)

$G(u, v, v) \leq k \max\{G(u, v, v), G(u, Tu, Tu), G(v, Tv, Tv), G(v, Tv, Tv), G(u, Tv, Tv), G(v, Tv, Tv),$
$G(v, Tu, Tu), G(u, Tv, Tv), G(v, Tu, Tu), G(v, Tv, Tv)\},$

$G(u, v, v) \leq k \max\{G(u, v, v), G(u, u, u), G(v, v, v), G(v, v, v), G(u, v, v), G(v, v, v),$
$G(v, u, u), G(u, v, v), G(v, u, u), G(v, v, v)\},$

$G(u, v, v) \leq k \max\{G(u, v, v), 0,0,0, G(u, v, v), 0, G(v, u, u), G(u, v, v), G(v, u, u), 0\},$

$G(u, v, v) \leq k \max\{G(u, v, v), G(v, u, u)\},$

So $G(u, v, v) \leq k\, G(v, u, u).$ (3.9)

Again by the same argument we will find $G(v, u, u) \leq k\, G(u, v, v)$ (3.10)

Substitute (3.10) in (3.9), we obtain

$G(u, v, v) \leq k.k\, G(u, v, v),$

$G(u, v, v) \leq k^2\, G(u, v, v).$

Which is contradiction since $0 \leq k^2 < \frac{1}{4}$.

Therefore u is unique fixed point of T.

Now, we show that, T is G-continuous at u. Let $\{y_n\}$ be a sequence in X, by completeness of X, the sequence $\{y_n\}$ converges to u in X.

$$\text{That is } \lim_{n\to\infty} y_n = u \tag{3.11}$$

$$G(Tu,Ty_n,Ty_n) \le k \max\{G(u,y_n,y_n), G(u,Tu,Tu), G(y_n,Ty_n,Ty_n), G(y_n,Ty_n,Ty_n), G(u,Ty_n,Ty_n), G(y_n,Ty_n,Ty_n), G(y_n,Tu,Tu), G(u,Ty_n,Ty_n), G(y_n,Tu,Tu), G(y_n,Ty_n,Ty_n)\},$$

$$G(u,Ty_n,Ty_n) \le k \max\{G(u,y_n,y_n), G(u,u,u), G(y_n,Ty_n,Ty_n), G(y_n,Ty_n,Ty_n), G(u,Ty_n,Ty_n), G(y_n,Ty_n,Ty_n), G(y_n,u,u), G(u,Ty_n,Ty_n), G(y_n,u,u), G(y_n,Ty_n,Ty_n)\},$$

$$G(u,Ty_n,Ty_n) \le k \max\{G(u,y_n,y_n), G(u,u,u), G(y_n,Ty_n,Ty_n), G(u,Ty_n,Ty_n), G(y_n,u,u)\},$$

$$G(u,Ty_n,Ty_n) \le k \max\{G(u,y_n,y_n), G(y_n,Ty_n,Ty_n), G(u,Ty_n,Ty_n), G(y_n,u,u)\},$$

by $(G5)$, we obtain

$$G(u,Ty_n,Ty_n) \le k \max\{G(u,y_n,y_n), G(y_n,u,u) + G(u,Ty_n,Ty_n), G(u,Ty_n,Ty_n), G(y_n,u,u)\},$$

$$G(u,Ty_n,Ty_n) \le k \max\{G(u,y_n,y_n), G(y_n,u,u) + G(u,Ty_n,Ty_n)\}, \tag{3.12}$$

And it leads the following two cases,

$$G(u,Ty_n,Ty_n) \le k\, G(u,y_n,y_n) \tag{3.12a}$$

$$G(u,Ty_n,Ty_n) \le k\,\{G(y_n,u,u) + G(u,Ty_n,Ty_n)\},$$

$$G(u,Ty_n,Ty_n) \le k\, G(y_n,u,u) + k\, G(u,Ty_n,Ty_n),$$

$$(1-k)\, G(u,Ty_n,Ty_n) \le k\, G(y_n,u,u),$$

$$G(u,Ty_n,Ty_n) \le \frac{k}{1-k} G(y_n,u,u) \tag{3.12b}$$

In all two cases [3.12(a) and 3.12(b)], letting $n \to \infty$, we obtain $G(u,Ty_n,Ty_n) \to 0$ as $n \to \infty$, and so the sequence $\{Ty_n\}$ is G-convergent to $u = Tu$.

This implies that T is G-continuous at u.

Remark 3.2: if the G-metric space is bounded (that is, for some $L > 0$ we have $G(x,y,z) \le L$ for all $x,y,z \in X$) then an argument similar to that used above establishes the result for $0 \le k < 1$.

Corollary 3.3 let (X,G) be a complete G-metric space and let $T\colon X \to X$ be a mapping which satisfies the following condition for some $m \in N$ and for all $x,y,z \in X$

$$G(T^m x, T^m y, T^m z) \le k \max\{G(x,y,z), G(x,T^m x,T^m x), G(y,T^m y,T^m y), G(z,T^m z,T^m z), G(x,T^m y,T^m y), G(y,T^m z,T^m z), G(z,T^m x,T^m x), G(x,T^m z,T^m z), G(y,T^m x,T^m x), G(z,T^m y,T^m y)\} \tag{3.13}$$

Where $0 \le k < \frac{1}{2}$, then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof: Given that $T\colon X \to X$ is self mapping, then for all $m \in N$, $T^m\colon X \to X$.

Therefore (X,G) be a complete G-metric space and $T^m\colon X \to X$ be a mapping which satisfies the given condition (3.13), then by theorem 3.1, T^m has a unique fixed point (say u), and T^m is G-continuous.

Now we shall prove that u is a unique fixed point T^m.

Consider $Tu = T(T^m u) = T^{m+1} u = T^m(Tu)$

Therefore $T^m(Tu) = Tu$, Tu is a fixed point of T^m.

Since T^m has a unique fixed u, $Tu = u$

Therefore u is a unique fixed point of T^m.

Theorem 3.4 let (X,G) be a complete G-metric space and $T\colon X \to X$ be a mapping which satisfies the following condition for all $x,y,z \in X$,

$$G(Tx,Ty,Tz) \le k \max\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz), G(x,y,z) + G(x,Ty,Ty) + G(y,Tx,Tx), G(x,y,z) + G(y,Tz,Tz) + G(z,Ty,Ty), G(x,y,z) + G(x,Tz,Tz) + G(z,Tx,Tx)\}. \tag{3.14}$$

Where $0 \le k < \frac{1}{4}$, then T has a unique fixed point (say$u$), and T is G-continuous at u.

Proof: suppose that T satisfies the condition (3.14), let $x_0 \in X$ be an ordinary point, and define the sequence $\{x_n\}$ by $x_n = T^n x_0$, then by (3.14)

$$G(x_n,x_{n+1},x_{n+1}) = G(Tx_{n-1},Tx_n,Tx_n)$$

$$G(x_n,x_{n+1},x_{n+1}) \le k \max\{G(x_{n-1},Tx_{n-1},Tx_{n-1}) + G(x_n,Tx_n,Tx_n) + G(x_n,Tx_n,Tx_n), G(x_{n-1},x_n,x_n) + G(x_{n-1},Tx_n,Tx_n) + G(x_n,Tx_{n-1},Tx_{n-1}), G(x_{n-1},x_n,x_n) + G(x_n,Tx_n,Tx_n) + G(x_n,Tx_n,Tx_n), G(x_{n-1},x_n,x_n) + G(x_{n-1},Tx_n,Tx_n) + G(x_n,Tx_{n-1},Tx_{n-1})\}.$$

$$G(x_n,x_{n+1},x_{n+1}) \le k \max\{G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}) + G(x_n,x_{n+1},x_{n+1}), G(x_{n-1},x_n,x_n) + G(x_{n-1},x_{n+1},x_{n+1}) + G(x_n,x_n,x_n), G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}) + G(x_n,x_{n+1},x_{n+1}),$$

$G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\}.$

$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n) + 2\, G(x_n, x_{n+1}, x_{n+1}),$
$G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1}),$
$G(x_{n-1}, x_n, x_n) + 2\, G(x_n, x_{n+1}, x_{n+1}),$
$G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})\}.$

$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n) + 2\, G(x_n, x_{n+1}, x_{n+1}),$
$G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})\}.$

$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n) + 2\, G(x_n, x_{n+1}, x_{n+1}),$
$G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}.$

$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n) + 2\, G(x_n, x_{n+1}, x_{n+1}),$
$2\, G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}.$ (3.15)

Inequality (3.15) leads to the following two cases

$G(x_n, x_{n+1}, x_{n+1}) \leq k\,\{G(x_{n-1}, x_n, x_n) + 2\, G(x_n, x_{n+1}, x_{n+1})\},$
$G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n) + 2k\, G(x_n, x_{n+1}, x_{n+1}),$
$(1-2k)\, G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n),$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{1-2k} G(x_{n-1}, x_n, x_n) \tag{3.15a}$$

$G(x_n, x_{n+1}, x_{n+1}) \leq k\{2\, G(x_{n-1}, x_n, x_n) + k\, G(x_n, x_{n+1}, x_{n+1})\},$
$G(x_n, x_{n+1}, x_{n+1}) \leq 2k\, G(x_{n-1}, x_n, x_n) + k\, G(x_n, x_{n+1}, x_{n+1}),$
$(1-k)\, G(x_n, x_{n+1}, x_{n+1}) \leq 2kG(x_{n-1}, x_n, x_n)\,,$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{2k}{1-k}\, G(x_{n-1}, x_n, x_n) \tag{3.15b}$$

Let $q = \left\{\frac{k}{1-2k}\ or\ \frac{2k}{1-k}\right\}$, ($q < 1$ since $0 \leq k < \frac{1}{4}$), from (3.15a) & (3.15b), we obtain that

$$G(x_n, x_{n+1}, x_{n+1}) \leq q\, G(x_{n-1}, x_n, x_n) \qquad (q < 1) \tag{3.16}$$

By repeated application of (3.16), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n\, G(x_0, x_1, x_1) \tag{3.17}$$

For all $m, n \in N$, $m > n$ and repeated use of rectangular inequality($G5$), we obtain

$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + - - - - + G(x_{m-1}, x_m, x_m).$
$G(x_n, x_m, x_m) \leq (q^n + q^{n+1} + q^{n+2} + - - - - - - + q^{m-1})\, G(x_0, x_1, x_1),$

$$G(x_n, x_m, x_m) \leq \frac{q^n}{q-1}\, G(x_0, x_1, x_1) \tag{3.18}$$

Letting $m, n \to \infty$ on both sides in (18), we obtain

$$G(x_n, x_m, x_m) \to 0 \text{ as } m, n \to \infty \tag{3.19}$$

For $n, m, l \in N$, by rectangular inequality of G-metric space implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l)$$

Taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$.

From (3.19) $\{x_n\}$ is G-cauchy sequence. By the completeness of (X, G), there exist $u \in X$ such that $\{x_n\}$ is G-convergent to u.

Suppose that $Tu \neq u$, then

$$G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$$

Then by (3.14), we have,

$G(x_n, Tu, Tu) \leq k \max\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(u, Tu, Tu) + G(u, Tu, Tu),$
$G(x_{n-1}, u, u) + G(x_{n-1}, Tu, Tu) + G(u, Tx_{n-1}, Tx_{n-1}),$
$G(x_{n-1}, u, u) + G(u, Tu, Tu) + G(u, Tu, Tu),$
$G(x_{n-1}, u, u) + G(x_{n-1}, Tu, Tu) + G(u, Tx_{n-1}, Tx_{n-1})\}.$

$G(x_n, Tu, Tu) \leq k \max\{G(x_{n-1}, x_n, x_n) + G(u, Tu, Tu) + G(u, Tu, Tu),$
$G(x_{n-1}, u, u) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n),$
$G(x_{n-1}, u, u) + G(u, Tu, Tu) + G(u, Tu, Tu),$
$G(x_{n-1}, u, u) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n)\}.$

$G(x_n, Tu, Tu) \leq k \max\{G(x_{n-1}, x_n, x_n) + 2\, G(u, Tu, Tu),$
$G(x_{n-1}, u, u) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n),$
$G(x_{n-1}, u, u) + 2\, G(u, Tu, Tu),$
$G(x_{n-1}, u, u) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n)\}.$ (3.20)

Taking the limit as $n \to \infty$ in (3.20), and using the fact that G is continuous in its variables, we get

$G(u, Tu, Tu) \leq k \max\{G(u, u, u) + 2\, G(u, Tu, Tu),$

$$G(u,u,u) + G(u,Tu,Tu) + G(u,u,u),$$
$$G(u,u,u) + 2\,G(u,Tu,Tu),$$
$$G(u,u,u) + G(u,Tu,Tu) + G(u,u,u)\}.$$
$$G(u,Tu,Tu) \le k\max\{2\,G(u,Tu,Tu), G(u,Tu,Tu), 2\,G(u,Tu,Tu), G(u,Tu,Tu)\},$$
$$G(u,Tu,Tu) \le 2k\,G(u,Tu,Tu) \quad (3.21)$$

The inequality (21) is contradiction since $2k < 1$. This implies that $Tu = u$.
Therefore u is a fixed point of T.

To prove the uniqueness of the fixed point, suppose that $v \ne u$ such that $Tv = v$. then
$$G(u,v,v) = G(Tu,Tv,Tv)$$
Then by (3.14),we have,
$$G(u,v,v) \le k\max\{G(u,Tu,Tu) + G(v,Tv,Tv) + G(v,Tv,Tv),$$
$$G(u,v,v) + G(u,Tv,Tv) + G(v,Tu,Tu),$$
$$G(u,v,v) + G(v,Tv,Tv) + G(v,Tv,Tv),$$
$$G(u,v,v) + G(u,Tv,Tv) + G(v,Tu,Tu)\}.$$
$$G(u,v,v) \le k\max\{G(u,u,u) + G(v,v,v) + G(v,v,v),$$
$$G(u,v,v) + G(u,v,v) + G(v,u,u),$$
$$G(u,v,v) + G(v,v,v) + G(v,v,v),$$
$$G(u,v,v) + G(u,v,v) + G(v,u,u)\}.$$
$$G(u,v,v) \le k\max\{0, 2\,G(u,v,v) + G(v,u,u), G(u,v,v), 2\,G(u,v,v) + G(v,u,u)\}.$$
$$G(u,v,v) \le k\,\{2\,G(u,v,v) + G(v,u,u)\},$$
$$G(u,v,v) \le 2k\,G(u,v,v) + k\,G(v,u,u),$$
$$(1-2k)\,G(u,v,v) \le k\,G(v,u,u),$$
$$G(u,v,v) \le \frac{k}{1-2k}\,G(v,u,u), \quad (3.22)$$

By repeated use of same argument in right side of (3.22), we obtain
$$G(u,v,v) \le \left(\frac{k}{1-2k}\right)\left(\frac{k}{1-2k}\right)\,G(u,v,v),$$
$$G(u,v,v) \le \left(\frac{k}{1-2k}\right)^2\,G(u,v,v),$$
$$G(u,v,v) \le q^2\,G(u,v,v), \quad (3.23)$$

The inequality (23) is contradiction since $q^2 < 1, \left(0 \le k < \frac{1}{4}\right)$ this implies that $u = v$. Therefore u is a unique fixed point of T.

Now, we show that, T is G-continuous at u. Let $\{y_n\}$ be a sequence in X, by completeness of X, the sequence $\{y_n\}$ converges to u in X.

That is $\lim_{n\to\infty} y_n = u$ (3.24)

Then by (3.14), we have,
$$G(Ty_n,Tu,Tu) \le k\max\{G(y_n,Ty_n,Ty_n) + G(u,Tu,Tu) + G(u,Tu,Tu),$$
$$G(y_n,u,u) + G(y_n,Tu,Tu) + G(u,Ty_n,Ty_n),$$
$$G(y_n,u,u) + G(u,Tu,Tu) + G(u,Tu,Tu),$$
$$G(y_n,u,u) + G(y_n,Tu,Tu) + G(u,Ty_n,Ty_n)\}.$$

$$G(Ty_n,u,u) \le k\max\{G(y_n,Ty_n,Ty_n) + G(u,u,u) + G(u,u,u),$$
$$G(y_n,u,u) + G(y_n,u,u) + G(u,Ty_n,Ty_n),$$
$$G(y_n,u,u) + G(u,u,u) + G(u,u,u),$$
$$G(y_n,u,u) + G(y_n,u,u) + G(u,Ty_n,Ty_n)\}.$$
$$G(Ty_n,u,u) \le k\max\{G(y_n,Ty_n,Ty_n), 2G(y_n,u,u) + G(u,Ty_n,Ty_n),\ G(y_n,u,u),$$
$$2G(y_n,u,u) + G(u,Ty_n,Ty_n)\}.$$
$$G(Ty_n,u,u) \le k\max\{G(y_n,Ty_n,Ty_n), 2G(y_n,u,u) + G(u,Ty_n,Ty_n)\}.$$
by $(G5)$ of definition 2.1,
$$G(Ty_n,u,u) \le k\max\{G(y_n,u,u) + G(u,Ty_n,Ty_n), 2G(y_n,u,u) + G(u,Ty_n,Ty_n)\}.$$
$$G(Ty_n,u,u) \le k\{2G(y_n,u,u) + G(u,Ty_n,Ty_n)\}.$$
$$G(Ty_n,u,u) \le k\{2\,G(y_n,u,u) +\ 2\,G(Ty_n,u,u)\} \quad (\text{Since } G(x,y,y) \le 2\,G(y,x,x)).$$
$$G(Ty_n,u,u) \le 2k\,G(y_n,u,u) + 2k\,G(Ty_n,u,u),$$
$$(1-2k)\,G(Ty_n,u,u) \le 2k\,G(y_n,u,u),$$
$$G(Ty_n,u,u) \le \frac{2k}{1-2k}G(y_n,u,u)$$
$$G(Ty_n,u,u) \le q\,G(y_n,u,u)\ \ \left(where\ q = \frac{2k}{1-2k}, q < 1, since\ 0 \le k < \frac{1}{4}\right) \quad (3.25)$$

Taking the limit as $n \to \infty$ in (3.25), we obtain that

$G(Ty_n, u, u) \to 0$ as $n \to \infty$.

Therefore $Ty_n \to Tu = u$ as $n \to \infty$.

This implies that T is G-continuous at u.

Corollary 3.5 let (X, G) be a complete G-metric space and let $T: X \to X$ be a self mapping which satisfies the following condition for some $m \in N$ and for all $x, y, z \in X$,

$$G(T^m x, T^m y, T^m z) \le k \max\{G(x, T^m x, T^m x) + G(y, T^m y, T^m y) + G(z, T^m z, T^m z), \\ G(x, y, z) + G(x, T^m y, T^m y) + G(y, T^m x, T^m x), \\ G(x, y, z) + G(y, T^m z, T^m z) + G(z, T^m y, T^m y), \\ G(x, y, z) + G(x, T^m z, T^m z) + G(z, T^m x, T^m x)\}. \quad (3.26)$$

Where $0 \le k < \frac{1}{4}$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof: Given that $T: X \to X$ is self mapping, then for all $m \in N$, $T^m: X \to X$.

Therefore (X, G) be a complete G-metric space and $T^m: X \to X$ be a mapping which satisfies the given condition (26), then by theorem 3.4, T^m has a unique fixed point (say u), and T^m is G-continuous.

Now we shall prove that u is a unique fixed point T^m.

Consider $Tu = T(T^m u) = T^{m+1} u = T^m(Tu)$

Therefore $T^m(Tu) = Tu$, Tu is a fixed point of T^m.

Since T^m has a unique fixed u, $Tu = u$

Therefore u is a unique fixed point of T^m.

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