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Application of Pontryagin's Maximum Principles and Runge-Kutta Methods in Optimal Control Problems

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***Abstract**: In this paper, we examine the application of Pontryagin's maximum principles and Runge-Kutta methods in finding solutions to optimal control problems. We formulated optimal control problems from Geometry, Economics and physics. We employed the Pontryagin's maximum principles in obtaining the analytical solutions to the optimal control problems. We further tested the numerical approach to these optimal control problems using Runge-Kutta methods. The results show that the Runge-Kutta method produced results that are comparable to analytic solutions. Therefore, we concluded that Runge-Kutta method gives error that is negligible.*

I. Introduction

There are choices available for decision making and the ability to pick the best, perfect and desirable way out of the possible alternatives or variables gives us the optimal control. [3], before commencing a search for such an optimal solution, the job must be well-defined; and must possessed the following features

(i) The nature of the system to be controlled,
(ii) The nature of the system constraints and possible alternatives,
(iii) The task to be accomplished,
(iv) The criteria for judging optimal performance.

The optimal control theory is very useful in the following fields, geometry, economics and physics. In geometry, it is interesting to see that by optimal control theory, the geometrical problems such as the problem of finding the shortest path from a given point A to another point B will be solved.

Continuous optimal control models provide a powerful tool for understanding the behavior of production/ inventory system where dynamic aspect plays an important role.

Some optimal control problems that are non-linear do not have solutions analytically. In fact, mostly all problems arise from real life are non- linear. As a result, it is necessary to employ numerical methods to solve optimal control problems. There are so many numerical techniques to optimal control problems like [3], [7], [10], recently contributed to the theory of optimal control. [3] used modified gradient method while [7] compared the Forward Backward Sweep, the Shooter Method, and an Optimization Method using the MATLAB Optimization Tool Box whereas [10] compared Euler, Trapezoidal and Runge-Kutta using Forward Backward Sweep method (FBSM). Looking at the work done by [7] and [10], we find out that both of them were not interested to compare their numerical approximations with analytic solutions and were only interested on problems with final value of the adjoint variable. We intend to give attention to problems with initial value of the state and adjoint variables and then generate the numerical approximations of both the state and adjoint variables forward instead of forward- backward using Runge-Kutta approach.

This work is concerned with the solution of the following three types of problems:

(a) Shortest distance between two points. These type of problems are typically of the form:

$$\text{Minimize} \int_{t_0}^{T} (1 + u^2)^{1/2}\, dt \qquad 1.1$$

subject to $\dot{x} = u$

and $x(t_0) = x_0, \lambda(T)\ free$ [5]

(b) The cocoa problem. This particular problem is of the form:

$$\text{Maximize} \quad P = \int_0^2 (u^2 + 2xu - x^2)dt \qquad 1.2$$

subject to $\dot{x} = -u$

and $x(0) = 1, u(0) = 0$

(c) Pendulum problem: this type of problem requires the description of the mechanical system of a point mass m constrained by a light wire of length l to swing in an arc. It is of the form:

Minimize $$S = \int_{t_1}^{t_2} \left(\frac{1}{2} m l^2 u^2 - m g l (1 - \cos\theta) \right) dt \qquad 1.3$$

subject to $\dot{\theta} = u$

and $\theta(t_1) = \theta_1$ $\theta(T)$ free.

II. Preliminaries

2.1. Derivation Of Euler- Langrage Equation

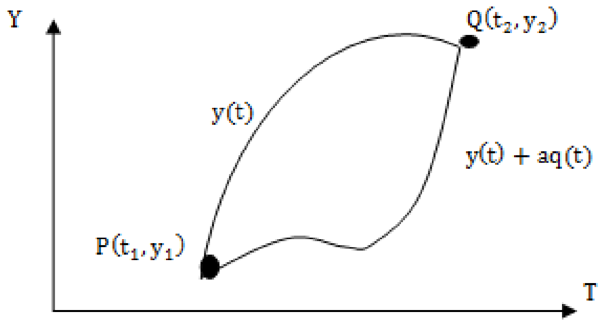
Consider the functional

$$S[y] = \int_{t_1}^{t_2} F(t, y, y')dt \qquad 2.1.1$$

$$W[y] = \int_{t_1}^{t_2} F(t, y, u)dt \qquad 2.1.1a$$

Where in equation (2.1.1) F is a differentiable function of the three variables $F = (t, y, y')$ and $F = F(t, y, u)$ in equation(2.1.1a).

Suppose $y(t)$ is any curve passing through the two points $P(t_1, y_1)$ and $Q(t_2, y_2)$.



Let the original shape of the curve be $y = y(t)$. Suppose there is a small variation due to certain disturbance, the curve changes shape to $y^*(t) = y(t) + aq(t)$, where 'a' is a small parameter, $q(t)$ is an arbitrary function, $y^*(t)$ is the new curve, $q(t_1) = 0$ and $q(t_2) = 0$, implies no variation in t.

This means that

$$y*'(t) = y'(t) + aq'(t)$$

Let the new functional denoted as S*[y]

$$=> S^*[y] = \int_{t_1}^{t_2} F[t, y + aq, y' + aq'] \, dt \qquad 2.1.2$$

Also let the variation on S denoted as δs,

$$=> \delta s = S^* - S = \int_{t_1}^{t_2} \{F[t, y + aq, y' + aq'] - F[t, y, y']\}dt \qquad 2.1.3$$

Since t is fixed, it implies that F varied in two variables y and y'. We recall that Taylor series of two variables is given by:

$$F[t+k, y+l] = F(t,y) + \left(k\frac{\partial}{\partial t} + l\frac{\partial}{\partial y}\right)(F(t,y) + \left(k\frac{\partial}{\partial t} + l\frac{\partial}{\partial y}\right)2\,\frac{F(t,y)}{2!} + \ldots$$

Taking the first order variation, equation (2.1.3) becomes

$$\delta s = a\int_{t_1}^{t_2} \left(q\frac{\partial F}{\partial y} + q'\frac{\partial F}{\partial y'}\right)dt \qquad 2.1.4$$

Where, $k = aq, l = aq'$

[4].For S[y] to be stationary (maximum or minimum),

$$\frac{ds}{da}\Big|_{a=0} = 0$$

We recall that,

$$\lim_{a\to 0}\frac{\partial s}{\partial a} = \frac{ds}{da}\Big|_{a=0} = 0$$

$$=> \int_{t_1}^{t_2}\left(q\frac{\partial F}{\partial t} + q'\frac{\partial F}{\partial y'}\right)dt = 0$$

$$=> \int_{t_1}^{t_2} q\frac{\partial F}{\partial y}dt + \int_{t_1}^{t_2} q'\frac{\partial F}{\partial y'}dt = 0$$

Integrating the second integral by part we have,

$$\int_{t_1}^{t_2} q\frac{\partial F}{\partial y}dt + q\frac{\partial F}{\partial y'} - \int_{t_1}^{t_2} q\frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)dt = 0$$

Thus,

$$\int_{t_1}^{t_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)\right]q(t)dt = 0 \qquad 2.1.5$$

Since $q(t)$ is an arbitrary function, the equation can only be satisfied if

$$\int_{t_1}^{t_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)\right]dt = 0 \qquad 2.1.6$$

Differentiating both sides equation (2.1.6) becomes

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 \qquad 2.1.7$$

If $F = F(t, y, u)$, then (2.1.7) becomes

$$\frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) = 0 \qquad 2.1.8$$

Equations (2.1.7)and (2.1.8) are known as Euler-Lagrange equations.

2.2. Pontryagin's Maximum Principle (Pmp)

This is a powerful method for the computation of optimal controls. It gives the fundamental necessary conditions for a controlled trajectory (x, u) to be optimal. For the solution of optimal control problems, the principal method resolves a set of necessary conditions that an optimal control and the consistent state equation must satisfy. The necessary conditions are derived from Hamiltonian, H, which is define as

$$H(t, x, u, \lambda, \mu_0) = \mu_0 f(t, x, u) + \lambda g(t, x, u)$$

Pontryagin's maximum principle states that: let (x^*, u^*) be a controlled trajectory defined over the interval $[t_0, T]$ with u^* piecewise continuous. If (x^*, u^*) is optimal, then there exist a constant $\mu_0 \geq 0$ and the adjoint $\lambda(t)$ such that the following conditions are satisfied

$H(t, x^*, u^*, \lambda, \mu_0) \geq H(t, x, u, \lambda, \mu_0)$ for all t $\in$ [0, T]

$\frac{\partial H}{\partial u} = 0$ (Optimality condition)

$\dot{x} = \frac{\partial H}{\partial \lambda}$ (State equation)

$\dot{\lambda} = -\frac{\partial H}{\partial x}$ (Adjoint equation) (2.2.1)

$\lambda(T)$ $free$ (Transversality condition). [5].

2.3. Derivation Of Pontryagin's Maximum Principle

Consider the basic optimal control problem of the form

$$J(u) = \int_{t_0}^{T} f(x(t), u(t), t) dt \qquad 2.3.1$$

subject to

$$\dot{x}_i(t) = g_i(x(t), u(t), t), \quad i = 1, 2, \ldots, n \qquad 2.3.2$$

where we wish to find the optimal control vector u that minimizes equation $(2.3.1)$.

In (2.3.1), there are three variables: time t, the state variable, x and the control variable u. We now introduce a new variable, known as adjoint variable and denoted by λ (t). Like the Lagrange multiplier, the adjoint variable is the shadow price of the state variable. The adjoint variable is introduced into the optimal control problem by a Hamiltonian function, $H(t, x, u, \lambda, \mu_0) \equiv \mu_0 f(t, x, u) + \lambda(t) g(t, x, u)$. Where H denotes the Hamiltonian and is a function of five variables t, x, u, λ, μ_0.

[11], for the ith constraint equation in $(2.3.1)$ we form an augmented functional J^* as

$$J^* = \int_0^T \left[f + \sum_{i=1}^{n} \lambda_i (g_i - \dot{x}_i) \right] dt \qquad 2.3.3$$

where the integrand

$$F = f + \sum_{i=1}^{n} \lambda_i (g_i - \dot{x}_i) \qquad 2.3.3a$$

The Hamiltonian functional, H is defined as

$$H = f + \sum_{i=1}^{n} \lambda_i g_i \qquad 2.3.4$$

such that

$$J^* = \int_0^T \left[H - \sum_{i=1}^{n} \lambda_i \dot{x}_i \right] dt \qquad 2.3.5$$

Now the new integrand $F = F(x, u, t)$ becomes

$$F = H - \sum_{i=1}^{n} \lambda_i \dot{x}_i \qquad 2.3.6$$

We recall Euler-Lagrange equations

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i} \right) = 0, i = 1, 2, ..., n \qquad 2.3.7$$

$$\frac{\partial F}{\partial u_j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}_j} \right) = 0, \; j = 1, 2, ..., m \qquad 2.3.8$$

If we relate equations $(2.3.3a), (2.3.7), (2.3.8)$, we have

$$\frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_i} + \dot{\lambda}_i = 0 \qquad 2.3.7a$$

$$\frac{\partial f}{\partial u_i} + \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial u_i} = 0 \qquad 2.3.8a$$

If we relate equations $(2.3.4)$, $(2.3.7a)$, $(2.3.8a)$, we have

$$-\frac{\partial H}{\partial x_i} = \dot{\lambda}_i, i = 1, 2, ..., n \qquad (2.3.9)$$

$$\frac{\partial H}{\partial u_i} = 0, \; j = 1, 2, ..., m \qquad (2.3.10)$$

where equation (2.3.9) is known as adjoint equation.

The optimum solutions for x, u, λ can be obtain by equation $(2.3.2), (2.3.9), (2.3.10)$.

We can now state the various components of the maximum principle for problem (2.3.1) as follows:

$H(t, x^*, u^*, \lambda, \mu_0) \geq H(t, x, u, \lambda, \mu_0)$ for all t∈ [0, T]

$\frac{\partial H}{\partial u} = 0$ (Optimality condition)

$\dot{x} = \frac{\partial H}{\partial \lambda}$ (State equation)

$\dot{\lambda} = -\frac{\partial H}{\partial x}$ (Adjoint equation) (2.3.11)

$\lambda(T)$ *free* (Transversality condition).

Condition one and two in (2.3.11) state that at every time t the value of $u(t)$, the optimal control, must be chosen so as to maximize the value of the Hamiltonian over all admissible values of $u(t)$. Condition three and four of the maximum principle, $\dot{x} = \frac{\partial H}{\partial \lambda}$ *and* $\dot{\lambda} = -\frac{\partial H}{\partial y}$, give us two equations of motion, referred to as the Hamiltonian systems for the given problem. Condition five, $\lambda(T)$ *free* is the transversality condition appropriate for the free terminal state problem only.

2.4 Hamilton's Principle In Mechanics

The evolution of many physical systems involved the minimization of certain physical quantities. The minimization approach to physical systems was formalized in detail by Hamilton, and resulted in Hamilton's principle which states that.

"Of all the possible paths along which a dynamical systems may move from one point to another within a specified time interval, the actual path is that which minimizes the time integral of the difference between the kinetic and potential energies" [9].

Expressing this principle in terms of the calculus of variations, we have

$$S = min\int_{t_1}^{t_2}(T - V)dt$$

Where S is the action to be minimized, T is the kinetic energy, V is the potential energy and the quantity $(T - V)$ is called the Lagrangian L. [13]. Applying the necessary condition to minimize the action, the Euler-Lagrange equations become

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = 0$$

Hence, in any dynamical system, we will first investigate the mechanical energy of the system and set up the Hamiltonian for the system. Then applying Hamiltonian equations, we will obtain an equation describing the motion of the system instead of using Newton's approach which will be more difficult to handle because it requires the total force on the system [12].

2.5 Procedures To Analytical Solution

- Form the Hamiltonian for the problem.
- Write the adjoint differential equation, transversality boundary condition, and the optimality condition in terms of three unknowns, $u*$, $x*$, and λ.
- Use the optimality equation $\frac{\partial H}{\partial u} = 0$ to solve for $u*$ in terms of $x*$ and λ.
- Solve the two differential equations for $x*$ and λ with two boundary conditions.
- After finding the optimal state and adjoint, solve for the optimal control using the formula derived by third procedure.

2.6. Runge- Kutta method of Order 4 (RK4)

This method is developed for solving ODE numerically and to avoid computation of derivatives. Since optimal control problems are described by a set of ODE, we shall use this technique to obtain the numerical approximations to optimal control problems.

2.6.1. Derivation Of Runge-Kutta Method Of Order 4,

Consider an ODE of the form

$$\dot{X} = f(t, x) \qquad 2.6.1a$$

In general form, RK4 is stated as

$$x_{i+1} = x_i + h\phi\left(t_i, x_i, h\right) \qquad 2.6.1b$$

Where $\phi\left(t_i, x_i, h\right) = a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4 + \ldots + a_nk_n$, while $a_i, i = 1, 2, \ldots, n$ are constants and $k_i, i = 1, 2, \ldots, n$ are functional relation given by

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \alpha_1 h, x_i + \beta_{11} k_1 h\right)$$

$$k_3 = f\left(t_i + \alpha_2 h, x_i + \beta_{21} k_1 h + \beta_{22} k_2 h\right)$$

.

.

.

$$k_n = f\left(t_i + \alpha_{n-1} h, x_i + \beta_{n-1,1} k_1 h + \beta_{n-2,2} k_2 h + \ldots + \beta_{n-1,n-1} k_{n-1} h\right)$$

Note that ϕ is called increment function.

Let $F_1 = f_t + ff_x, F_2 = f_{tt} + 2ff_{tx} + f^2 f_{xx}, F_3 = f_{ttt} + 3ff_{ttx} + 3f^2 f_{txx} + f^3 f_{xxx}$

Now differentiating the equation (2.6.1a), we have

$$\ddot{x} = f_t + f_x \dot{x} = f_t + f_x f = F_1$$

$$\dddot{x} = f_{tt} + 2ff_{tx} + f^2 f_{xx} + f_x(f_t + ff_x) = F_2 + f_x F_1$$

$$x^{(iv)} = f_{ttt} + 3ff_{ttx} + 3f^2 f_{txx} + f^3 f_{xxx} + f_x(f_{tt} + 2ff_{tx} + f^2 f_{xx}) + 3(f_t + ff_x)(f_{tx} + ff_{xx}) + f_x^{\,2}(f_t + ff_x)$$

$$= F_3 + f_x F_2 + 3F_1(f_{tx} + ff_{xx}) + f_x^{\,2} F_1$$

Now the Taylor series con be written as

$$x_{i+1} = x_i + hf + \frac{h^2}{2} F_1 + \frac{h^3}{6}(F_2 + f_x F_1) + \frac{h^4}{24}\left[F_3 + f_x F_2 + 3F_1(f_{tx} + ff_{xx}) + f_x^{\,2} F_1\right] + \ldots \quad 2.6.1c$$

And the functional at n=4 is given by

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \alpha_1 h, x_i + \beta_{11} k_1 h\right)$$

$$k_3 = f\left(t_i + \alpha_2 h, x_i + \beta_{21} k_1 h + \beta_{22} k_2 h\right)$$

$$k_4 = f\left(t_i + \alpha_3 h, x_i + \beta_{3,1} k_1 h + \beta_{3,2} k_2 h + \beta_{3,3} k_3 h\right)$$

If we substitute into (2.6.1b), we have

$$x_{i+1} = x_i + h\left[a_1 f(t_i, x_i) + a_2 f\left(t_i + \alpha_1 h, x_i + \beta_{11} k_1 h\right) + a_3 f\left(t_i + \alpha_2 h, x_i + \beta_{21} k_1 h + \beta_{22} k_2 h\right) + a_4 f\left(t_i + \alpha_3 h, x_i + \beta_{3,1} k_1 h + \beta_{3,2} k_2 h + \beta_{3,3} k_3 h\right)\right] \quad 2.6.1d$$

If we compare (2.6.1c) with (2.6.1d) and the classical Runge-Kutta vales for the constants

$$\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}, \alpha_3 = 1$$

$$\beta_{11} = \frac{1}{2}, \beta_{21} = 0, \beta_{22} = \frac{1}{2}, \beta_{31} = 0, \beta_{32} = 0, \beta_{33} = 1$$

$$a_1 = \frac{1}{6}, a_2 = \frac{1}{3}, a_3 = \frac{1}{3}, a_4 = \frac{1}{6}$$

[6]. Therefore, the RK4 becomes

$$k_1 = f(t_i, x_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2} k_2\right)$$

$$k_4 = f\left(t_i + h, x_i + hk_3\right)$$

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), i = 0,1,2,3 \ldots N$$

The algorithm for Classical Runge-Kutta method of order four for $\dot{X} = f(t, x, u)$ is given by

$$k_1 = f(t_i, x_i, u_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1, u_i + \frac{h}{2}\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_2, u_i + \frac{h}{2}\right)$$

$$k_4 = f\left(t_i + h, x_i + hk_3, u_i + h\right)$$

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), i = 0,1,2,3\ldots N \qquad 2.6.1e$$

The algorithm for Classical Runge-Kutta method of order four for $\dot{\lambda} = f(t, \lambda, x, u)$ is given by

$$k_1 = f(t_i, \lambda_{i,} x_i, u_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_1, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_2, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) \qquad 2.6.1f$$

$$k_4 = f\left(t_i + h, \lambda_i + hk_3, u_i + h, x_i + h\right)$$

$$\lambda_{i+1} = \lambda_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), i = 1,2,3\ldots N$$

$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), i = 0,1,2,3\ldots N$ is the iterative method for generating the next value for x; it is calculated using the current value of x_i plus the weighted average of four values of $K_j, j = 1,2\ldots4$. Where $K_j, j = 1,2\ldots4$ are functional relations. Note that h is the step size.

III. Results

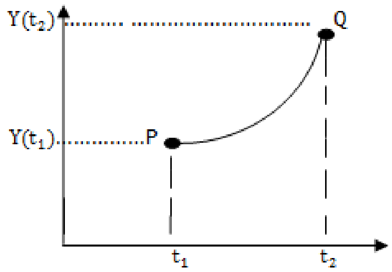
3.1. Analytical Solutions

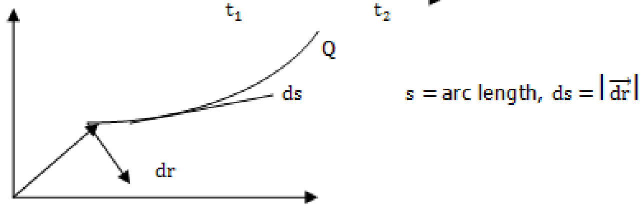
3.1a Geometrical Problems; Shortest Distance Between Two Points

PROBLEM**:** What curve joining two different points P and Q has the shortest length? [8].

SOLUTION:

Let the curve be $y = y(t)$ and the two points be $P[t_1, y_1]$ and $Q[t_2, y_2]$





where,

$$\vec{r} = t\,\vec{i} + y\,\vec{j},\quad \vec{r}(s) = t(s)\,\vec{i} + y(s)\,\vec{j},\quad d(\vec{r}) = dt\,\vec{i} + dy\,\vec{j},\quad ds = \left|d\vec{r}\right| = \sqrt{(dt)^2 + (dy)^2}$$

$$= \sqrt{\left((dt)^2 + (dy)^2\right)\frac{(dt)^2}{(dt)^2}}$$

$$= \left(\sqrt{1 + (\dot{y})^2}\right)dt$$

Where s is arc length parameter and ds small element of arc length from P The distance between the two points is "L"

$$L = \int ds = \int_{t_1}^{t_2} \left(\sqrt{1 + (y')^2}\right)dt \qquad 3.1a.1$$

If we let $\dot{y} = u$ be the control variable, (3.1a. 1) can be expressed as

$$L = \int_{t_1}^{t_2} \{\sqrt{1 + u^2}\}dt \qquad 3.1a.2$$

Thus, the shortest-path problem is

$$\text{Minimize} \int_{t_0}^{T} (1 + u^2)^{1/2}\,dt \qquad 3.1a.3$$

subject to $\dot{y} = u$

and $y(t_0) = y_0$, $\quad$ y(T) free

where, $\quad F = \sqrt{1 + u^2}$

The Hamiltonian for the problem is,

$$H = (1 + u^2)^{1/2} + \lambda u \qquad 3.1a.4$$

Recall that the necessary condition for optimal control is given by

$$-\frac{\partial H}{\partial y} = \dot{\lambda} \qquad 3.1a.5$$

$$\frac{\partial H}{\partial u} = 0 \qquad 3.1a.6$$

$(3.1a.5)$ & $(3.1a.6) \Rightarrow$

$$\dot{\lambda} = 0$$

$$\frac{1}{2}(1+u^2)^{-\frac{1}{2}} 2u + \lambda = 0$$

$$\Rightarrow (1+u^2)^{-\frac{1}{2}} u + \lambda = 0$$

$$\Rightarrow (1+u^2)^{-\frac{1}{2}} u = -\lambda$$

$$\Rightarrow \frac{1}{1+u^2} u^2 = \lambda^2$$

$$\Rightarrow u^2 = \lambda^2 (1+u^2)$$

$$\Rightarrow u^2 (1-\lambda^2) = \lambda^2$$

$$\Rightarrow u(t) = \pm \frac{\lambda}{\sqrt{1-\lambda^2}}$$

From,

$$\dot{\lambda} = 0,$$

$$\lambda(t) = c$$

But,

$$\dot{y} = u$$

$$\Rightarrow \dot{y} = \pm \frac{\lambda}{\sqrt{1-\lambda^2}}$$

$$\Rightarrow y(t) = \pm \frac{\lambda}{\sqrt{1-\lambda^2}} t + k$$

If we substitute the value of $\lambda = c$, we have the control variable as

$$u(t) = \pm \frac{c}{\sqrt{1-c^2}}$$

and the corresponding state variable is

$$y(t) = \pm \frac{c}{\sqrt{1-c^2}} t + k$$

3.1b. The Economic Problems; Cocoa Production Problem

Problem: a farmer who owns a cocoa plantation and has a problem to decide at what rate to produce cocoa from his plantation. He is meant to manage the plantation from date 0 to a period of 2years. At date 0, there is x_0 cocoa in the farm, and the instantaneous stock of cocoa $x(t)$ declines at the rate the farmer produces $u(t)$. The plantation owner produces cocoa at cost x^2 and sells $u^2 + 2xu$ of cocoa at constant price \$1. He does not value the cocoa remaining in the farm at the end of the period (there is no scrap value). At what rate of production in time $u(t)$ will maximize his profits over the period of ownership with no discount time.

SOLUTION: Let P represents the farmer's profits, if he produces cocoa at cost x^2 and sells $u^2 + 2xu$ at constant price \$1 then, P is given by

$$P = (\$1)(u^2 + 2xu) - x^2 \qquad 3.1b.1$$

The total profit over the period of ownership is

$$P = \int_0^2 (u^2 + 2xu - x^2)dt \qquad 3.1b.2$$

Then the cost functional is given by

Maximizes $$P = \int_0^2 (u^2 + 2xu - x^2)dt \qquad 3.1b.3$$

subject to $\dot{x} = -u$

and $x(0) = 1, \dot{x}(0) = 0$

Taking the Hamiltonian equation, we have

$$H = f + \lambda g = u^2 + 2xu - x^2 - \lambda u \qquad 3.1b.4$$

Recall that the necessary condition for optimal control is given by

$$-\frac{\partial H}{\partial y} = \dot{\lambda} \qquad 3.1b.5$$

$$\frac{\partial H}{\partial u} = 0 \qquad 3.1b.6$$

$(3.1b.5)$ & $(3.1b.6) \Rightarrow$

$$-2u + 2x = \dot{\lambda} \qquad 3.1b.7$$

$$2u + 2x - \lambda = 0 \qquad 3.1b.8$$

If we differentiate equation (4.2a.8), we have

$$2\dot{u} + 2\dot{x} = \dot{\lambda}$$

$$\Rightarrow 2\dot{u} + 2\dot{x} = -2u + 2x$$

$$\Rightarrow \dot{u} + \dot{x} = -u + x \qquad 3.1b.9$$

But, $\dot{x} = -u$, equation (3.1b.9) becomes

$$-\ddot{x} + \dot{x} = \dot{x} + x$$

$$\Rightarrow \ddot{x} + x = 0$$

If we take the auxiliary equation, we have

$$r^2 + 1 = 0$$

$$\Rightarrow r^2 = -1$$

$$\Rightarrow r = \pm i$$

$$\therefore x(t) = A\cos t + B\sin t$$

Where A, B are constants applying the initial conditions, we have

$$x(0) = 1 = A\cos 0 + B\sin 0$$

$$\Rightarrow\ A = 1,$$

$$\dot{x}(t) = \left[-A\sin t + B\cos t\right]$$

$$\dot{x}(0) = 0 = \left[-A\sin 0 + B\cos 0\right]$$

$$\Rightarrow\ B = 0$$

We recall that, from equation (3.1b.3) we have

$$\dot{x} = -u \Rightarrow u(t) = -\left[-\sin t\right]$$

$$\therefore u(t) = A\sin t$$

Substituting the values our constants, the control becomes

$$u(t) = \sin t$$

The corresponding state trajectory is given by

$$x(t) = \cos t$$

3.1c. Applications To Physical Principles

3.1c.1 The Pendulum Problem.

PROBLEM: Describe the mechanical system of a point mass m constrained by a light wire of length l to swing in an arc.

SOLUTION:

Let figure $(3.1c.1)$ be the diagrammatical illustration of the system.

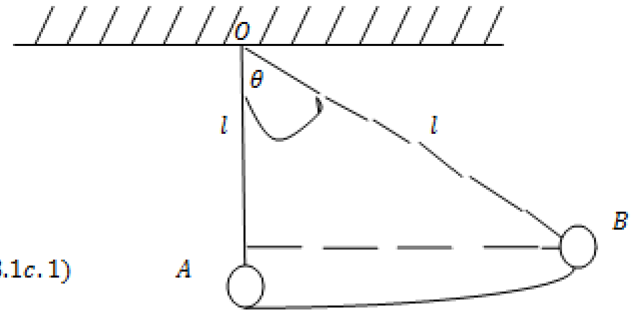


Figure $(3.1c.1)$

Let s be the displacement of the bob from its mean position.

Let $\overline{OA} = l$ be the length of the light wire before swing.

Let $\overline{OB}$ be the length after swing.

$$=> \ \overline{OA} = \overline{OB} = l$$

Let the angle θ (in radian) be the angle made by $\overline{OB}$ and the vertical $\overline{OA}$.

From the diagram above, at point A the potential energy $V = 0$ and the kinectic energy $T = T$, but at point B, kinetic energy $T = 0$ and potential energy $V = V$

In order to set up the Lagrangian for the system, we first define the work done by the system as

$$W = \int F.dr,$$

where r is the position and F is the force

$$=> \int m\frac{dv}{dt}.dr \ = m\int_0^v \frac{dr}{dt}.dv$$

$$= \; m\int_0^v v.dv \; = \; \frac{1}{2}mv^2$$

Since the energy of the particle is its capacity to do work, we have
Kinetic energy

$$T = \frac{1}{2}mv^2 \qquad\qquad 3.1c.1a$$

Similarly potential energy

$$V = mgh \qquad\qquad 3.1c.1b$$

The circular measure of an angle in radian is equal to the ratio of the arc which the angle subtends when at the centre of the circle to the radius of that circle, [1]. That is

$$\theta = \frac{arcAB}{\overline{OA}} = \frac{S}{l}$$

$$=> S = l\theta$$

The velocity is

$$\frac{ds}{ds} = \frac{d(l\theta)}{dt} = \frac{ld\theta}{dt} = l\theta'$$

This implies that the kinetic energy becomes

$$T = \frac{1}{2}m(l\theta')^2 = \frac{1}{2}ml^2\theta'^2$$

Also from the diagram above

$$\overline{OA} - \overline{OC} = \overline{AC} = h$$

But,

$$\overline{OA} = l, \; cos\,\theta = \frac{\overline{OC}}{\overline{OB}} = \frac{\overline{OC}}{l}$$

$$=> \; \overline{OC} = l\cos\theta$$

$$\therefore \qquad \overline{OA} - \overline{OC} = h = l - l\cos\theta = l(1 - cos\,\theta)$$

Substitute into equation $(3.1c.1b)$, we have potential energy

$$V = mg(l - l\cos\theta)$$

$$V = mgl(1 - cos\,\theta)$$

Now that we have both kinetic energy and potential energy of the system, then we can define the Lagrangian L of system as L = kinetic energy – potential energy [12].

$$L = \frac{1}{2}ml^2(\dot{\theta})^2 \; - mgl(1 - cos\,\theta)$$

[2], using Hamilton principle, the action S becomes

$$S = \int_{t_1}^{t_2} (\frac{1}{2}ml^2(\dot{\theta})^2 \; - mgl(1 - cos\,\theta))dt \qquad\qquad 3.1c.1c$$

If we let $\dot{\theta} = u$ be the control variable, (4.1c.2.1c) can be expressed as

$$S = \int_{t_1}^{t_2} (\frac{1}{2}ml^2u^2 - mgl(1 - cos\,\theta))dt$$

Thus, the pendulum problem is

Minimize

$$S = \int_{t_1}^{t_2} (\frac{1}{2}ml^2u^2 - mgl(1 - cos\,\theta))dt$$

subject to $\dot{\theta} = u$
and $\theta(t_1) = \theta_1 \qquad \theta(T)$ free
where, $\quad F = \frac{1}{2}ml^2u^2 - mgl(1 - \cos\theta)$

The Hamiltonian for the problem is,

$$H = \frac{1}{2}ml^2u^2 - mgl(1 - \cos\theta) + \lambda u$$

Recall that the necessary condition for optimality is given by

$$-\frac{\partial H}{\partial x} = \dot{\lambda}$$

$$\frac{\partial H}{\partial u} = 0$$

$$\Rightarrow \dot{\lambda} = mgl\sin\theta$$

$$ml^2u + \lambda = 0$$

From,

$$ml^2u + \lambda = 0,$$

$$ml^2u = -\lambda$$

$$\Rightarrow ml^2\dot{u} = -\dot{\lambda} = -mgl\sin\theta$$

But,

$$\dot{\theta} = u$$

$$\Rightarrow \ddot{\theta} = \dot{u}$$

$$\Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \ldots (3.1c.1d)$$

The equation $(3.1c.2.1d)$ above constitutes the Newton's second law of motion which describes the motion of pendulum oscillation to arbitrary angles.
Now solving for equation $(3.1c.2.1d)$ above, we have

$$\theta(t) = A\cos\sqrt{\frac{g}{l}}t + B\sin\sqrt{\frac{g}{l}}t$$

and

$$u(t) = \dot{\theta} = \sqrt{\frac{g}{l}}\{-A\sin\sqrt{\frac{g}{l}}t + B\cos\sqrt{\frac{g}{l}}t\}$$

where A and B are constants.

If we let $w = \sqrt{\frac{g}{l}}$, we have

$$\theta(t) = A\cos wt + B\sin wt \qquad 3.1c.1e$$

$$u(t) = w\{-A\cos wt + B\sin wt\}$$

Therefore the equation $(3.1c.1e)$ above shows that the system is a Harmonic oscillator (SHM).

3.2. The Numerical Results.

Here, we want to apply the results we have in (2.6.1e) and (2.6.1f) to obtain the numerical approximations to problems (3.1a), (3.1b) and (3.1c).

1. the shortest-path problem:

$$\text{Minimize} \int_0^5 (1+u^2)^{1/2}\,dt$$

$$\text{subject to } \dot{x} = u$$

$$\text{and } x(0) = 2, \qquad x(T) \text{ free}$$

where, $F = \sqrt{1+u^2}$

Taken the first four procedures in (2.5), we have

$$u_i = \pm \frac{\lambda_i}{\sqrt{1-\lambda_i^2}}, \lambda \angle 1$$

$$\dot{\lambda} = 0$$

$$\dot{x} = u, x_0 = 2$$

Employing Runge-Kutta method of order 4, with initial guess of

$$u_0 = 1$$

$$\Rightarrow \lambda_0 = \sqrt{\frac{1}{2}}$$

we have,

$$k_1 = f(t_i, x_i, u_i) = u_i$$

$$k_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1, u_i + \frac{h}{2}\right) = u_i + \frac{h}{2}$$

$$k_3 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_2, u_i + \frac{h}{2}\right) = u_i + \frac{h}{2}$$

$$k_4 = f\left(t_i + h, x_i + hk_3, u_i + h\right) = u_i + h$$

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = x_i + \frac{h}{6}\left(u_i + 2u_i + h + 2u_i + h + u_i + h\right)$$

$$x_{i+1} = x_i + \frac{h}{6}\left(6u_i + 3h\right), i = 0,1,2,3\ldots N, (3.2.1a)$$

Also, for iterative formula λ

$$k_1 = f(t_i, \lambda_{i,} x_i, u_i) = 0$$

$$k_2 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_1, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) = 0$$

$$k_3 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_2, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) = 0$$

$$k_4 = f\left(t_i + h, \lambda_i + hk_3, u_i + h, x_i + h\right) = 0$$

$$\lambda_{i+1} = \lambda_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \lambda_i$$

$$\lambda_{i+1} = \lambda_i, i = 1,2,3\ldots N, (3.2.1b)$$

The table below shows the results of problem 1
h=0.1

S/N	Time	u .	x .	λ	u .exact	x .exact
1	0	1	2	0.707107	1	2
2	0.1	1	2.105	0.707107	1	2.1
3	0.2	1	2.21	0.707107	1	2.2
4	0.3	1	2.315	0.707107	1	2.3
5	0.4	1	2.42	0.707107	1	2.4
6	0.5	1	2.525	0.707107	1	2.5
7	0.6	1	2.63	0.707107	1	2.6
8	0.7	1	2.735	0.707107	1	2.7
9	0.8	1	2.84	0.707107	1	2.8
10	0.9	1	2.945	0.707107	1	2.9
11	1	1	3.05	0.707107	1	3

Table 1

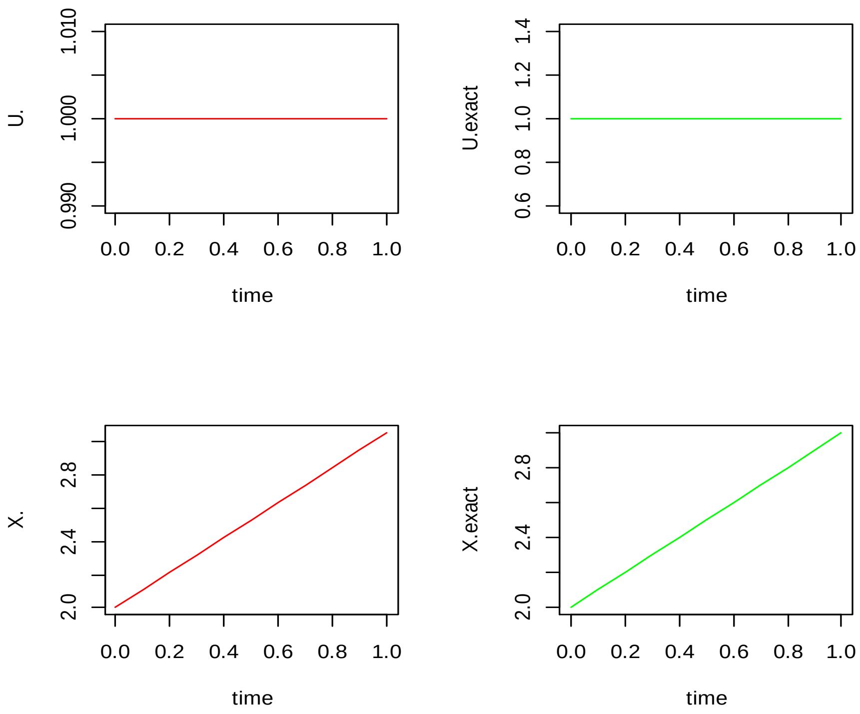


Fig. 3.2.a, optimal state and control values at h = 0.1

(2). **The cocoa problem**:

Maximizes $$P = \int_0^2 (u^2 + 2xu - x^2)dt$$

subject to $$\dot{x} = -u$$

and $$x(0) = 1, u(0) = 0$$

Taken the first four procedures in (2.5), we have

$$u_i = \frac{\lambda_i - 2x_i}{2}$$

$$\dot{x} = -u$$

$$\dot{\lambda} = -2u + 2x$$

Employing Runge-Kutta method of order 4, with initial guess of

$$u_0 = 0$$

$$\Rightarrow \lambda_0 = 2$$

we have,

$$k_1 = f(t_i, x_i, u_i) = -u_i$$

$$k_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1, u_i + \frac{h}{2}\right) = -u_i - \frac{h}{2}$$

$$k_3 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_2, u_i + \frac{h}{2}\right) = -u_i - \frac{h}{2}$$

$$k_4 = f\left(t_i + h, x_i + hk_3, u_i + h\right) = -u_i - h$$

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = x_i + \frac{h}{6}\left(-u_i - 2u_i - h - 2u_i - h - u_i - h\right)$$

$$x_{i+1} = x_i + \frac{h}{6}\left(-6u_i - 3h\right), i = 0,1,2,3\ldots N, (3.2.1c)$$

Also, the iterative formula for λ

$$k_1 = f(t_i, \lambda_{i,} x_i, u_i) = -2u_i + 2x_i$$

$$k_2 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_1, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) = -2u_i - h + 2x_i + h$$

$$k_3 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_2, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) = -2u_i - h + 2x_i + h$$

$$k_4 = f\left(t_i + h, \lambda_i + hk_3, u_i + h, x_i + h\right) = -2u_i - h + 2x_i + h$$

$$\lambda_{i+1} = \lambda_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \lambda_i + 2h\left(-12u_i + 12x_i\right)$$

$$\lambda_{i+1} = \lambda_i + 2h\left(-12u_i + 12x_i\right), i = 1,2,3\ldots N, (3.2.1d)$$

The table below shows the numerical results to problem 2

h=0.02

S/N	Time	u	x	λ	u. exact	x. exact
1	0	0	1	2	0	1
2	0.02	0.0202	0.9998	2.04	0.019999	0.9998
3	0.04	0.040396	0.999196	2.079184	0.039989	0.9992
4	0.06	0.06058	0.998188	2.117536	0.059964	0.998201
5	0.08	0.080744	0.996776	2.15504	0.079915	0.996802
6	0.1	0.100879	0.994962	2.191682	0.099833	0.995004
7	0.12	0.120978	0.992744	2.227445	0.119712	0.992809
8	0.14	0.141033	0.990124	2.262316	0.139543	0.990216
9	0.16	0.161036	0.987104	2.296279	0.159318	0.987227
10	0.18	0.180978	0.983683	2.329322	0.17903	0.983844
11	0.2	0.200852	0.979864	2.36143	0.198669	0.980067

12	0.22	0.220649	0.975646	2.392591	0.21823	0.975897
13	0.24	0.240362	0.971034	2.422791	0.237703	0.971338
14	0.26	0.259982	0.966026	2.452017	0.257081	0.96639
15	0.28	0.279503	0.960627	2.480259	0.276356	0.961055
16	0.3	0.298915	0.954837	2.507504	0.29552	0.955336
17	0.32	0.318212	0.948658	2.533741	0.314567	0.949235
18	0.34	0.337385	0.942094	2.558959	0.333487	0.942755
19	0.36	0.356427	0.935146	2.583147	0.352274	0.935897
20	0.38	0.37533	0.927818	2.606296	0.37092	0.928665
21	0.4	0.394087	0.920111	2.628395	0.389418	0.921061

Table 2

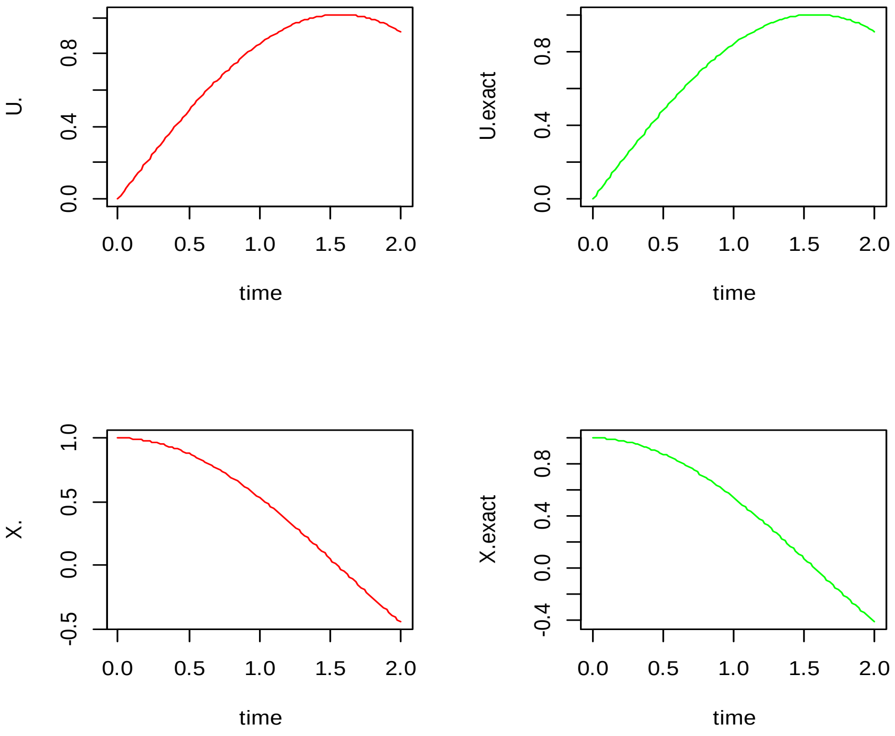


Fig. 3.2.b. Optimal state and control values at h = 0.02

(3) The pendulum problem:
Minimize

$$S = \int_{t_1}^{t_2} (\frac{1}{2} ml^2 u^2 - mgl(1 - \cos\theta))dt$$

subject to $\dot{\theta} = u$
and $\theta(t_1) = \theta_1$ $\theta(T)$ free
where, $F = \frac{1}{2} ml^2 u^2 - mgl(1 - \cos\theta)$
Taken the first four procedures in (2.5), we have

$$u_i = -\lambda_i$$

$$\dot{\lambda} = mgl\sin\theta$$

$$\dot{x} = u$$

$$m = l = g = 1$$

Employing Runge-Kutta method of order 4, with initial guess of

$$\theta_0 = 1$$
$$u_0 = 2$$
$$\Rightarrow \lambda_0 = -2$$

we have,

$$k_1 = f(t_i, \theta_i, u_i) = u_i$$

$$k_2 = f\left(t_i + \frac{h}{2}, \theta_i + \frac{h}{2}k_1, u_i + \frac{h}{2}\right) = u_i + \frac{h}{2}$$

$$k_3 = f\left(t_i + \frac{h}{2}, \theta_i + \frac{h}{2}k_2, u_i + \frac{h}{2}\right) = u_i + \frac{h}{2}$$

$$k_4 = f\left(t_i + h, \theta_i + hk_3, u_i + h\right) = u_i + h$$

$$\theta_{i+1} = \theta_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \theta_i + \frac{h}{6}\left(u_i + 2u_i + h + 2u_i + h + u_i + h\right)$$

$$\theta_{i+1} = \theta_i + \frac{h}{6}\left(6u_i + 3h\right), i = 0,1,2,3 \ldots N, (3.2.1e)$$

Also, for iterative formula λ

$$k_1 = f(t_i, \lambda_{i,} x_i, u_i) = \sin\theta_i$$

$$k_2 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_1, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) = \sin(\theta_i + \frac{h}{2})$$

$$k_3 = f\left(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_2, u_i + \frac{h}{2}, x_i + \frac{h}{2}\right) = \sin(\theta_i + \frac{h}{2})$$

$$k_4 = f\left(t_i + h, \lambda_i + hk_3, u_i + h, x_i + h\right) = \sin(\theta_i + h)$$

$$\lambda_{i+1} = \lambda_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \lambda_i + \frac{h}{6}\left(\sin\theta_i + 2\sin(\theta_i + \frac{h}{2}) + 2\sin(\theta_i + \frac{h}{2}) + \sin(\theta_i + h)\right)$$

$$\lambda_{i+1} = \lambda_i + \frac{h}{6}\left(\sin\theta_i + 4\sin(\theta_i + \frac{h}{2}) + \sin(\theta_i + h)\right), i = 1,2,3 \ldots N, (3.2.1f)$$

The table below shows the numerical approximations to problem 3.
h=0.01

S/N	Time	u .	θ .	λ	u .exact	θ .exact
1	0	2.	1	-2	2	1
2	0.01	1.991563	1.02005	-1.99156	1.9899	1.019949667
3	0.02	1.98302	1.040016	-1.98302	1.979601	1.03979734
4	0.03	1.974375	1.059896	-1.97437	1.969105	1.059541034
5	0.04	1.965631	1.07969	-1.96563	1.958411	1.079178775
6	0.05	1.956794	1.099396	-1.95679	1.947521	1.098708599
7	0.06	1.947866	1.119014	-1.94787	1.936437	1.118128553
8	0.07	1.938851	1.138542	-1.93885	1.925159	1.137436695
9	0.08	1.929753	1.157981	-1.92975	1.913689	1.156631094
10	0.09	1.920577	1.177329	-1.92058	1.902027	1.175709831
11	0.1	1.911325	1.196584	-1.91132	1.890175	1.194670999
12	0.11	1.902002	1.215748	-1.902	1.878134	1.2135127
13	0.12	1.892611	1.234818	-1.89261	1.865905	1.23223305
14	0.13	1.883157	1.253794	-1.88316	1.85349	1.250830179
15	0.14	1.873642	1.272675	-1.87364	1.840889	1.269302226
16	0.15	1.864071	1.291462	-1.86407	1.828104	1.287647343
17	0.16	1.854447	1.310152	-1.85445	1.815136	1.305863697
18	0.17	1.844774	1.328747	-1.84477	1.801987	1.323949465
19	0.18	1.835056	1.347245	-1.83506	1.788658	1.34190284
20	0.19	1.825296	1.365645	-1.8253	1.77515	1.359722025
21	0.2	1.815497	1.383948	-1.8155	1.761464	1.377405239

Table 3.

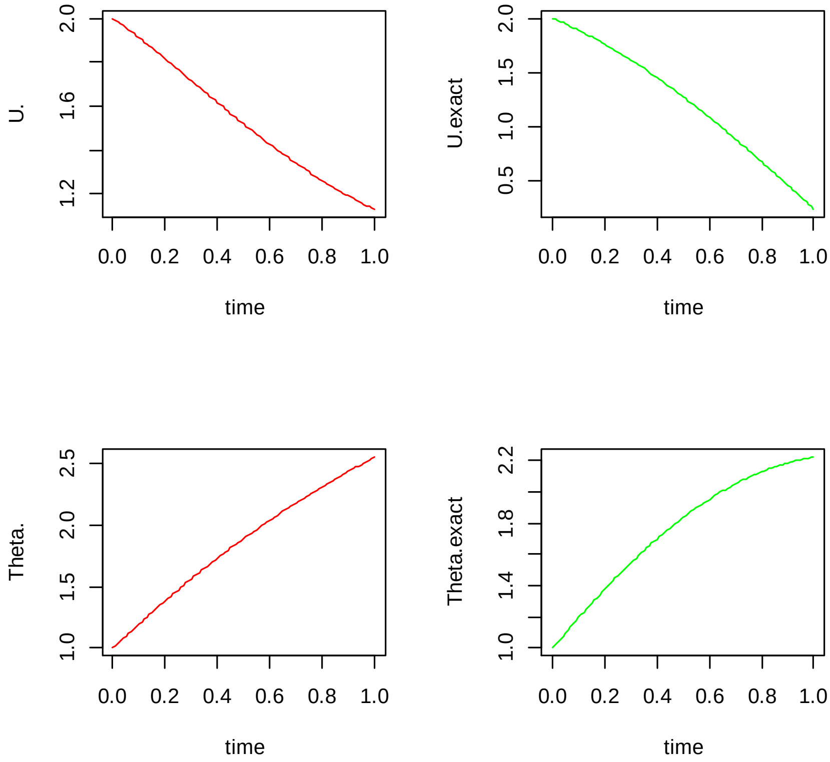


Fig. 3.2.c, optimal state and control values at h = 0.01

IV. Discussion

We use Pontryagin's maximum principles to obtain analytic solutions to problems formulated from three different fields. We derived Runge-Kutta algorithm that generates the numerical approximations to the problems considered. Both the state and adjoint variables are solved with forward Runge-Kutta iterative formulas. It is observed that the Runge-Kutta scheme produces results that are comparable with analytic results as it is shown in above tables. The errors occur are highly infinitesimal.

V. Conclusion

We have formulated optimal control problems from three different fields. The Pontryagin's maximum principles were employed for obtaining analytic solutions to optimal control problems. A Runge-Kutta method of order four for numerical approximations to optimal control problems has been developed. From the numerical experiments, the results show that the Runge-Kutta method of order four produced results that are comparable to analytic solutions to the problems considered. Therefore, we conclude that Runge-Kutta method gives error that is negligible.

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