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Numerical Investigation of linear first order Fuzzy Differential Equations using He's Homotopy Perturbation Method

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Abstract:*In this paper, He's Homotopy Perturbation Method (HHPM) is used to study the linear first order fuzzy differential equations (FDE). The results obtained using He's Homotopy Perturbation Method and the methods taken from the literature [9] were compared with the exact solutions of the linear first order fuzzy differential equations. It is found that the solution obtained using the He's Homotopy Perturbation Method is closer to the exact solutions of the linear first order fuzzy differential equations. Error graphs for discrete and exact solutions are presented in a graphical form to highlight the efficiency of this method.*

Keywords:*Fuzzy Differential Equations, Differential Equations, Initial Value Problems, He's Homotopy Perturbation Method, Leapfrog Method.*

I. Introduction

This paper presents a comparative study between He's homotopy perturbation method (HHPM) [11-28] and one traditional method, namely the Leapfrog method [9], for solving linear first order fuzzy differential equations. The linear first order fuzzy differential equations which we study here are given by the following, respectively. S. Abbasbandy and T. Allahviranloo [1] addressed knowledge about dynamical systems modelled by differential equations is often incomplete or vague. It concerns, for example, parameter values, functional relationships, or initial conditions. The well-known methods for solving analytically or numerically initial value problems can only be used for finding a selected system behavior, e.g., by fixing the unknown parameters to some plausible values.

The topics of fuzzy differential equations, which attracted a growing interest for some time, in particular, in relation to the fuzzy control, have been rapidly developed recent years. The concept of a fuzzy derivative was first introduced by S. L. Chang, L. A. Zadeh in [2]. It was followed up by D. Dubois, H. Prade in [3], who defined and used the extension principle. Other methods have been discussed by M. L. Puri, D. A. Ralescu in [4] and R. Goetschel, W. Voxman in [5]. Fuzzy differential equations and initial value problems were regularly treated by O. Kaleva in [6] and [7], S. Seikkala in [8]. A numerical method for solving fuzzy differential equations has been introduced by M. Ma, M. Friedman, A. Kandel in [10] via the standard Euler method.

The structure of this paper is organized as follows. In section 2, the proposed He's homotopy perturbation method is explained in detailed. In section 3, we define the problem that is a fuzzy initial value problem. Its numerical solution is of the main interest of this work. Sekar *et al* [9] discussed the linear first order fuzzy differential equations using Leapfrog method. The aim of this paper is to extend the He's homotopy perturbation method to find the solution of linear first order fuzzy differential equations. In this paper, the same linear first order fuzzy differential equations was considered (Sekar *et al* [9]) but present a different approach using He's homotopy perturbation method for finding the numerical solution of linear first order fuzzy differential equations with more accuracy. Furthermore, we use some examples to demonstrate the efficiency and effectiveness of the proposed method. Solving numerically the fuzzy differential equation by the He's homotopy perturbation method is discussed in section 4. The proposed algorithm is illustrated by some examples in section 4 and the conclusion is in section 5.

II. He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [11-28]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t, p) : \Omega \times [0,1] \rightarrow \Re$, which satisfies:

$$H(U,p) = (1- p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \quad (2)$$

or

$$H(U,p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \quad (3)$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2) or (3) we will have $H(U,0) = LU(t) - Lu_0(t) = 0, H(U,1) = AU(t) - f(t) = 0$

The changing process of p from zero to unity is just that of $U(t, p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the HHPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2) or (3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + ... \quad (4)$$

Setting $p = 1$, results in the approximate solution of Eq. (1)

$$u(t) = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + ...$$

Applying the inverse operator $L^{-1} = \int_0^t (.)dt$ to both sides of Eq. (3), we obtain

$$U(t) = U(0) + \int_0^t Lu_0(t)dt - p\int_0^t Lu_0(t)dt - p\left[\int_0^t (NU(t) - f(t))dt\right] \quad (5)$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \quad (6)$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), \ldots$ are specific functions.

Substituting (4) and (6) into (5) and equating the coefficients of p with the same power leads to

$$\begin{cases} p^0 : U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ p^1 : U_1(t) = -\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ p^2 : U_2(t) = -\int_0^t NU_1(t)dt \\ \vdots \\ \vdots \\ p^j : U_j(t) = -\int_0^t NU_{j-1}(t)dt \end{cases} \quad (7)$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (7) results in

$U_1(t) = U_2(t) = U_3(t) = ... = 0$.

and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \quad (8)$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

can be used in Eq. (11), where $a_0, a_1, a_2, ...$ are known coefficients and α_n are unknown ones, which must be computed.

We explain this method by considering three examples in the following.

III. Fuzzy Initial Value Problems

Consider a first-order fuzzy initial value differential equation is given by

$$\left.\begin{aligned} y'(t) &= f(t, y(t)),\ t \in [t_0, T], \\ y(t_0) &= y_0, \end{aligned}\right\} \tag{3}$$

where *y* is a fuzzy function of *t*, *f (t, y)* is a fuzzy function of the crisp variable t and the fuzzy variable *y*, y' is the fuzzy derivative of *y* and $y(t_0) = y_0$ is a parallelogram or a parallelogram shaped fuzzy number. We denote the fuzzy function *y* by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of *y(t)* for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)], [y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)], r \in (0,1]$$

we write $f(t;y) = [\underline{f}(t;y), \overline{f}(t;y)]$ and $\underline{f}(t;y) = F[t, \underline{y}, \overline{y}], \overline{f}(t;y) = G[t, \underline{y}, \overline{y}]$.

Because of $y'(t) = f(t, y)$ we have

$$\underline{f}(t; y(t;r)) = F[t; \underline{y}(t;r), \overline{y}(t;r)] \tag{4}$$

$$\overline{f}(t; y(t;r)) = G[t; \underline{y}(t;r), \overline{y}(t;r)] \tag{5}$$

By using the extension principle, we have the membership function

$$f(t; y(t))(s) = Sup\{y(t)(\tau)\backslash s = f(t, \tau)\},\ s \in R \tag{6}$$

so fuzzy number *f(t; y(t))*. From this it follows that

$$[f(t; y(t))]_r = [\underline{f}(t, y(t;r)), \overline{f}(t, y(t;r))], r \in [0;1] \tag{7}$$

where $\underline{f}(t, y(t;r)) = \min\{f(t,u) \backslash u \in [y(t)]_r\}$ (8)

$$\overline{f}(t, y(t;r)) = \max\{f(t,u) \backslash u \in [y(t)]_r\} \tag{9}$$

Definition 4.1

A function f: R → R_F is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\in > 0, \delta > 0$ such that $|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \in$ exists.

Throughout this paper we also consider fuzzy functions which are continuous in metric *D*. Then the continuity of *f(t, y(t); r)* guarantees the existence of the definition of *f(t, y(t); r)* for $t \in [t_0, T]$ and $r \in [0,1]$ [1]. Therefore, the functions *G* and *F* can be definite too.

IV. Numerical Examples

Consider a first-order fuzzy initial value differential equation is given by In this section, the exact solutions and approximated solutions obtained by He's homotopy perturbation method (HHPM) and Leapfrog method. To show the efficiency of the He's homotopy perturbation method (HHPM), we have considered the following problem taken from [9], along with the exact solutions.

The discrete solutions obtained by the two methods, He's homotopy perturbation method (HHPM) and Leapfrog method; the absolute errors between them are tabulated and are presented in Table 1 and Table 2. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of "r" and are presented in Fig. 1 to Fig. 6 for the following problem, using three dimensional effects.

Example 5.1

Consider the initial value problem [9]

$$\left.\begin{aligned} y'(t) &= tf(t),\ t \in [0,1], \\ y(0) &= (1.01 + 0.1r\sqrt{e}, 1.5 + 0.1r\sqrt{e}) \end{aligned}\right\}$$

The exact solution at *t* = 0.1 is given by

$$Y(0.1;r) = [(1.01 + 0.1r\sqrt{e})e^{0.005}, (1.5 + 0.1r\sqrt{e})e^{0.005}],\ 0 \le r \le 1$$

Example 5.2

Consider the fuzzy initial value problem [9]

$$\left.\begin{aligned} y'(t) &= y(t),\ t \in I = [0,1], \\ y(0) &= (0.75 + 0.25r, 1.125 - 0.125r), 0 < r \le 1. \end{aligned}\right\}$$

The exact solution is given by

$Y_1(t;r) = y_1(0;r)e^t,\ Y_2(t;r) = y_2(0;r)e^t$ which at $t = 1$

$$Y_1(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e],\ 0 < r \le 1.$$

Example 5.3

Consider the fuzzy initial value problem [9]

$$y'(t) = c_1 y^2(t) + c_2,\ y(0) = 0$$

where $c_i > 0$, for $i = 1,2$ are triangular fuzzy numbers.

The exact solution is given by

$$Y_1(t;r) = l_1(r)\tan(w_1(r)t),$$
$$Y_2(t;r) = l_2(r)\tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)},\quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)}$$
$$w_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)},\quad w_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)}$$

where

$[c_1]_r = [c_{1,1}(r),\ c_{1,2}(r)]$ and $[c_2]_r = [c_{2,1}(r),\ c_{2,2}(r)]$

$$c_{1,1}(r) = 0.5 + 0.5r,\ c_{1,2}(r) = 1.5 - 0.5r,$$
$$c_{2,1}(r) = 0.75 + 0.25r,\ c_{2,2}(r) = 1.25 - 0.25r,$$

The r-level sets of $y'(t)$ are

$$Y_1'(t;r) = c_{2,1}(r)\sec^2(w_1(r)t),\ Y_2'(t;r) = c_{2,2}(r)\sec^2(w_2(r)t),$$

Which defines a fuzzy number. We have

$$f_1(t,y;r) = \min\{c_1.u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)],\ c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$
$$f_2(t,y;r) = \max\{c_1.u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)],\ c_2 \in [c_{2,1}(r), c_{2,2}(r)]\}.$$

Table 1: Error Calculations

	Leapfrog Method Error					
	Example 5.1		Example 5.2		Example 5.3	
r	y_1	y_2	y_1	y_2	y_1	y_2
0.1	1.00E-07	1.00E-07	6.00E-07	6.00E-07	1.00E-07	1.00E-08
0.2	2.00E-07	2.00E-07	7.00E-07	7.00E-07	2.00E-07	2.00E-08
0.3	3.00E-07	3.00E-07	8.00E-07	8.00E-07	3.00E-07	3.00E-08
0.4	4.00E-07	4.00E-07	9.00E-07	9.00E-07	4.00E-07	4.00E-08
0.5	5.00E-07	5.00E-07	1.00E-06	1.00E-06	5.00E-07	5.00E-08
0.6	6.00E-07	6.00E-07	1.10E-06	1.10E-06	6.00E-07	6.00E-08
0.7	7.00E-07	7.00E-07	1.20E-06	1.20E-06	7.00E-07	7.00E-08
0.8	8.00E-07	8.00E-07	1.30E-06	1.30E-06	8.00E-07	8.00E-08
0.9	9.00E-07	9.00E-07	1.40E-06	1.40E-06	9.00E-07	9.00E-08
1	1.00E-06	1.00E-06	1.50E-06	1.50E-06	1.00E-06	9.90E-08

Table 2: Error Calculations

	He's Homotopy Perturbation Method Error					
r	Example 5.1		Example 5.2		Example 5.3	
	y_1	y_2	y_1	y_2	y_1	y_2
0.1	1.00E-09	1.00E-09	6.00E-09	6.00E-09	1.00E-09	1.00E-11
0.2	2.00E-09	2.00E-09	7.00E-09	7.00E-09	2.00E-09	2.00E-11
0.3	3.00E-09	3.00E-09	8.00E-09	8.00E-09	3.00E-09	3.00E-11
0.4	4.00E-09	4.00E-09	9.00E-09	9.00E-09	4.00E-09	4.00E-11
0.5	5.00E-09	5.00E-09	1.00E-08	1.00E-08	5.00E-09	5.00E-11
0.6	6.00E-09	6.00E-09	1.10E-08	1.10E-08	6.00E-09	6.00E-11
0.7	7.00E-09	7.00E-09	1.20E-08	1.20E-08	7.00E-09	7.00E-11
0.8	8.00E-09	8.00E-09	1.30E-08	1.30E-08	8.00E-09	8.00E-11
0.9	9.00E-09	9.00E-09	1.40E-08	1.40E-08	9.00E-09	9.00E-11
1	1.00E-08	1.00E-08	1.50E-08	1.50E-08	1.00E-08	9.90E-11

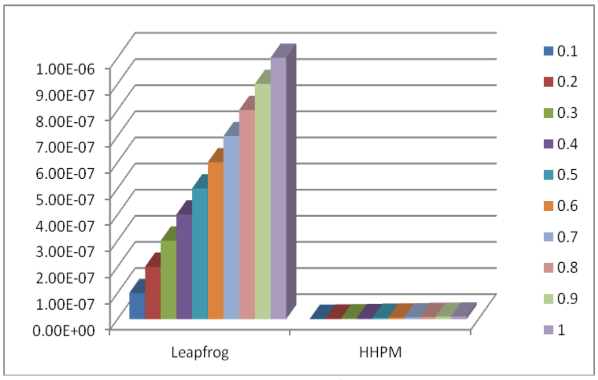

Fig. 1 Error estimation of Example 5.1 at y_1

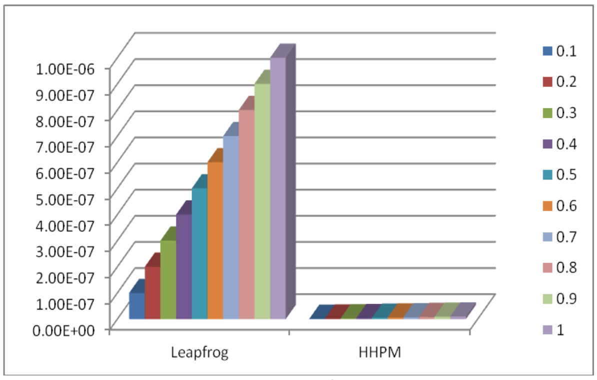

Fig. 2 Error estimation of Example 5.1 at y_2

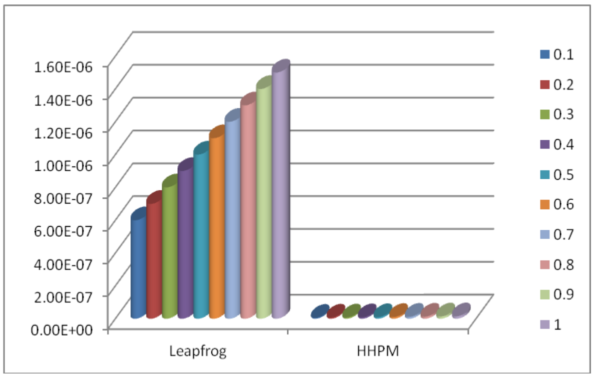

Fig. 3 Error estimation of Example 5.2 at y_1

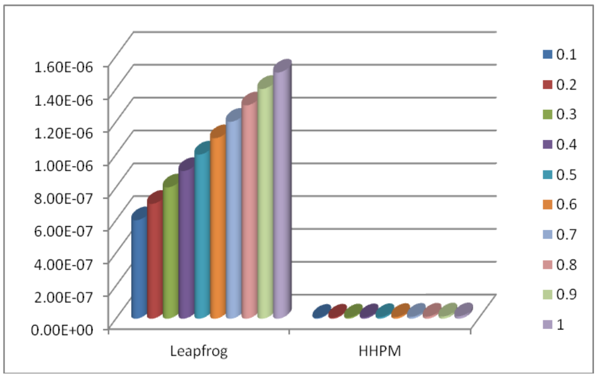

Fig. 4 Error estimation of Example 5.2 at y_2

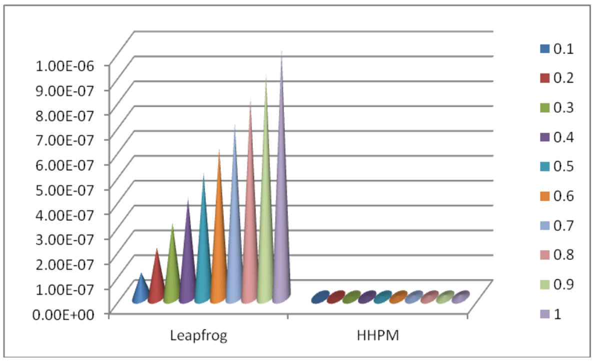

Fig. 5 Error estimation of Example 5.3 at y_1

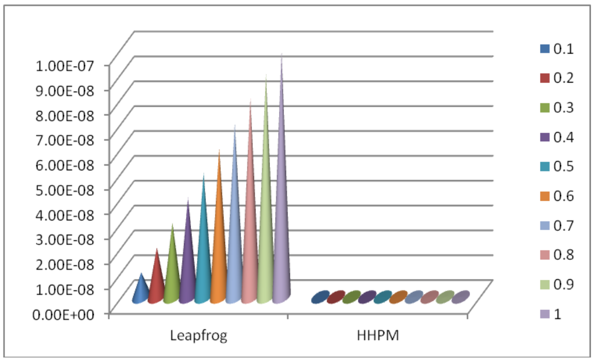

Fig. 6 Error estimation of Example 5.3 at y_2

V. Conclusion

In this work, the He's homotopy perturbation method (HHPM) has been successfully applied for solving linear first order fuzzy differential equations. Examples 1, 2 and 3 show that we can solve linear first order fuzzy differential equations and achieve a very good approximation to the actual solution of the equations by using only one iteration of the He's homotopy perturbation method (HHPM). As we can see this method will be useful for linear first order fuzzy differential equations. From the Fig. 1-6, it can be predicted that the error is very less in He's homotopy perturbation method when compared to the Leapfrog method [5]. .

Since the real world problems lead to the solution of linear first order fuzzy differential equations, it would be very interesting to extend this method to such problems. Research in this matter is one of our future goals.

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References

[1] S. Abbasbandy and T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor method, Journal of Computational Methods in Applied Mathematics. 2, 2002, 113-124.
[2] S. S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, IEEE Transactions on Systems, Man, and Cybernetics, 2, 1972, 30–34.
[3] D. Dubois and H. Prade, Towards fuzzy differential calculus.III. Differentiation, Fuzzy Sets and Systems, 8(3), 1982, 225–233.
[4] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications, 91(2), 1983, 552–558.
[5] R. Goetschel and Woxman, Elementary fuzzy calculus, Fuzzy Sets and Systems, 18, 1986, 31-43.
[6] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24(3), 1987, 301–317.
[7] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems, 35(3), 1990, 389–396.
[8] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24(3), 1987, 319–330.
[9] S. Sekar and K. Prabhavathi, "Numerical solution of first order linear fuzzy differential equations using Leapfrog method", IOSR Journal of Mathematics (IOSR-JM), Vol. 10, Issue 5 Ver. I, (Sep-Oct. 2014), pp. 07-12.
[10] M. Ma, M. Friedman and A. kandel, Numerical Solutions of Fuzzy Differential Equations, Fuzzy Sets and Systems, 105, 1999, 133-138.
[11] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method, Appl. Math. Comput. Vol 172, pp. 485-490, 2006.
[12] O. Abdul Aziz, I. Hashim, S.Momani, Application of homotopy perturbation method to fractional IVPs, J. Comput. Appl. Math. Vol 216(2), pp. 574-584, 2008.
[13] H. Aminikhah, M. Hemmatnezhad, An effective modification of the homotopy perturbation method for stiff systems of ordinary differential equations, Appl. Math. Lett., Vol 24, pp. 1502-1508, 2011.
[14] M.A. Asadi, F. Salehi, S.T. Mohyud-Din, M.M. Hosseini, Modified homotopy perturbation method for stiff delay differential equations (DDEs), Int. J. Phys. Sci., Vol 7(7), pp. 1025-1034, 2012.
[15] D. Baleanu, Alireza K. Golmankhaneh, Ali K. Golmankhaneh, Solving of the fractional non-linear and linear Schrodinger equations by homotopy perturbation method, Romanian J. Phys., Vol 54, Issue: 9-10, pp. 823-832, 2009.
[16] J. Biazar, H. Ghazvini, He's homotopy perturbation method for solving systems of Volterra integral equations, Chaos Solitons Fractals, 2007, doi:10.1016/j.chaos.2007.01.108.
[17] M.S.H. Chowdhury, I. Hashim, Solutions of class of singular second-order IVPs by homotopy perturbation method, Phys. Lett. A Vol 365, pp. 439-447, 2007.
[18] A. Golbabai, B. Keramati, Modified homotopy perturbation method for solving Fredholm integral equations, Chaos Solitons Fractals, Vol 37(5), pp. 1528-1537, 2008.
[19] M. Ghasemi, M. Tavassoli Kajani, E. Babolian, Numerical solutions of nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method, Appl. Math. Comput. Vol 188, pp. 446-449, 2007.
[20] J.H. He, Homotopy perturbation technique, Comp. Meth. Appl. Mech. Eng. Vol 178, pp. 257-262, 1999.
[21] J.H. He, A coupling method of homotopy technique and a perturbation technique for non linear problems, Int. J. Non linear Mech. Vol 35, pp. 37-43, 2000.
[22] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fractals, Vol 26, pp. 295-700, 2005.
[23] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A, Vol 350, pp. 87-88, 2006.
[24] A. Kadem, D. Baleanu, Homotopy perturbation method for the coupled fractional Lotka-Volterra equations, Romanian J. Phys., Vol 56 Issue: 3-4, pp. 332-338, 2011.
[25] F. Soltanian, M. Dehghan, S.M. Karbassi, Solution of the differential algebraic equations via homotopy perturbation method and their engineering applications, Inter. J. Comput. Math., Vol 87 (9), pp. 1950-1960, 2010.
[26] A. Yıldırım, Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method, Comput. Math. Appl. Vol 56(12), pp. 3175-3180, 2008.
[27] A. Yıldırım, He's Homotopy Perturbation Method for nonlinear differential-difference equations, Int. J. Comput. Math. doi:10.1080/ 00207160802247646, 2008.
[28] A. Yıldırım, The homotopy perturbation method for approximate solution of the modified KdV equation, Z. Naturforsch. Vol 63a, pp. 621-626, 2008.