IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 6 Ver. III (Nov. - Dec. 2015), PP 71-73
www.iosrjournals.org

A Treat from Topology

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Abstract: *In this article, we present Hillel Furstenberg's proof on the infinity of primes in a way that it can be read and understood even by a freshman of mathematics or engineering with no background in topology*

I. Introduction

The aim of this paper is to provide to the applied scientists (mature or budding) a glimpse of a beautiful area from abstract mathematics. It is well known from the time of Euclid that primes are infinitely many. But what might not be so well known that Hillel Furstenberg while a junior at Yeshiva Universityin New York City, provided an elegant proof of the this fact borrowing concepts from topology. As applied scientists or even applied mathematicians rarely take a course on mathematical topology, this proof by Furstenberg remains unknown to most of them, though there are many among them who have read, mastered and memorized Euclid's proof [1] on the infiniteness of primes during their student days in a course on discrete mathematics or theory of computation. It has been the intention of the author of this document to present Hillel's proof on the infinity of primes in a way that it can be read and understood even by a freshman of mathematics or engineering with no background in topology.

II. Preliminaries

Let X be a set and τ be a subset of power set $P(X)$. Then τ is a topology on X if

i. $\phi,\ X \in \tau$
ii. If $U_\alpha \in \tau$ for $\alpha \in I$, where I is an arbitrary index set, then $\bigcup_{\alpha\in I} U_\alpha \in \tau$
iii. If $U, V \in \tau$, then $U \cap V \in \tau$.

If $U \in \tau$, then U is called open. A set C is called closed if $X \setminus C \in \tau$. DeMorgan's law states that $X \setminus (C_1 \cup C_2 \cup ... \cup C_n) = (X \setminus C_1) \cap (X \setminus C_2) \cap ... \cap (X \setminus C_n)$. Hence if each of $C_1, ..., C_n$ is closed, then each of $X \setminus C_1, \ldots, X \setminus C_n$ is open and byiii. of definition of topology above $(X \setminus C_1) \cap (X \setminus C_2) \cap ... \cap (X \setminus C_n) = X \setminus (C_1 \cup C_2 \cup ... \cup C_n)$ is open. Thus a finite union $C_1 \cup C_2 \cup ... \cup C_n$ of closed sets is closed.

We now define a collection τ of subsets U of Z such that $U = \phi$ or $U = \bigcup_{\alpha\in I} S_\alpha$ where $S_\alpha = aZ + b$ for some $a, b \in Z,\ a \neq 0$. Notice that each S_α is characterized by a and b.

Theorem (2.1) $U \in \tau$ ifffor every $x \in U\ \exists\ a = a(x) \in Z,\ a \neq 0$ such $x \in aZ + x \subseteq U$.

Proof. Let $U \in \tau$ and $x \in U$. As $U = \bigcup_{\alpha\in I} S_\alpha$, $x \in S_\alpha = aZ + b$ for some $a, b \in Z,\ a \neq 0$. Then as $x \in aZ + b$, we have $x = am + b,\ m \in Z$. Then $aZ + b = a(Z + m) + b = aZ + (am + b) = aZ + x$. Thus if $U \in \tau$, then for every $x \in U$, $\exists\ aZ + x$ such that $x \in aZ + x \subseteq U$.

We now prove the converse. Let $U \subseteq Z$ and for every $x \in U,\ \exists\ a = a(x) \in Z,\ a \neq 0$ such $x \in aZ + x \subseteq U$. As $x \in aZ + x = a(x)Z + x,\ U \subseteq \bigcup_{x\in U} a(x)Z + x$. On the other hand, for each $x \in U$, $a(x)Z + x \subseteq U$. Hence $\bigcup_{x\in U} a(x)Z + x \subseteq U$. Thus $U = \bigcup_{x\in U} a(x)Z + x$ and therefore $U \in \tau$. ■

Theorem (2.2) Let $U, V \in \tau$. Then $U \cap V \in \tau$.

Proof. Let $x \in U \cap V$. Then $x \in U$ implies $\exists a \neq 0,$ such that $x \in aZ + x \subseteq U$ and $x \in V$ impels $\exists b \neq 0,$ $x \in bZ + x \subseteq V$. We will now show that $abZ + x \subseteq aZ + x \cap bZ + x.$

Let $y \in abZ + x.$ Then $y = abz + x$ for some $z \in Z.$ As $bz \in Z,$ $y = abz + x = a(bz) + x \in aZ + x.$ Similarly as $az \in Z,$ $y = abz + x = b(az) + x \in bZ + x.$ Hence if $y \in abZ + x,$ then $y \in aZ + x \cap bZ + x.$ Thus $abZ + x \subseteq aZ + x \cap bZ + x.$ Thus for every $x \in U \cap V,$ $x = ab.0 + x \in abZ + x$ and $abZ + x \subseteq aZ + x \cap bZ + x \subseteq U \cap V.$ Hence by Theorem (2.1) above, $U \cap V \in \tau.$ ■

Hence the collection τ (of subsets U of Z) defined earlier in our discussion is in fact a topology as:

i. $\phi \in \tau$ by definition and $Z \in \tau$ as $Z = 1Z + 0$ has the form of $aZ + b$

ii. if $U_\alpha \in \tau$ for $\alpha \in I,$ then $\bigcup_{\alpha \in I} U_\alpha \in \tau$ since if each U_α is an arbitrary union of sets of form $aZ + b$, then $\bigcup_{\alpha \in I} U_\alpha$ too is an arbitrary union of sets of form $aZ + b$.

iii. by Theorem (2.2), if $U, V \in \tau,$ then $U \cap V \in \tau.$

III. The Topological Proof

We state and prove a decomposition theorem for $Z.$

Theorem (3.1) Let $a \in Z$ and $a > 0.$ Then $Z = \bigcup_{j=0}^{a-1} (aZ + j)$ and for any $b \in \{0,1,\ldots, a-1\},$ $aZ + b = Z \setminus \bigcup_{j=0, j \neq b}^{a-1} (aZ + j)$.

Proof. Let $z \in Z$. As $a > 0,$ we can divide z by a. Hence $z = aq + r$ for some $q, r \in Z$ where $0 \leq r \leq a - 1.$ Thus $z \in aZ + r$. Since z was arbitrary, $Z \subseteq \bigcup_{j=0}^{a-1} (aZ + j)$. On the other hand, since $aZ + j \subseteq Z,$ we have $\bigcup_{j=0}^{a-1} (aZ + j) \subseteq Z.$ Thus $Z = \bigcup_{j=0}^{a-1} (aZ + j)$.

We now prove the other equality i.e. $aZ + b = Z \setminus \bigcup_{j=0, j \neq b}^{a-1} (aZ + j)$ for any $b \in \{0,1,\ldots, a-1\}.$ First note that for $j_1 \neq j_2,$ $(aZ + j_1) \cap (aZ + j_2) = \phi$ where $0 \leq j_1, j_2 \leq a - 1.$ Let $x \in (aZ + j_1) \cap (aZ + j_2)$. Then $x = az_1 + j_1$ as well as $x = az_2 + j_2$ for some $z_1, z_2 \in Z$. Assuming without loss that $j_1 > j_2,$ we have then $0 = a(z_1 - z_2) + (j_1 - j_2).$ Thus $a \mid (j_1 - j_2),$ a contradiction as $0 < j_1 - j_2 \leq a - 1.$ Hence for $j_1 \neq j_2,$ $(aZ + j_1) \cap (aZ + j_2) = \phi.$ As $Z = \bigcup_{j=0}^{a-1} (aZ + j)$, then $aZ + b = Z \setminus \bigcup_{j=0, j \neq b}^{a-1} (aZ + j)$ follows immediately. ■

Corollary (3.2). For $a, b \in Z,$ $a \neq 0,$ $aZ + b$ is closed.

Proof. As $aZ + b = Z \setminus \bigcup_{j=0, j \neq b}^{a-1} (aZ + j)$ by theorem above and $\bigcup_{j=0, j \neq b}^{a-1} (aZ + j)$ is open by axiom (ii) of topology, $aZ + b$ is closed as complement of an open set. ■

Finally we are ready to prove our master theorem.

Theorem (3.3). Primes are infinite.
Proof. Note that $Z \setminus \{\pm 1\} = \bigcup_p pZ$ where p ranges over all primes. If primes were finitely many, then $\bigcup_p pZ$ as finite union of closed sets would be a closed set. Then $\{\pm 1\} = Z \setminus \bigcup_p pZ$ as complement of closed set would be a open set, which is a contradiction, as every open set in τ contains a set of form $aZ + b$, hence infinitely many elements.■

References

[1] Ralph P. Grimaldi, Discrete and Combinatorial Mathematics- An Applied Introduction, Fifth Edition, Pearson (2004)