IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 6 Ver. IV (Nov. - Dec. 2015), PP 01-11
www.iosrjournals.org

Numerical Solutions of Second Order Boundary Value Problems by Galerkin Residual Method on Using Legendre Polynomials

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Abstract: *In this paper, an analysis is presented to find the numerical solutions of the second order linear and nonlinear differential equations with Robin, Neumann, Cauchy and Dirichlet boundary conditions. We use the Legendre piecewise polynomials to the approximate solutions of second order boundary value problems. Here the Legendre polynomials over the interval [0,1] are chosen as trial functions to satisfy the corresponding homogeneous form of the Dirichlet boundary conditions in the Galerkin weighted residual method. In addition to that the given differential equation over arbitrary finite domain [a,b] and the boundary conditions are converted into its equivalent form over the interval [0,1]. Numerical examples are considered to verify the effectiveness of the derivations. The numerical solutions in this study are compared with the exact solutions and also with the solutions of the existing methods. A reliable good accuracy is obtained in all cases.*
Keywords: *Galerkin Method, Linear and Nonlinear VBP, Legendre polynomials*

I. Introduction

In order to find out the numerical solutions of many linear and nonlinear problems in science and engineering, namely second order differential equations, we have seen that there are many methods to solve analytically but a few methods for solving numerically with various types of boundary conditions. In the literature of numerical analysis solving a second order boundary value problem of differential equations, many authors have attempted to obtain higher accuracy rapidly by using numerical methods. Among various numerical techniques, finite difference method has been widely used but it takes more computational costs to get higher accuracy. In this method, a large number of parameters are required and it can not be used to evaluate the value of the desired points between two grid points. For this reason, Galerkin weighted residual method is widely used to find the approximate solutions to any point in the domain of the problem.

Continuous or piecewise polynomials are incredibly useful as mathematical tools since they are precisely defined and can be differentiated and integrated easily. They can be approximated any function to any accuracy desired [1], spline functions have been studied extensively in [2-9]. Solving boundary value problems only with Dirichlet boundary conditions has been attempted in [4] while Bernstein polynomials [10, 11] have been used to solve the two point boundary value problems very recently by the authors Bhatti and Bracken [1] rigorously by the Galerkin method. But it is limited to the second order boundary value problems with Dirichlet boundary conditions and to first order nonlinear differential equation. On the other hand, Ramadan et al. [2] has studied linear boundary value problems with Neumann boundary conditions using quadratic cubic polynomial splines and nonpolynomial splines. We have also found that the linear boundary value problems with Robin boundary conditions have been solved using finite difference method [12] and Sinc-Collocation method [13], respectively. Thus except [9], little concentration has been given to solve the second order nonlinear boundary value problems with dirichlet, Neumann and Robin boundary conditions. Therefore, the aim of this paper is to present the Galerkin weighted residual method to solve both linear and nonlinear second order boundary value problems with all types of boundary conditions. But none has attempted, to the knowledge of the present authors, using these polynomials to solve the second order boundary value problems. Thus in this paper, we have given our attention to solve some linear and nonlinear boundary value problems numerically with different types of boundary conditions though it is originated in [1].

However, in this paper, we have solved second order differential equations with various types of boundary conditions numerically by the technique of very well-known Galerkin method [15] and Legendre piecewise polynomials [14] are used as trial function in the basis. Individual formulas for each boundary value problem consisting of Dirichlet, Neumann, Robin and Cauchy boundary conditions are derived respectively. Numerical examples of both linear and nonlinear boundary value problems are considered to verify the effectiveness of the derived formulas and are also compared with the exact solutions. All derivations are performed by MATLAB programming language.

II. Legendre Polynomials

The solution of the Legendre's equation is called the Legendre polynomial of degree n and is denoted by $p_n(x)$.

Then $p_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$

where $N = \frac{n}{2}$ for n even

and $N = \frac{n-1}{2}$ for n odd

The first few Legendre polynomials are

$p_1(x) = x$

$p_2(x) = \frac{1}{2}(3x^2 - 1)$

$p_3(x) = \frac{1}{2}(5x^3 - 3x)$

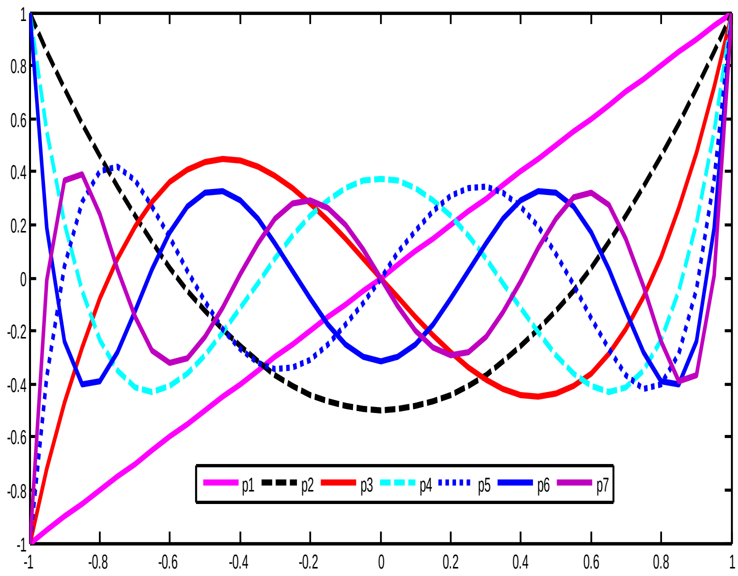
$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$p_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$p_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

$p_7 = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$ etc

Graphs of first few Legendre polynomials



Shifted Legendre polynomials

Here the "shifting" function (in fact, it is an affine transformation) is chosen such that it bijectively maps the interval [0, 1] to the interval [−1, 1], implying that the polynomials are
An explicit expression for the shifted Legendre polynomials is given by orthogonal on [0, 1]:

$$\tilde{p}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k$$

The analogue of Rodrigues' formula for the shifted Legendre polynomials is

$$\tilde{p}(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^2 - x)^n$$

To satisfy the condition $p_n(0) = p_n(1) = 0,\ n \geq 1$, we modified the shifted Legendre polynomials given above in the following form

$$p_n(x) = \left[\frac{1}{n!}\frac{d^n}{dx^n}(x^2 - x)^n - (-1)^n\right] \times (x-1).$$

Since Legendre polynomials have special properties at $x = 0$ and $x = 1$: $p_n(0) = 0$ and $p_n(1) = 0,\ n \geq 1$ respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the Dirichlet boundary conditions to derive the matrix formulation of second order BVP over the interval [0,1].

III. Formulation Of Second Order Bvp

We consider the general second order linear BVP [15]:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = r(x), \quad a < x < b \tag{1a}$$

$$\alpha_0 u(a) + \alpha_1 u'(a) = c_1,\ \beta_0 u(b) + \beta_1 u'(b) = c_2 \tag{1b}$$

where $p(x), q(x)$ and r are specified continuous functions and $\alpha_0, \alpha_1, \beta_0, \beta_1, c_1, c_2$ are specified numbers. Since our aim is to use the Legendre polynomials as trial functions which are derived over the interval [0,1], so the BVP (1) is to be converted to an equivalent problem on [0,1] by replacing x by $(b-a)x + a$, and thus we have:

$$-\frac{d}{dx}\left(\tilde{p}(x)\frac{du}{dx}\right) + \tilde{q}(x)u = r(x), 0 < x < 1 \tag{2a}$$

$$\alpha_0 u(0) + \frac{\alpha_1}{b-a}u'(0) = c_1,\ \beta_0 u(1) + \frac{\beta_1}{b-a}u'(1) = c_2 \tag{2b}$$

where $\tilde{p}(x) = \dfrac{1}{(b-a)^2} p((b-a)x + a), \quad \tilde{q}(x) = q((b-a)x + a), \tilde{r}(x) = r((b-a)x + a)$

Using Legendre polynomials, $p_i(x)$ we assume an approximate solution in a form,

$$\tilde{u}(x) = \sum_{i=1}^{n} a_i p_i(x), \quad n \geq 1 \tag{3}$$

Now the Galerkin weighted residual equations corresponding to the differential equation (1a) is given by

$$\int_0^1 \left[-\frac{d}{dx}\left(\tilde{p}(x)\frac{d\tilde{u}}{dx}\right) + \tilde{q}(x)\tilde{u} - \tilde{r}(x)\right] p_j(x)dx = 0,\ j = 1,2,\ldots,n \tag{4}$$

After minor simplification, from (2) we can obtain

$$\sum_{i=0}^{n}\left[\int_0^1\left[\tilde{p}(x)\frac{dp_i}{dx}\frac{dp_j}{dx} + \tilde{q}(x)p_i(x)p_j(x)\right]dx + \frac{\beta_0(b-a)\tilde{p}(1)p_i(1)p_j(1)}{\beta_1} - \frac{\alpha_0(b-a)\tilde{p}(0)p_i(0)p_j(0)}{\alpha_1}\right]a_i$$

$$= \int_0^1 \tilde{r}(x)p_j(x)dx + \frac{c_2(b-a)\tilde{p}(1)p_j(1)}{\beta_1} - \frac{c_1(b-a)\tilde{p}(0)p_j(0)}{\alpha_1} \tag{5}$$

Or, equivalently in matrix form

$$\sum_{i-1}^{n} K_{i,j} a_i = F_j\,, \quad j = 1,2,3,\cdots\cdots\cdots, n \tag{6a}$$

where $$K_{i,j} = \int_0^1 \left[\tilde{p}(x) \frac{dp_i}{dx}\frac{dp_j}{dx} + \tilde{q}(x) p_i(x) p_j(x) \right] dx + \frac{\beta_0 (b-a)\tilde{p}(1) p_i(1) p_j(1)}{\beta_1} - \frac{\alpha_0 (b-a)\tilde{p}(0) p_i(0) p_j(0)}{\alpha_1} \tag{6b}$$

$$F_j = \int_0^1 \tilde{r}(x) p_j(x) dx + \frac{c_2 (b-a)\tilde{p}(1) p_j(1)}{\beta_1} - \frac{c_1 (b-a)\tilde{p}(0) p_j(0)}{\alpha_1}\,, \; j = 1,2,\cdots, n \tag{6c}$$

Solving the system (6a), we find the values of the parameters $a_i \; (i = 1,2,\ldots, n)$ and then substituting these parameters into eqn. (3), we get the approximate solution of the boundary value problem (2). If we replace x by $\frac{x-a}{b-a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the boundary value problem (1).

Now we discuss the different types of boundary value problems using various types of boundary conditions as follows:

Case 1: The matrix formulation with the Robin boundary conditions $(\alpha_0 \neq 0, \alpha_1 \neq 0, \beta_0 \neq 0, \beta_1 \neq 0)$, are already discussed in equation (6).

Case 2: The matrix formulation of the differential equation (1a) with the Dirichlet boundary conditions $(i.e., \alpha_0 \neq 0, \alpha_1 = 0, \beta_0 \neq 0, \beta_1 = 0)$ is given by

$$\sum_{i=1}^{n} K_{i,j}\, a_i = F_j\,, \; j = 1,2,\cdots, n \tag{7a}$$

where $$K_{i,j} = \int_0^1 \left[\tilde{p}(x) \frac{dp_i}{dx}\frac{dp_j}{dx} + \tilde{q}(x) p_i(x) p_j(x) \right] dx,\; i, j = 1,2,\cdots, n \tag{7b}$$

$$F_j = \int_0^1 \left[\tilde{r}(x) p_j(x) - \tilde{p}(x) \frac{d\theta_0}{dx}\frac{dp_j}{dx} + \tilde{q}(x)\theta_0(x) p_j(x) \right] dx,\; j = 1,2,\cdots n \tag{7c}$$

Case 3: The approximate solution of the differential equation (1a) consisting of Neumann boundary conditions $(i.e., \alpha_0 = 0, \alpha_1 \neq 0, \beta_0 = 0, \beta_1 \neq 0)$ is given by

$$\sum_{i=1}^{n} K_{i,j}\, a_i = F_j\,, \; j = 1,2,\cdots, n \tag{8a}$$

where $$K_{i,j} = \int_0^1 \left[\tilde{p}(x) \frac{dp_i}{dx}\frac{dp_j}{dx} + \tilde{q}(x) p_i(x) p_j(x) \right] dx, i, j = 1,2,\cdots, n \tag{8b}$$

$$F_j = \int_0^1 \tilde{r}(x) p_j(x) dx + \frac{c_2 (b-a)\tilde{p}(1) p_j(1)}{\beta_1} - \frac{c_1 (b-a)\tilde{p}(0) p_j(0)}{\alpha_1}\,, \; j = 1,2,\cdots, n \tag{8c}$$

Case 4(i): The approximate solution of the differential equation (1a) consisting of Cauchy boundary conditions $(i.e.,\; \alpha_1 \neq 0, \beta_1 = 0)$ is given by

$$\sum_{i=1}^{n} K_{i,j}\, a_i = F_j\,, \; j = 1,2,\cdots, n \tag{9a}$$

where $$K_{i,j} = \int_0^1 \left[\tilde{p}(x) \frac{dp_i}{dx}\frac{dp_j}{dx} + \tilde{q}(x) p_i(x) p_j(x) \right] dx - \frac{\alpha_0 \tilde{p}(0) p_i(0) p_j(0)}{\alpha_1}, i, j = 1,2,\cdots, n \tag{9b}$$

$$F_j = \int_0^1 \left[\tilde{r}(x) p_j(x) - \tilde{p}(x) \frac{d\theta_0}{dx}\frac{dp_j}{dx} - \tilde{q}(x)\theta_0(x) p_j(x) \right] dx - \frac{c_1 \tilde{p}(0) p_j(0)}{\alpha_1} + \frac{\alpha_0 \tilde{p}(0)\theta_0(0) p_j(0)}{\alpha_1} \tag{9c}$$

Case 4(ii): The matrix formulation with the Cauchy boundary conditions $(i.e., \alpha_1 = 0, \beta_1 \neq 0)$ is given by

$$\sum_{i=1}^{n} K_{i,j}\, a_i = F_j\,,\ j = 1,2,\cdots,n \tag{10a}$$

where $$K_{i,j} = \int_0^1 \left[\tilde{p}(x) \frac{dp_i}{dx} \frac{dp_j}{dx} + \tilde{q}(x) p_i(x) p_j(x) \right] dx + \frac{\beta_0 \tilde{p}(1) p_i(1) p_j(1)}{\beta_1},\ i, j = 1,2,\cdots,n \tag{10b}$$

$$F_j = \int_0^1 \left[\tilde{r}(x) p_j(x) - \tilde{p}(x) \frac{d\theta_0}{dx} \frac{dp_j}{dx} - \tilde{q}(x)\theta_0(x) p_j(x) \right] dx + \frac{c_2 \tilde{p}(1) p_j(1)}{\beta_1} - \frac{\beta_0 \tilde{p}(1)\theta_0(1) p_j(1)}{\beta_1},\ j = 1,2,\cdots n \tag{10c}$$

Similar calculation for nonlinear boundary value problems using the Legendre polynomials can be derived which will be discussed through numerical examples in next portion.

IV. Numerical Examples

In this section, we explain four linear and two nonlinear boundary value problems which are available in the existing literatures, considering four types of boundary conditions to verify the effectiveness of the present formulations described in the previous sections. The convergence of each linear boundary value problem is calculated by

$$E = \left| u_{n+1}(x) - u_n(x) \right| < \delta$$

where $u_n(x)$ represents the approximate solution by the proposed method using n-th degree polynomial approximation. The convergence of nonlinear boundary value problem is assumed when the absolute error of two consecutive iterations is recorded below the convergence criterion δ such that

$$\left| \tilde{u}^{N+1} - \tilde{u}^{N} \right| < \delta$$

where N is the Newton's iteration number and δ varies from 10^{-8}.

Example1. First we consider the boundary value problem with Robin boundary conditions [15]:

$$-\frac{d^2u}{dx^2} + u = 2\cos x,\ \frac{\pi}{2} < x < \pi \tag{11a}$$

$$u'\left(\frac{\pi}{2}\right) + 3u\left(\frac{\pi}{2}\right) = -1,\quad u'(\pi) + 4u(\pi) = -4 \tag{11b}$$

The exact solution is $u(x) = \cos x$.

The boundary value problem (11) over [0, 1] is equivalent to the BVP

$$-\frac{1}{\left(\frac{\pi}{2}\right)^2} \frac{d^2u}{dx^2} + u = 2\cos\left(\frac{\pi}{2}x + \frac{\pi}{2}\right),\ 0 < x < 1$$

$$\frac{2}{\pi} u'(0) + 3u(0) = -1,\quad \frac{2}{\pi} u'(1) + 4u(1) = -4$$

Using the formula derived in **Case-1,** equation (6) and using different number of Legendre polynomials, the approximate solutions are shown in **Table 1**. It is observe that the accuracy is found nearly the order 10^{-5}, 10^{-6}, 10^{-6}, 10^{-7} on using 6, 7, 8, and 9 Legendre polynomials respectively.

Table 1: Exact, approximate solutions and absolute differences for the example 1

x	Exact	Approximate	Error	Approximate	Error
		6 Legendre polynomials		7 Legendre polynomials	
π/2	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
11π/20	-0.1564344650	-0.1563952198	3.9245266397E-005	-0.1564361741	1.7090241794E-006
3π/5	-0.3090169944	-0.3090826905	6.5696082539E-005	-0.3090293644	1.2370074109E-005
13π/20	-0.4539904997	-0.4540545634	6.4063642331E-005	-0.4539871936	3.3061242330E-006
7π/10	-0.5877852523	-0.5877674749	1.7777435319E-005	-0.5877722257	1.3026583116E-005
3π/4	-0.7071067812	-0.7070318602	7.4920958162E-005	-0.7071038639	2.9172939944E-006
4π/5	-0.8090169944	-0.8089692169	4.7777521913E-005	-0.8090274888	1.0494427349E-005
17π/20	-0.8910065242	-0.8910363629	2.9838679610E-005	-0.8910135046	6.9803926110E-006
9π/10	-0.9510565163	-0.9511166917	6.0175450790E-005	-0.9510489959	7.5203691469E-006
19π/20	-0.9876883406	-0.9876784226	9.9179536835E-006	-0.9876846996	3.6409813292E-006
π	-1.0000000000	-1.0000000000	0.0000000000	-1.0000000000	0.0000000000
		8 Legendre polynomials		9 Legendre polynomials	
π/2	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
11π/20	-0.1564344650	-0.1564359285	1.4634359017E-006	-0.1564348025	3.3747539643E-007
3π/5	-0.3090169944	-0.3090176234	6.2907488652E-007	-0.3090167598	2.3457270337E-007
13π/20	-0.4539904997	-0.4539882914	2.2082959843E-006	-0.4539903134	1.8630984755E-007
7π/10	-0.5877852523	-0.5877849061	3.4624056477E-007	-0.5877856122	3.5985972813E-007
3π/4	-0.7071067812	-0.7071088461	2.0649209902E-006	-0.7071068440	6.2781989829E-008
4π/5	-0.8090169944	-0.8090175407	5.4633655033E-007	-0.8090166411	3.5332033022E-007
17π/20	-0.8910065242	-0.8910047081	1.8160907849E-006	-0.8910065885	6.4299355618E-008
9π/10	-0.9510565163	-0.9510563902	1.2606701094E-007	-0.9510567651	2.4875544480E-007
19π/20	-0.9876883406	-0.9876895985	1.2579438687E-006	-0.9876880856	2.5495029321E-007
π	-1.0000000000	-1.0000000000	0.0000000000	-1.0000000000	0.0000000000

Example2. We consider the boundary value problem with Dirichlet boundary conditions [1]:

$$\frac{d^2u}{dx^2} + u = x^2 e^{-x}, \quad 0 < x < 10 \tag{12a}$$

$$u(0) = 0, u(10) = 0 \tag{12b}$$

The exact solution is: $\frac{1}{2}e^{-x}\left[1 + 2x + x^2 - e^x \cos x - 2e^x\left\{-\frac{\cot 10}{2} + \frac{121\cos ec10}{2e^{10}}\right\}\sin x\right]$

The boundary value problem (12) is similar to the VBP

$$\frac{1}{100}\frac{d^2u}{dx^2} + u = 100x^2 e^{-10x}, \quad 0 < x < 1 \tag{13a}$$

$$u(0) = u(1) = 0 \tag{13b}$$

Using the formula calculated in **Case-2**, equation (7), the approximate solutions are summarized in **Table 2.** It is shown that the accuracy up to 3, 5, 6 and 8 significant digits are obtained for 8, 10, 12 and 15 Legendre polynomials respectively.

Table 2: Exact, approximate solutions and absolute differences for the example 2

x	Exact	Approximate	Error	Approximate	Error
		8 Legendre Polynomials		10 Legendre Polynomials	
0.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000
0.1	1.1187780396	1.1170936907	1.6843488272E-003	1.1187696958	8.3437592517E-006
0.2	1.5229010422	1.5275119317	4.6108895155E-003	1.5229254770	2.4434769731E-005
0.3	1.0028335888	0.9991649652	3.6686235984E-003	1.0028621730	2.8584174599E-005
0.4	-0.0316812662	-0.0335749590	1.8936928166E-003	-0.0317936975	1.1243135195E-004
0.5	-0.7648884663	-0.7601267725	4.7616938417E-003	-0.7647484694	1.3999696859E-004
0.6	-0.6362448424	-0.6381149360	1.8700936538E-003	-0.6363187413	7.3898882488E-005
0.7	0.1621981240	0.1584875515	3.7105725450E-003	0.1621711868	2.6937180385E-005
0.8	0.8543002645	0.8589605755	4.6603109949E-003	0.8543785180	7.8253508555E-005
0.9	0.7816320450	0.7799092631	1.7227819094E-003	0.7815831186	4.8926372264E-005
1.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000
		12 Legendre Polynomials		15 Legendre Polynomials	

0.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000
0.1	1.1187780396	1.1187868241	8.7845183310E-006	1.1187780208	1.8750267117E-008
0.2	1.5229010422	1.5228940750	6.9672726579E-006	1.5229010732	3.0956165631E-008
0.3	1.0028335888	1.0028413857	7.7969029351E-006	1.0028335785	1.0297521058E-008
0.4	-0.0316812662	-0.0316917502	1.0484051623E-005	-0.0316812686	2.4126875567E-009
0.5	-0.7648884663	-0.7648764993	1.1967000683E-005	-0.7648884575	8.7905381863E-009
0.6	-0.6362448424	-0.6362555845	1.0742144593E-005	-0.6362448562	1.3803168386E-008
0.7	0.1621981240	0.1622062945	8.1704803209E-006	0.1621981449	2.0844985821E-008
0.8	0.8543002645	0.8542931007	7.1637940648E-006	0.8543002369	2.7584562634E-008
0.9	0.7816320450	0.7816403137	8.2686651681E-006	0.7816320520	6.9470443842E-009
1.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000

Example3. In this case we consider the boundary value problem with Neumann boundary conditions [2]:

$$\frac{d^2u}{dx^2}+u=-1, \quad 0<x<1 \tag{14a}$$

$$u'(0)=\frac{1-\cos 1}{\sin 1}, \quad u'(1)=\frac{1-\cos 1}{\sin 1} \tag{14b}$$

whose exact solution is, $u(x)=\cos x+\frac{1-\cos 1}{\sin 1}\sin x-1$.

Applying the formula derived in **Case-3**,equation (8), the approximate solutions, given in **Table 3**, are obtained on using 5, 7, 8 and 10 Legendre polynomials with the remarkable accuracy nearly the order of 10^{-11}, 10^{-13}, 10^{-13} and 10^{-17}. On the other hand, Ramadan et al. [6] has found nearly the accuracy of order 10^{-6} and 10^{-6}, and 10^{-8} on using quadratic and cubic polynomial splines, and nonpolynomial spline respectively with h=1/128 where h= (b-a)/N, a and b are the endpoints of the domain and N is number of subdivision of intervals [a,b].

Table 3: Exact, approximate solutions and absolute differences for the example 3

x	Exact	Approximate	Error	Approximate	Error
		5 Legendre Polynomials		7 Legendre Polynomials	
0.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000
0.1	0.0495434094	0.0495434085	8.1779601840E-010	0.0495434094	5.3554383150E-013
0.2	0.0886001279	0.0886001278	8.6266188637E-011	0.0886001279	7.7055029024E-013
0.3	0.1167799138	0.1167799150	1.2220623125E-009	0.1167799138	6.1556315600E-013
0.4	0.1338012040	0.1338012039	9.0318336143E-011	0.1338012040	2.8521629503E-013
0.5	0.1394939273	0.1394939260	1.2807547800E-009	0.1394939273	8.7904683532E-013
0.6	0.1338012040	0.1338012039	9.0318585944E-011	0.1338012040	2.8493873927E-013
0.7	0.1167799138	0.1167799150	1.2220623263E-009	0.1167799138	6.1554927822E-013
0.8	0.0886001279	0.0886001278	8.6266216393E-011	0.0886001279	7.7057804582E-013
0.9	0.0495434094	0.0495434085	8.1779603922E-010	0.0495434094	5.3552995372E-013
1.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000
		8 Legendre Polynomials		10 Legendre Polynomials	
0.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000
0.1	0.0495434094	0.0495434094	5.3554383150E-013	0.0495434094	2.7755575616E-016
0.2	0.0886001279	0.0886001279	7.7055029024E-013	0.0886001279	8.3266726847E-017
0.3	0.1167799138	0.1167799138	6.1556315600E-013	0.1167799138	8.3266726847E-017
0.4	0.1338012040	0.1338012040	2.8521629503E-013	0.1338012040	3.3306690739E-016
0.5	0.1394939273	0.1394939273	8.7904683532E-013	0.1394939273	3.3306690739E-016
0.6	0.1338012040	0.1338012040	2.8493873927E-013	0.1338012040	8.3266726847E-017
0.7	0.1167799138	0.1167799138	6.1554927822E-013	0.1167799138	8.3266726847E-017
0.8	0.0886001279	0.0886001279	7.7057804582E-013	0.0886001279	9.7144514655E-017
0.9	0.0495434094	0.0495434094	5.3552995372E-013	0.0495434094	2.7061686225E-016
1.0	0.0000000000	0.0000000000	0.0000000000E+000	0.0000000000	0.0000000000E+000

Example4. We consider the Cauchy (mixed) boundary value problem [4]:

$$\frac{d^2u}{dx^2}-3(u-25)=0, \quad 0\le x\le 2 \tag{15a}$$

with mixed boundary conditions

$$u(0)=150\ (Dirichlet), u'(2)=0\,(Neumann) \tag{15b}$$

whose exact solution is $u(x) = 25 + 125e^{-\sqrt{3}x} - \frac{125e^{-\sqrt{3}x}}{1+e^{4\sqrt{3}}} + \frac{125e^{\sqrt{3}x}}{1+e^{4\sqrt{3}}}$

Using the formula illustrated in **Case-4(ii),** equation (10), and using different number of Legendre polynomials, the approximate solutions are shown in **Table 4.** It is observe that the accuracy is found nearly the order 10^{-5}, 10^{-7}, 10^{-9} and 10^{-14} on using 8, 10, 12, and 15 Legendre polynomials respectively.

Table 4: Exact, approximate solutions and absolute differences for the example 4

x	Exact	Approximate	Error	Approximate	Error
		8 Legendre polynomials		**10 Legendre polynomials**	
0.0	150.0000000000	150.0000000000	0.0000000000E+000	150.0000000000	0.0000000000E+000
0.1	130.1632369724	130.1632361028	8.6964047341E-007	130.1632367772	1.9518458316E-007
0.2	113.4892661882	113.4892227060	4.3482237473E-005	113.4892661238	6.4393191224E-008
0.3	99.4766167249	99.4766067840	9.9408884751E-006	99.4766170265	3.0160821041E-007
0.4	87.7038570984	87.7038954397	3.8341331972E-005	87.7038572221	1.2375100766E-007
0.5	77.8169206793	77.8169675873	4.6908003981E-005	77.8169204267	2.5263204861E-007
0.6	69.5184571131	69.5184703563	1.3243194047E-005	69.5184568118	3.0132471807E-007
0.7	62.5588894853	62.5588587219	3.0763410692E-005	62.5588894927	7.3751351692E-009
0.8	56.7289082771	56.7288570188	5.1258324511E-005	56.7289085840	3.0687399288E-007
0.9	51.8531763663	51.8531411416	3.5224774393E-005	51.8531766501	2.8373816008E-007
1.0	47.7850557499	47.7850603820	4.6321808611E-006	47.7850557239	2.5985485763E-008
1.1	44.4021973948	44.4022380045	4.0609714048E-005	44.4021970899	3.0494597070E-007
1.2	41.6028615824	41.6029098061	4.8223707843E-005	41.6028613102	2.7216093912E-007
1.3	39.3028580779	39.3028800573	2.1979341483E-005	39.3028581183	4.0339642737E-008
1.4	37.4330141035	37.4329943677	1.9735765207E-005	37.4330144044	3.0088499159E-007
1.5	35.9370939611	35.9370491681	4.4793006822E-005	35.9370941670	2.0580901605E-007
1.6	34.7701077404	34.7700776491	3.0091356948E-005	34.7701075952	1.4517870284E-007
1.7	33.8969582431	33.8969721439	1.3 900810046E-005	33.8969579798	2.6333222536E-007
1.8	33.2913854326	33.2914230933	3.7660648488E-005	33.2913855119	7.9248636098E-008
1.9	32.9351766612	32.9351748738	1.7874420521E-006	32.9351768223	1.6105848744E-007
2.0	32.8176189233	32.8176189248	0.0000000000E+000	32.8176189233	0.0000000000E+000
		12 Legendre polynomials		**15 Legendre polynomials**	
0.0	150.0000000000	150.0000000000	0.0000000000E+000	150.0000000000	0.0000000000E+000
0.1	130.1632369724	130.1632369713	1.1265512967E-009	130.1632369724	2.8421709430E-014
0.2	113.4892661882	113.4892661894	1.1613110473E-009	113.4892661882	1.4210854715E-014
0.3	99.4766167249	99.4766167252	2.8325075618E-010	99.4766167249	2.5579538487E-013
0.4	87.7038570984	87.7038570969	1.5066063952E-009	87.7038570984	2.7000623959E-013
0.5	77.8169206793	77.8169206790	2.9929481116E-010	77.8169206793	5.6843418861E-014
0.6	69.5184571131	69.5184571146	1.4935892523E-009	69.5184571131	3.4106051316E-013
0.7	62.5588894853	62.5588894862	9.1058893759E-010	62.5588894853	1.4210854715E-014
0.8	56.7289082771	56.7289082761	1.0389911154E-009	56.7289082771	3.6237679524E-013
0.9	51.8531763663	51.8531763648	1.5080843241E-009	51.8531763663	2.8421709430E-014
1.0	47.7850557499	47.7850557500	1.0601297618E-010	47.7850557499	3.6948222260E-013
1.1	44.4021973948	44.4021973964	1.5544259213E-009	44.4021973948	7.1054273576E-014
1.2	41.6028615824	41.6028615833	8.4782669774E-010	41.6028615824	2.9132252166E-013
1.3	39.3028580779	39.3028580769	1.0288943031E-009	39.3028580779	8.5265128291E-014
1.4	37.4330141035	37.4330141021	1.3430536683E-009	37.4330141035	2.9842794902E-013
1.5	35.9370939611	35.9370939616	4.4136072574E-010	35.9370939611	8.5265128291E-014
1.6	34.7701077404	34.7701077418	1.3676100252E-009	34.7701077404	1.8474111130E-013
1.7	33.8969582431	33.8969582428	3.7826453081E-010	33.8969582431	3.6948222260E-013
1.8	33.2913854326	33.2913854316	9.9556984878E-010	33.2913854326	4.2632564146E-014
1.9	32.9351766612	32.9351766622	1.0126584016E-009	32.9351766612	1.0658141036E-013
2.0	32.8176189233	32.8176189233	0.0000000000E+000	32.8176189233	0.0000000000E+000

We now also apply the procedure described in section 3, formulation of second order linear BVP, to find the numerical solutions of one nonlinear second order boundary value problem.

Example5. We consider a nonlinear BVP with Dirichlet boundary conditions [16]

$$\frac{d^2u}{dx^2} + \frac{1}{8}u\frac{du}{dx} = \left(4 + \frac{1}{4}x^3\right), \quad 1 < x < 3 \tag{16a}$$

$$u(1) = 17 \text{ and } u(3) = \frac{43}{3} \tag{16b}$$

The exact solution of the problem is given by $u(x) = x^2 + \frac{16}{x}$

To implement the Legendre polynomials, first we convert the BVP (16) to an equivalent BVP on the interval [0,1] by replacing x by $2x+1$ such that

$$\frac{d^2u}{dx^2}+\frac{1}{4}u\frac{du}{dx}=16+(2x+1)^3,\ 0<x<1 \tag{17a}$$

$$u(0)=17 \text{ and } u(1)=\frac{43}{3} \tag{17b}$$

Suppose that the approximate solution of the boundary value problem (17) applying the Legendre polynomials is given by

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}a_i p_i(x),\ n>1 \tag{18}$$

Where $\theta_0(x)=17-\frac{8x}{3}$ is specified by the dirichlet boundary conditions at $x=0$ and $x=1$ and $p_i(0)=p_i(1)=0$ for each $i=1,2,\cdots,n.$

The weighted residual equations of (17a) corresponding to the approximation solution (18), given by

$$\int_0^1\left(\frac{d^2\tilde{u}}{dx^2}+\frac{1}{4}\tilde{u}\frac{d\tilde{u}}{dx}-\left[16+(2x+1)^3\right]\right)p_k(x)dx=0,k=1,2,\cdots,n \tag{19}$$

Exploiting integration by parts with minor simplifications, we have

$$\sum_{i=1}^{n}\left[\int_0^1\left[-\frac{dp_i}{dx}\frac{dp_k}{dx}+\frac{1}{4}\left(\theta_0\frac{dp_i}{dx}p_k+\frac{d\theta_0}{dx}p_ip_k\right)+\frac{1}{4}\sum_{j=1}^{n}a_j\left(p_i\frac{dp_j}{dx}p_k\right)\right]dx\right]a_i$$
$$=\int_0^1\left[\left(16+(2x+1)^3\right)p_k+\frac{d\theta_0}{dx}\frac{dp_k}{dx}-\frac{1}{4}\theta_0\frac{d\theta_0}{dx}p_k\right]dx,k=1,2,\cdots,n \tag{20}$$

The above equation (20) is equivalent to the matrix form

$$(D+C)A=G \tag{21a}$$

where the elements of the matrix A, C, D and G are $a_i, c_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[-\frac{dp_i}{dx}\frac{dp_k}{dx}+\frac{1}{4}\left(\theta_0\frac{dp_i}{dx}p_k+\frac{d\theta_0}{dx}p_ip_k\right)\right]dx \tag{21b}$$

$$c_{i,k}=\frac{1}{4}\sum_{j=1}^{n}a_j\int_0^1\left(p_i\frac{dp_j}{dx}p_k\right)dx \tag{21c}$$

$$g_k=\int_0^1\left[\left(16+(2x+1)^3\right)p_k+\frac{d\theta_0}{dx}\frac{dp_k}{dx}-\frac{1}{4}\theta_0\frac{d\theta_0}{dx}p_k\right]dx,k=1,2,\cdots,n \tag{21d}$$

The initial values of these coefficients a_i are obtained by applying the Galerkin method to the BVP neglecting the nonlinear term in (17a). Therefore, to find the initial coefficients, we will solve the system

$$DA=G \tag{22a}$$

where the elements of the matrices are given by

$$d_{i,k}=-\int_0^1\frac{dp_i}{dx}\frac{dp_k}{dx}dx \tag{22b}$$

$$g_k=\int_0^1\left[\left(16+(2x+1)^3\right)p_k+\frac{d\theta_0}{dx}\frac{dp_k}{dx}\right]dx,k=1,2,\cdots,n \tag{22c}$$

Once the initial values of the parameters a_i are obtained from equation (22a), they are substituted into equation (21a) to obtain new estimates for the values of a_i. This iteration process continues until the converged values of the unknowns are obtained. Putting the final values of coefficients into equation (18), we obtain an approximate solution of the BVP (17), and if we replace x by $\frac{(x-1)}{2}$ in this solution we will get the approximate solution of the given BVP (16).

Using first 10 and 15 Legendre polynomials with 10 iterations, the absolute differences between exact and the approximate solutions are given in **Table 5**. It is observed that the accuracy is found of the order nearly 10^{-6} and 10^{-9} on using 10 and 15 Legendre polynomials respectively.

Table 5: Exact, approximate solutions and absolute differences of example 5 using 10 iterations

x	Exact	Approximate	Error	Approximate	Error
		10 Legendre polynomials		15 Legendre polynomials	
1.0	17.0000000000	17.0000000000	0.000000E+000	17.0000000000	0.000000E+000
1.1	15.7554545455	15.7554513780	3.167443E-006	15.7554545479	2.436138E-009
1.2	14.7733333333	14.7733354460	2.112697E-006	14.7733333301	3.184004E-009
1.3	13.9976923077	13.9976948749	2.567226E-006	13.9976923079	2.160068E-010
1.4	13.3885714286	13.3885686469	2.781699E-006	13.3885714325	3.968140E-009
1.5	12.9166666667	12.9166634176	3.249050E-006	12.9166666616	5.087697E-009
1.6	12.5600000000	12.5600012904	1.290388E-006	12.5599999987	1.277906E-009
1.7	12.3017647059	12.3017686614	3.955476E-006	12.3017647109	5.053868E-009
1.8	12.1288888889	12.1288904019	1.512994E-006	12.1288888877	1.141489E-009
1.9	12.0310526316	12.0310499612	2.670420E-006	12.0310526261	5.513813E-009
2.0	12.0000000000	11.9999964073	3.592663E-006	12.0000000012	1.181718E-009
2.1	12.0290476190	12.0290472659	3.531724E-007	12.0290476234	4.306418E-009
2.2	12.1127272727	12.1127304365	3.163751E-006	12.1127272702	2.555799E-009
2.3	12.2465217391	12.2465245358	2.796678E-006	12.2465217349	4.242077E-009
2.4	12.4266666667	12.4266656697	9.969229E-007	12.4266666696	2.950340E-009
2.5	12.6500000000	12.6499967192	3.280754E-006	12.6500000021	2.129884E-009
2.6	12.9138461538	12.9138454470	7.068234E-007	12.9138461496	4.234208E-009
2.7	13.2159259259	13.2159287029	2.776953E-006	13.2159259274	1.455382E-009
2.8	13.5542857143	13.5542862145	5.002600E-007	13.5542857152	8.964705E-010
2.9	13.9272413793	13.9272392744	2.104884E-006	13.9272413784	9.028049E-010
3.0	14.3333333333	14.3333333333	0.000000E-000	14.3333333333	0.000000E-000

V. Conclusions

In this paper, the formulation of one dimensional linear and nonlinear second order boundary value problems have been discussed in details by the Galerkin weighted residual method applying Legendre polynomials as the basis functions in the approximation. The proposed method is applied to solve some numerical examples both linear and nonlinear. The computed results are compared with the exact solutions and we have found a good agreement with the exact solution. All the mathematical formulations and numerical computations have been done by **MATLAB-10** code.

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