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Sums of Squares of Jacobsthal Numbers

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Abstract: *There are many identities on Jacobsthal sequence of numbers. Here we try tofind some more identities on Sums of Squares of Consecutive Jacobsthal numbers using Binet forms.*
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I. Introduction

The Jacobsthal and Jacobsthal –Lucas sequences J_n and j_n are defined by the recurrence relations

$$J_0 = 0, J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2. \text{---------------(1)}$$

$$j_0 = 2, j_1 = 1,\ j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2. \text{---------------(2)}$$

Applications of these two sequences to curves are given in [1]. Sequence (1) appears in [2] but (2) does not. From (1) and (2) we thus have the following tabulation for the Jacobsthal numbers J_n and the Jacobsthal –Lucas sequences j_n.

n	0	1	2	3	4	5	6	7	8	9	10	…
J_n	0	1	1	3	5	11	21	43	85	171	341	…
j_n	2	1	5	7	17	31	65	127	257	511	1025	…

When required we can extend these sequences through negative values of n by means of the recurrence (1) and (2). Observe that all the J_n and j_n except j_0 are odd by virtue of the definitions.

In [3], the Binet forms of Jacobsthal form are given as

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{1}{3}[(2)^n - (-1)^n]$$

$$j_n = \alpha^n + \beta^n = [(2)^n + (-1)^n]$$

Considering Binet forms, Jacobsthal sequences can also be represented as

$$J_n = \frac{1}{3}\left[\left(2(\alpha+\beta)\right)^n - (\alpha\beta)^n\right] = \frac{1}{3}[(2)^n - (-1)^n]$$

$$j_n = \left[\left(2(\alpha+\beta)\right)^n + (\alpha\beta)^n\right] = [(2)^n + (-1)^n]$$

Based on this construction, we obtain some identities.

Proposition 1.

For every $n \geq 0$ the following equality holds

$$J_n^2 + J_{n+1}^2 = \frac{1}{9}[(\alpha-\beta)^2 2^{2n} + 2^{n+1}(\alpha\beta)^n + 2(\alpha+\beta)]$$

Proof.

$$J_n^2 = \frac{1}{9}\left[\left(2(\alpha+\beta)\right)^n - (\alpha\beta)^n\right]^2$$

$$J_{n+1}^2 = \frac{1}{9}\left[\left(2(\alpha+\beta)\right)^{n+1} - (\alpha\beta)^{n+1}\right]^2$$

$$J_n^2 + J_{n+1}^2 = \frac{1}{9}\left\{\begin{matrix} 2^{2n}(\alpha+\beta)^{2n}\left[1 + \left(2(\alpha+\beta)\right)^2\right] + (\alpha\beta)^{2n}[1 + (\alpha\beta)^2] - \\ 2\left(2(\alpha+\beta)\right)^n(\alpha\beta)^n\left[1 + \left(2(\alpha+\beta)\right)(\alpha\beta)\right] \end{matrix}\right\}$$

$$J_n^2 + J_{n+1}^2 = \frac{1}{9}[(\alpha-\beta)^2 2^{2n} + 2^{n+1}(\alpha\beta)^n + 2(\alpha+\beta)].$$

Remark:

For every $n \geq 0$ the following equality holds

$$j_n^2 + j_{n+1}^2 = [(\alpha-\beta)^2 2^{2n} - 2^{n+1}(\alpha\beta)^n + 2(\alpha+\beta)].$$

Proposition 2.

For every $n \geq 0$

$$J_n^2 + J_{n+1}^2 + j_n^2 + j_{n+1}^2 = \frac{1}{9}[5.2^{2n+1}(\alpha-\beta)^2 - 2^{n+4}(\alpha\beta)^n + 5.2^2(\alpha+\beta)]$$

Proof.

$$J_n^2 + J_{n+1}^2 + j_n^2 + j_{n+1}^2$$
$$= \frac{1}{9}[(\alpha-\beta)^2 2^{2n} + 2^{n+1}(\alpha\beta)^n + 2(\alpha+\beta)] + [(\alpha-\beta)^2 2^{2n} - 2^{n+1}(\alpha\beta)^n + 2(\alpha+\beta)]$$
$$= \frac{1}{9}[5.2^{2n+1}(\alpha-\beta)^2 - 2^{n+4}(\alpha\beta)^n + 5.2^2(\alpha+\beta)]$$

Proposition 3:

For every $n \geq 0$

$$\sum_{n=1}^{n}(J_{n+1}^2 - J_n^2) = J_{n+1}^2 - 1.$$

Proof .

$$\sum_{n=1}^{n}(J_{n+1}^2 - J_n^2) = \frac{1}{9}\left[\left(2(\alpha+\beta)\right)^{2n+2} + (\alpha\beta)^{2n+2} - 2\left(2(\alpha+\beta)\right)^{n+1}(\alpha\beta)^{n+1}\right]$$
$$-\frac{1}{9}\left[\left(2(\alpha+\beta)\right)^2 + (\alpha\beta)^2 - 2\left(2(\alpha+\beta)\right)^1(\alpha\beta)^1\right]$$
$$= \frac{1}{9}\left[\left(2(\alpha+\beta)\right)^{n+1} - (\alpha\beta)^{n+1}\right]^2 - \frac{1}{9}\left[\left(2(1)\right)^1 - (-1)^1\right]^2$$
$$\sum_{n=1}^{n}(J_{n+1}^2 - J_n^2) = J_{n+1}^2 - 1.$$

Proposition 4.

For every $n \geq 0$

$$\sum_{n=1}^{n} J_n^2 = \frac{1}{9}\left[\frac{4}{3}(2^{2n}-1)n + 2\sum_{n=1}^{n}(-2)^n\right]$$

Proof .

By the formula $J_k = \frac{2^k - (-1)^k}{3}$ we have

$$\sum_{n=1}^{n} J_n^2 = \frac{1}{9}\{[(2^2)^1 + (2^2)^2 + (2^2)^3 + \cdots + (2^2)^n] + n - 2[2(-1) + 2^2(-1)^2 + 2^3(-1)^3 + \cdots + 2^n(-1)^n]\}$$
$$= \frac{1}{9}\left[\frac{(2^2)^1((2^2)^n - 1)}{2^2 - 1} + n + 2\sum_{n=1}^{n}(-2)^n\right]$$
$$\sum_{n=1}^{n} J_n^2 = \frac{1}{9}\left[\frac{4}{3}(2^{2n}-1) + n + 2\sum_{n=1}^{n}(-2)^n\right]$$

Proposition 5.

For every $n \geq 0$,$i \geq 0$ the following equality holds

$$J_{2n+2i} = 2^{2i+2} J_{2n-2} + \frac{2^{2i+2}-1}{3}.$$

Proof.

By the formula $J_k = \frac{2^k - (-1)^k}{3}$ we have

$$RHS = 2^{2i+2}\left(\frac{(2)^{2n-2} - (-1)^{2n-2}}{3}\right) + \frac{2^{2i+1} - 1}{3}$$

$$= \frac{2^{2n+2i} - 2^{2i+2}(-1)^{2n-2}}{3} + \frac{2^{2i+2} - 1}{3}$$

$$= \frac{2^{2n+2i} - (-1)^{2n+2i}}{3}$$

$$J_{2n+2i} = 2^{2i+1} J_{2n-2} + \frac{2^{2i+1} - 1}{3}.$$

Proposition 6.

For every $n \geq 0$, $i \geq 0$ the following equality holds

$$J_{2n+2i+1} = 2^{2i+3} J_{2n-2} + \frac{2^{2i+3} + 1}{3}.$$

Proof.

By the formula $J_k = \frac{2^k - (-1)^k}{3}$ we have

$$LHS = \frac{2^{2n+2i+1} - (-1)^{2n+2i+1}}{3}$$

$$= \frac{2^{2i+3}.2^{2n-2} - (-1)^{2i+3}(-1)^{2n-2}}{3}$$

$$= \frac{2^{2i+3}}{3}[2^{2n-2} - (-1)^{2n-2}] + \frac{2^{2i+3}}{3}(-1)^{2n-2} - \frac{(-1)^{2i+3}(-1)^{2n-2}}{3}$$

$$J_{2n+2i+1} = 2^{2i+3} J_{2n-2} + \frac{2^{2i+3} + 1}{3}.$$

Proposition 7.

For every $n \geq 0, i \geq 0$ the following equality holds

$$J_{2n+2i} + J_{2n+2i+1} = 2^{2i+2}[3J_{2n-2} + 1].$$

Proof.

$$J_{2n+2i} + J_{2n+2i+1} = 2^{2i+2} J_{2n-2} + \frac{2^{2i+2} - 1}{3} + 2^{2i+3} J_{2n-2} + \frac{2^{2i+3} + 1}{3}$$

$$= 2^{2i+2} J_{2n-2}[1+2] + \frac{2^{2i+2}}{3}[1+2]$$

$$= 3.2^{2i+2}.J_{2n-2} + 2^{2i+2}$$

$$J_{2n+2i} + J_{2n+2i+1} = 2^{2i+2}[3J_{2n-2} + 1].$$

Proposition 8.

For every $n \geq 0, k \geq 0$ the following equality holds

$$j_{n+k+2}\, j_{n+k+3} - j_{n+k}\, j_{n+k+1} = 3.2^{n+k}[2^{n+k+3} + 3J_{n+k+1}]$$

Proof .

$$j_{n+k}\, j_{n+k+1} = 2^{2n+2k+1} + 2^{n+k}(-1)^{n+k} + (-1)^{2n+2k+1}$$

$$j_{n+k+2}\, j_{n+k+3} = 2^{2n+2k+5} + 2^{n+k+2}(-1)^{n+k+2} + (-1)^{2n+2k+5}$$

$$j_{n+k+2}\, j_{n+k+3} - j_{n+k}\, j_{n+k+1} = 2^{2n+2k+1}(2^4 - 1) + 2^{n+k}(-1)^{n+k}(2^2(-1)^2 - 1) + (-1)^{2n+2k+1}((-1)^4 - 1)$$

$$= 15.2^{2n+2k+1} + 3.2^{n+k}(-1)^{n+k}$$

$$= 12.2^{2n+2k+1} + 3.2^{n+k}[2^{n+k+1} - (-1)^{n+k+1}]$$

$$= 12.2^{2n+2k+1} + 9.2^{n+k} J_{n+k+1}$$

$$j_{n+k+2}\, j_{n+k+3} - j_{n+k}\, j_{n+k+1} = 3.2^{n+k}[2^{n+k+3} + 3J_{n+k+1}]$$

References

[1] A.F.Horadam, "Jacobsthal and pell curves" The Quartely 26.I(1988):79-83.

[2] N.J.A.Sloane, A Handbook of integer Sequences.NewYork:Academic Press,1973.

[3] A.F.Horadam, Jacobsthal Representation Numbers,Unversity of New England,Armidale,2351,Australia.

[4] Zvonko Cerin, Sums of Squares and Products of Jacobsthal Numbers,Journal of integer Sequences,Vol. 10 (2007)