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Modeling the Simultaneous Effect of Two Toxicants Causing Deformity in a Subclass of Biological Population

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Abstract: *In this paper, we have proposed and analyzed a mathematical model to study the simultaneous effect of two toxicants on a biological population, in which a subclass of biological population is severely affected and exhibits abnormal symptoms like deformity, fecundity, necrosis, etc. On studying the qualitative behavior of model, it is shown that the density of total population will settle down to an equilibrium level lower than the carrying capacity of the environment. In the model, we have assumed that a subclass of biological population is not capable in further reproduction and it is found that the density of this subclass increases as emission rates of toxicants or uptake rates of toxicants increase. For large emission rates it may happen that the entire population gets severely affected and is not capable in reproduction and after a time period all the population may die out. The stability analysis of the model is determined by variational matrix and method of Lyapunov's function. Numerical simulation is given to illustrate the qualitative behavior of model.*
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I. Introduction

The dynamics of effect of toxicants on biological species using mathematical models ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]) have been studied by many researchers. These studies have been carried out for different cases such as: Rescigno ([9]) proposed a mathematical model to study the effect of a toxicant on a biological species when toxicant is being produced by the species itself, Hallam *et. al.*([6], [7]) proposed and analyzed a mathematical model to study the effect of a toxicant on the growth rate of biological species, Shukla *et. al.* [11] proposed a model to study the simultaneous effect of two different toxicants, emitted from some external sources, etc. In all of these studies, it is assumed that the toxicants affect each and every individual of the biological species uniformly. But it is observed that some members of biological species get severely affected by toxicants and show change in shape, size, deformity, etc. These changes are observed in the biological species living in aquatic environment ([12], [13], [14], [15], [16], [17], [18], [19], [20]) and in terrestrial environment, in plants ([21], [22]) and in animals ([23], [24], [25], [26]).

The study of such very important observable fact where a subclass of the biological species is adversely affected by the toxicant and shows abnormal symptoms such as deformity, incapable in reproduction etc. using mathematical models is very limited. Agrawal and Shukla [2] have studied the effect of a single toxicant (emitted from some external sources) on a biological population in which a subclass of biological population is severely affected and shows abnormal symptoms like deformity, fecundity, necrosis, etc. using mathematical model. However, no study has been done for this phenomenon under the simultaneous effect of two toxicants. Therefore, in this paper we have proposed a dynamical model to study the simultaneous effect of two toxicants (both toxicants are constantly emitted from some external sources) on a biological species in which a subclass of biological population is severely affected and shows abnormal symptoms like deformity, fecundity, necrosis, etc.

II. Mathematical Model

We consider a logistically growing biological population with density $N(t)$ in the environment and simultaneously affected by two different types of toxicants with environment concentrations $T_1(t)$ and $T_2(t)$ (both toxicants are constantly emitted in the environment at the rates Q_1 and Q_2 respectively, from some external sources). These toxicants are correspondingly uptaken by the biological population at different concentration rates $U_1(t)$ and $U_2(t)$. These toxicants decrease the growth rate of biological population as well as they also adversely affect a subclass of biological population with density $N_D(t)$ and decay the capability of reproduction. Here, $N_A(t)$ is the density of biological population which is capable in reproduction. Keeping these views in mind, we have proposed the following model:

$$\begin{aligned}
\frac{dN_A}{dt} &= (b-d)N_A - (r_1U_1 + r_2U_2)N_A - \frac{rN_AN}{K(T_1,T_2)} \\
\frac{dN_D}{dt} &= (r_1U_1 + r_2U_2)N_A - \frac{rN_DN}{K(T_1,T_2)} - (\alpha+d)N_D \\
\frac{dT_1}{dt} &= Q_1 - \delta_1T_1 - \gamma_1T_1N + \pi_1\nu_1NU_1 \\
\frac{dT_2}{dt} &= Q_2 - \delta_2T_2 - \gamma_2T_2N + \pi_2\nu_2NU_2 \\
\frac{dU_1}{dt} &= \gamma_1T_1N - \beta_1U_1 - \nu_1NU_1 \\
\frac{dU_2}{dt} &= \gamma_2T_2N - \beta_2U_2 - \nu_2NU_2
\end{aligned} \tag{2.1}$$

$$N_A(0), N_D(0) \geq 0, \quad T_i(0) \geq 0, \quad U_i(0) \geq c_iN(0), \quad c_i > 0, \quad 0 < \pi_i < 1 \quad \text{for} \quad i = 1,2$$

All the parameters used in the model (2.1) are positive and defined as follows:

- b − the birth rate of logistically growing biological population,
- d − the death rate of logistically growing biological population,
- r − the growth rate of biological population in toxicants free environment, i.e. $r = (b-d)$
- α − the decay rate of the deformed population due to high toxicity,
- r_1 & r_2 − the decreasing rates of the growth rate associated with the uptakes of environmental concentration of toxicants T_1 and T_2 respectively,
- δ_1 & δ_2 − the natural depletion rate coefficients of T_1 and T_2 respectively,
- β_1 & β_2 − the natural depletion rate coefficients of U_1 and U_2 respectively,
- γ_1 & γ_2 − the depletion rate coefficients due to uptake by the population respectively, $(i.e.\ \gamma_1T_1N$ & $\gamma_2T_2N)$
- ν_1 & ν_2 − the depletion rate coefficients of U_1 and U_2 respectively due to decay of some members of N, $(i.e.\ \nu_1NU_1$ & $\nu_2NU_2)$
- π_1 & π_2 − the fractions of the depletion of U_1 and U_2 respectively due to decay of some members of N which may reenter into the environment, $(i.e.\ \pi_1\nu_1NU_1$ & $\pi_2\nu_2NU_2)$

In the above model (2.1), total density of logistically growing biological population N is equal to the sum of density of biological population without deformity N_A and with deformity N_D, $(i.e.\ N = N_A + N_D)$.

So, the above system can be written in terms of N, N_D, T_1, T_2, U_1 and U_2 as follows:

$$\begin{aligned}
\frac{dN}{dt} &= rN - \frac{rN^2}{K(T_1,T_2)} - (\alpha+b)N_D \\
\frac{dN_D}{dt} &= (r_1U_1 + r_2U_2)(N - N_D) - \frac{rN_DN}{K(T_1,T_2)} - (\alpha+d)N_D \\
\frac{dT_1}{dt} &= Q_1 - \delta_1T_1 - \gamma_1T_1N + \pi_1\nu_1NU_1 \\
\frac{dT_2}{dt} &= Q_2 - \delta_2T_2 - \gamma_2T_2N + \pi_2\nu_2NU_2 \\
\frac{dU_1}{dt} &= \gamma_1T_1N - \beta_1U_1 - \nu_1NU_1 \\
\frac{dU_2}{dt} &= \gamma_2T_2N - \beta_2U_2 - \nu_2NU_2
\end{aligned} \tag{2.2}$$

$$N(0) \geq 0, \quad N_D(0) \geq 0, \quad T_i(0) \geq 0, \quad U_i(0) \geq c_i N(0), \quad 0 \leq \pi_i \leq 1, \quad \text{for} \quad i = 1,2$$

where $c_1, c_2 > 0$ are constants relating to the initial uptake concentration $U_i(0)$ with the initial density of biological population $N(0)$.

In the model (2.2), the function $K(T_1, T_2) > 0$ (for all values of T_1 & T_2) denotes the carrying capacity of the environment for the biological population N and it decreases when T_1 or T_2 or both increase.

we have,

$$\text{initial carrying capacity, } K_0 = K(0,0) \text{ and } \frac{\partial K}{\partial T_i} < 0 \quad \text{for} \quad T_i > 0, \ (i = 1,2) \tag{2.3}$$

III. Equilibrium points and stability analysis

The model (2.2) has two non – negative equilibrium points $E_1 = \left(0, 0, \frac{Q_1}{\delta_1}, \frac{Q_2}{\delta_2}, 0, 0\right)$ and $E_2 = (N^*, N_D^*, T_1^*, T_2^*, U_1^*, U_2^*)$. It is obvious that equilibria E_1 exist, hence existence of E_1 is not discussed.

Existence of E_2: The value of N^*, N_D^*, T_1^*, T_2^*, U_1^* and U_2^* are the positive solutions of the following system of equations:

$$N = \frac{1}{r}(r - r_1 U_1 - r_2 U_2) K(T_1, T_2) \tag{3.1}$$

$$N_D = \frac{(r_1 U_1 + r_2 U_2) N K(T_1, T_2)}{rN + (r_1 U_1 + r_2 U_2 + \alpha + d) K(T_1, T_2)} \tag{3.2}$$

$$T_1 = \frac{Q_1(\beta_1 + \nu_1 N)}{f_1(N)} = g_1(N) \tag{3.3}$$

$$T_2 = \frac{Q_2(\beta_2 + \nu_2 N)}{f_2(N)} = g_2(N) \tag{3.4}$$

$$U_1 = \frac{Q_1 \gamma_1 N}{f_1(N)} = h_1(N) \tag{3.5}$$

$$U_2 = \frac{Q_2 \gamma_2 N}{f_2(N)} = h_2(N) \tag{3.6}$$

where, $$f_1(N) = \delta_1 \beta_1 + (\gamma_1 \beta_1 + \delta_1 \nu_1) N + \gamma_1 \nu_1 (1 - \pi_1) N^2 \tag{3.7}$$

$$f_2(N) = \delta_2 \beta_2 + (\gamma_2 \beta_2 + \delta_2 \nu_2) N + \gamma_2 \nu_2 (1 - \pi_2) N^2 \tag{3.8}$$

Using equations (3.1-3.8), we can assume a function

$$F(N) = rN - \left(r - r_1 h_1(N) - r_2 h_2(N)\right) K\left(g_1(N), g_2(N)\right) \tag{3.9}$$

From (3.9), we can say that

$$F(0) < 0 \text{ and } F(K_0) > 0$$

this implies there must exist a root between 0 and K_0 for the equation $F(N) = 0$, says N^*.

Uniqueness of E_2:

For N^* to be unique root of $F(N) = 0$, we must have

$$\frac{dF}{dN} = \left[r + K\left(g_1(N), g_2(N)\right)\left\{r_1 \frac{dh_1}{dN} + r_2 \frac{dh_2}{dN}\right\} - \left(r - r_1 h_1(N) - r_2 h_2(N)\right)\left\{\frac{\partial K}{\partial T_1}\frac{dg_1}{dN} + \frac{\partial K}{\partial T_2}\frac{dg_2}{dN}\right\}\right] > 0$$

where

$$\frac{dh_1}{dN} = \frac{Q_1 \gamma_1}{f_1^2(N)}\{\delta_1 \beta_1 - \gamma_1 \nu_1 (1 - \pi_1) N^2\} \tag{3.10}$$

$$\frac{dh_2}{dN} = \frac{Q_2 \gamma_2}{f_2^2(N)}\{\delta_2 \beta_2 - \gamma_2 \nu_2 (1 - \pi_2) N^2\} \tag{3.11}$$

$$\frac{dg_1}{dN} = -\frac{Q_1\gamma_1}{f_1^2(N)}\{\beta_1^2 + 2\beta_1\nu_1(1-\pi_1)N + \nu_1^2(1-\pi_1)N^2\} < 0 \tag{3.12}$$

$$\frac{dg_2}{dN} = -\frac{Q_2\gamma_2}{f_2^2(N)}\{\beta_2^2 + 2\beta_2\nu_2(1-\pi_2)N + \nu_2^2(1-\pi_2)N^2\} < 0 \tag{3.13}$$

Since, $\frac{\partial K}{\partial T_1}, \frac{\partial K}{\partial T_2} < 0$ (from eq. (2.3)) and $\frac{dg_1}{dN}, \frac{dg_2}{dN} < 0$ (from eq. (3.12-3.13)), this implies that:

$$\left[\left(r - r_1h_1(N) - r_2h_2(N)\right)\left\{\frac{\partial K}{\partial T_1}\frac{dg_1}{dN} + \frac{\partial K}{\partial T_2}\frac{dg_2}{dN}\right\}\right] > 0$$

then $\frac{dF}{dN} > 0$, only when

$$\left[r + K\left(g_1(N), g_2(N)\right)\left\{r_1\frac{dh_1}{dN} + r_2\frac{dh_2}{dN}\right\}\right] > \left[\left(r - r_1h_1(N) - r_2h_2(N)\right)\left\{\frac{\partial K}{\partial T_1}\frac{dg_1}{dN} + \frac{\partial K}{\partial T_2}\frac{dg_2}{dN}\right\}\right] \tag{3.14}$$

Hence, if the conditions (3.14) is satisfied, the root N^* of $F(N) = 0$ is unique and lower than the carrying capacity of the environment.

After that, we can compute the value of $N_D^*,\ T_1^*,\ T_2^*,\ U_1^*$ and U_2^* with the help of N^* and equations (3.2-3.8).

3.1 Local stability analysis

To study the local stability behavior of the equilibrium points $E_1 = \left(0, 0, \frac{Q_1}{\delta_1}, \frac{Q_2}{\delta_2}, 0, 0\right)$ and $E_2 = (N^*, N_D^*, T_1^*, T_2^*, U_1^*, U_2^*)$, we compute the variational matrices M_1 and M_2 corresponding to the equilibrium points E_1 and E_2 such as:

$$M_1 = \begin{bmatrix} r & -(\alpha+b) & 0 & 0 & 0 & 0 \\ 0 & -(\alpha+d) & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1 Q_1}{\delta_1} & 0 & -\delta_1 & 0 & 0 & 0 \\ -\frac{\gamma_2 Q_2}{\delta_2} & 0 & 0 & -\delta_2 & 0 & 0 \\ \frac{\gamma_1 Q_1}{\delta_1} & 0 & 0 & 0 & -\beta_1 & 0 \\ \frac{\gamma_2 Q_2}{\delta_2} & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}$$

From M_1, it is obvious that E_1 is a saddle point unstable locally only in the $N-$direction and with stable manifold locally in the $N_D - T_1 - T_2 - U_1 - U_2$ space.

And

$$M_2 = \begin{bmatrix} -r\left(\frac{2N^*}{K(T_1^*,T_2^*)} - 1\right) & -(\alpha+b) & rN^{*2}K_1(T_1^*,T_2^*) & rN^{*2}K_2(T_1^*,T_2^*) & 0 & 0 \\ r_1U_1^* + r_2U_2^* - \frac{rN_D^*}{K(T_1^*,T_2^*)} & -(r_1U_1^* + r_2U_2^*)\frac{N^*}{N_D^*} & rN^*N_D^*K_1(T_1^*,T_2^*) & rN^*N_D^*K_2(T_1^*,T_2^*) & r_1(N^* - N_D^*) & r_2(N^* - N_D^*) \\ -\gamma_1T_1^* + \pi_1\nu_1U_1^* & 0 & -(\delta_1 + \gamma_1N^*) & 0 & \pi_1\nu_1N^* & 0 \\ -\gamma_2T_2^* + \pi_2\nu_2U_2^* & 0 & 0 & -(\delta_2 + \gamma_2N^*) & 0 & \pi_2\nu_2N^* \\ \gamma_1T_1^* - \nu_1U_1^* & 0 & \gamma_1N^* & 0 & -(\beta_1 + \nu_1N^*) & 0 \\ \gamma_2T_2^* - \nu_2U_2^* & 0 & 0 & \gamma_2N^* & 0 & -(\beta_2 + \nu_2N^*) \end{bmatrix}$$

Here,

$$K_1(T_1^*,T_2^*) = \frac{1}{K^2(T_1^*,T_2^*)}\cdot\left[\frac{\partial K}{\partial T_1}\right]_{(T_1^*,T_2^*)} < 0 \quad \text{and} \quad K_2(T_1^*,T_2^*) = \frac{1}{K^2(T_1^*,T_2^*)}\cdot\left[\frac{\partial K}{\partial T_2}\right]_{(T_1^*,T_2^*)} < 0$$

According to the ***Gershgorin's disc***, all the eigenvalues of variational matrix M_2 are negative or having negative real parts if

$$K(T_1^*, T_2^*) < 2N^* \tag{3.15}$$

$$|(\alpha + b)| + \left|rN^{*2}K_1(T_1^*, T_2^*)\right| + \left|rN^{*2}K_2(T_1^*, T_2^*)\right| \; < \; r\left(\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right) \tag{3.16}$$

$$\left|r_1U_1^* + r_2U_2^* - \frac{rN_D^*}{K(T_1^*, T_2^*)}\right| + |rN^*N_D^*K_1(T_1^*, T_2^*)| + |rN^*N_D^*K_2(T_1^*, T_2^*)| + |r_1(N^* - N_D^*)| + |r_2(N^* - N_D^*)| < (r_1U_1^* + r_2U_2^*)\frac{N^*}{N_D^*} \tag{3.17}$$

$$|-\gamma_1T_1^* + \pi_1\nu_1U_1^*| + |\pi_1\nu_1N^*| < (\delta_1 + \gamma_1N^*) \tag{3.18}$$

$$|-\gamma_2T_2^* + \pi_2\nu_2U_2^*| + |\pi_2\nu_2N^*| < (\delta_2 + \gamma_2N^*) \tag{3.19}$$

$$|\gamma_1T_1^* - \nu_1U_1^*| + |\gamma_1N^*| < (\beta_1 + \nu_1N^*) \tag{3.20}$$

$$|\gamma_2T_2^* - \nu_2^*U_2^*| + |\gamma_2N^*| < (\beta_2 + \nu_2N^*) \tag{3.21}$$

Hence, we can state the following theorem.

Theorem 1: The equilibrium point E_2 is locally asymptotically stable if the conditions (3.15-3.21) are satisfied.

3.2 Global stability analysis

To found a set of sufficient conditions for globally asymptotically stable behavior of the equilibria E_2, we need a lemma which establishes the region of attraction of E_2.

Lemma 1: The region

$$\Omega = \left\{(N, N_D, T_1, T_2, U_1, U_2): 0 \le N \le K_0, \quad 0 \le N_D \le \frac{(r_1 + r_2)(Q_1 + Q_2)K_0}{(r_1 + r_2)(Q_1 + Q_2) + \delta_m(\alpha + d)}, \; 0 \le T_1 + T_2 + U_1 + U_2 \le \frac{(Q_1 + Q_2)}{\delta_m}\right\}$$

where $\delta_m = \min(\delta_1, \delta_2, \beta_1, \beta_2)$

attracts all solution initiating in the interior of the positive orthant.

Proof: From the first equation of model (2.2),

we have, $\dfrac{dN}{dt} \le rN - \dfrac{rN^2}{K_0} = r\left(1 - \dfrac{N}{K_0}\right)N$

Thus, $\limsup_{t\to\infty} N(t) \le K_0$.

From the last four equations of model (2.2),

we have,
$$\frac{dT_1}{dt} + \frac{dT_2}{dt} + \frac{dU_1}{dt} + \frac{dU_2}{dt} = (Q_1 + Q_2) - (\delta_1T_1 + \delta_2T_2 + \beta_1U_1 + \beta_2U_2) - (1 - \pi_1)\nu_1NU_1 - (1 - \pi_2)\nu_2NU_2$$
$$\le (Q_1 + Q_2) - \delta_m(T_1 + T_2 + U_1 + U_2)$$

where $\delta_m = \min(\delta_1, \delta_2, \beta_1, \beta_2)$

Thus, $\limsup_{t\to\infty}(T_1 + T_2 + U_1 + U_2) \le \dfrac{Q_1 + Q_2}{\delta_m}$

From the second equation of model (2.2),

we have,
$$\frac{dN_D}{dt} = (r_1U_1 + r_2U_2)(N - N_D) - \frac{rN_DN}{K(T_1, T_2)} - (\alpha + d)N_D$$
$$\le \frac{(r_1 + r_2)(Q_1 + Q_2)}{\delta_m}(K_0 - N_D) - (\alpha + d)N_D$$

Thus, $\limsup_{t\to\infty} N_D(t) \le \dfrac{(r_1 + r_2)(Q_1 + Q_2)K_0}{(r_1 + r_2)(Q_1 + Q_2) + \delta_m(\alpha + d)}$

proving the lemma.□

The following theorem establishes global asymptotic stability conditions for the equilibrium point E_2.
Theorem 2: Let $K(T)$ satisfies the following inequalities in Ω with the assumptions in equation (2.3):

$$K_m \leq K(T) \leq K_0, \qquad 0 \leq -\frac{\partial K}{\partial T_1}(T_1, T_2) \leq \kappa_1, \qquad 0 \leq -\frac{\partial K}{\partial T_2}(T_1, T_2) \leq \kappa_2$$

where K_m, κ_1 & κ_2 are positive constants.

Then E_2 is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant, if the following conditions hold in Ω:

$$\left[r_1 U_1^* + r_2 U_2^* - (\alpha + b) + \frac{rK_0(r_1 + r_2)(Q_1 + Q_2)}{K(T_1^*, T_2^*)\{(r_1 + r_2)(Q_1 + Q_2) + \delta_m(\alpha + d)\}}\right]^2 < \frac{4r}{25}(r_1 U_1^* + r_2 U_2^*)\frac{N^*}{N_D^*}\left[\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right] \quad (3.22)$$

$$\left[(\gamma_1 + \pi_1 \nu_1)\frac{(Q_1 + Q_2)}{\delta_m} + \frac{rK_0^2 \kappa_1}{K_m^2}\right]^2 < \frac{4r}{15}(\delta_1 + \gamma_1 N^*)\left[\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right] \quad (3.23)$$

$$\left[(\gamma_2 + \pi_2 \nu_2)\frac{(Q_1 + Q_2)}{\delta_m} + \frac{rK_0^2 \kappa_2}{K_m^2}\right]^2 < \frac{4r}{15}(\delta_2 + \gamma_2 N^*)\left[\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right] \quad (3.24)$$

$$\left[(\gamma_1 + \nu_1)\frac{(Q_1 + Q_2)}{\delta_m}\right]^2 < \frac{4r}{15}(\beta_1 + \nu_1 N^*)\left[\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right] \quad (3.25)$$

$$\left[(\gamma_2 + \nu_2)\frac{(Q_1 + Q_2)}{\delta_m}\right]^2 < \frac{4r}{15}(\beta_2 + \nu_2 N^*)\left[\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right] \quad (3.26)$$

$$\left[\frac{rK_0 \kappa_1 (r_1 + r_2)(Q_1 + Q_2)}{K_m^2\{(r_1 + r_2)(Q_1 + Q_2) + (\alpha + d)\delta_m\}}\right]^2 < \frac{4}{15}(\delta_1 + \gamma_1 N^*)(r_1 U_1^* + r_2 U_2^*)\frac{N^*}{N_D^*} \quad (3.27)$$

$$\left[\frac{rK_0 \kappa_2 (r_1 + r_2)(Q_1 + Q_2)}{K_m^2\{(r_1 + r_2)(Q_1 + Q_2) + (\alpha + d)\delta_m\}}\right]^2 < \frac{4}{15}(\delta_2 + \gamma_2 N^*)(r_1 U_1^* + r_2 U_2^*)\frac{N^*}{N_D^*} \quad (3.28)$$

$$[r_1 K_0]^2 < \frac{4}{15}(\beta_1 + \nu_1 N^*)(r_1 U_1^* + r_2 U_2^*)\frac{N^*}{N_D^*} \quad (3.29)$$

$$[r_2 K_0]^2 < \frac{4}{15}(\beta_2 + \nu_2 N^*)(r_1 U_1^* + r_2 U_2^*)\frac{N^*}{N_D^*} \quad (3.30)$$

$$[(\gamma_1 + \pi_1 \nu_1)N^*]^2 < \frac{4}{9}(\delta_1 + \gamma_1 N^*)(\beta_1 + \nu_1 N^*) \quad (3.31)$$

$$[(\gamma_2 + \pi_2 \nu_2)N^*]^2 < \frac{4}{9}(\delta_2 + \gamma_2 N^*)(\beta_2 + \nu_2 N^*) \quad (3.32)$$

The proof of **Theorem 2** is given in **Appendix A.**

IV. Numerical simulation

To make the qualitative results more clear, we give here numerical simulation of model (2.2) by defining the function:

$$K(T_1, T_2) = K_0 - \frac{b_{11} T_1}{1 + b_{12} T_1} - \frac{b_{21} T_2}{1 + b_{22} T_2} \quad (4.1)$$

and assuming a set of parameters

$$\left\{\begin{array}{cccccc} b = 0.005, & d = 0.00001, & r_1 = 0.0007, & r_2 = 0.0005, & Q_1 = 0.001, & Q_2 = 0.0004 \\ \delta_1 = 0.004, & \delta_2 = 0.001, & \gamma_1 = 0.0005, & \gamma_2 = 0.0003, & \pi_1 = 0.0004, & \pi_2 = 0.0006 \\ \nu_1 = 0.005, & \nu_2 = 0.003, & \beta_1 = 0.006, & \beta_2 = 0.004, & K_0 = 10.0, & b_{11} = 0.0002, \\ b_{12} = 1.0, & b_{21} = 0.0001, & b_{22} = 2.0, & \kappa_1 = 0.001, & \kappa_2 = 0.001, & K_m = 3.0 \end{array}\right\} \quad (4.2)$$

For the above function and set of values of parameters (4.1-4.2), we have obtained equilibrium point $E_2(N^*, N_D^*, T_1^*, T_2^*, U_1^*, U_2^*)$ with values $N^* = 9.7771,\ N_d^* = 0.0227,\ T_1^* = 0.1113,\ T_2^* = 0.1002,\ U_1^* = 0.0099$

and $U_2^* = 0.0088$. Here, condition (3.14) satisfies which shows that the values $N^*, N_D^*, T_1^*, T_2^*, U_1^*$ and U_2^* are unique in the region Ω. The eigenvalues of variational matrix M_2 corresponding to the equilibrium point E_2 for the model (2.2) are obtained as -0.0559, -0.0339, -0.0090, -0.0039, $-0.0050 + 0.0002i$ and $-0.0050 - 0.0002i$. We note that four eigenvalues of variational matrix are negative and remaining two eigenvalues have negative real parts which show that equilibrium point E_2 is locally asymptotically stable. Also, the equilibrium point E_2 satisfies all the conditions of global asymptotic stability (3.22-3.32). (see Fig.1)

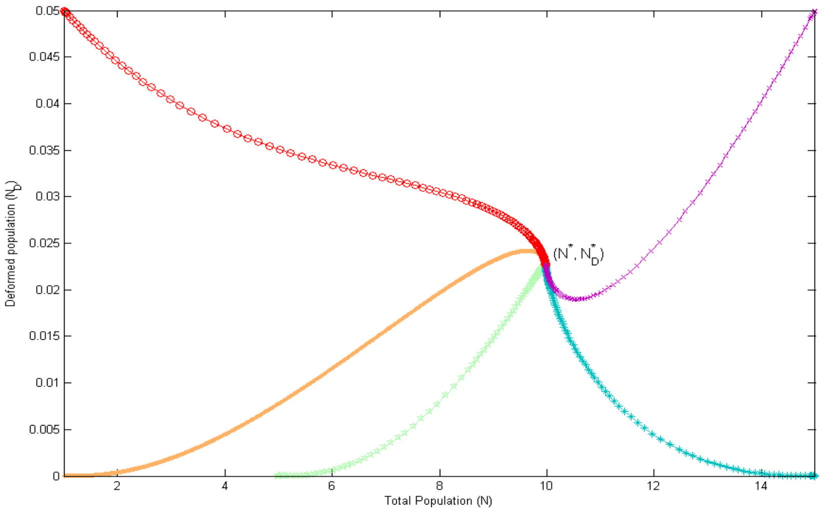

Fig.1: Nonlinear stability of (N^*, N_D^*) in $N - N_D$ plane for different initial starts

In Fig.2 & Fig.3, we have shown the changes in density of deformed population with respect to time for different values of emission rates of toxicant in the environment Q_1 and Q_2 respectively. Here, we take all the parameters same as eq. (4.2) except Q_1 and Q_2. In both figures, we can see that when emission rate of toxicant Q_1 as well as emission rate of toxicant Q_2 increases the density of the deformed population also increases, which shows that more members of the population will get deformed if the rate of toxicant emission increases.

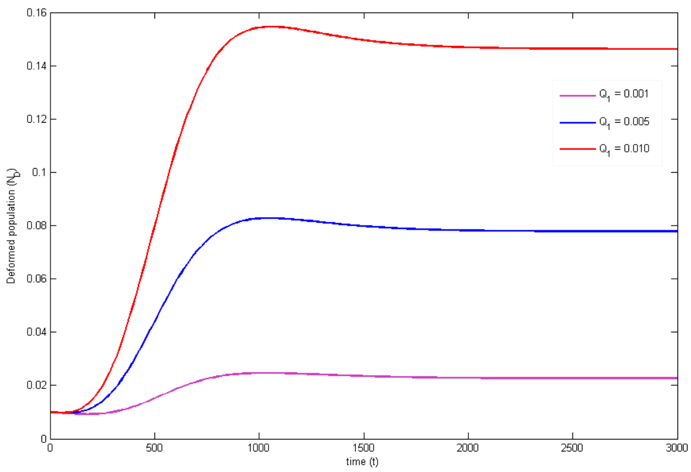

Fig.2: Variation of deformed population N_D with time for different values of Q_1

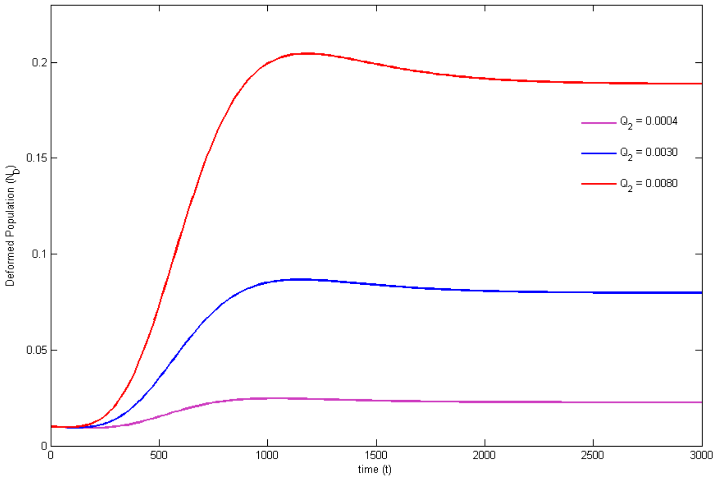

Fig.3: Variation of deformed population N_D with time for different values of Q_2

In Fig.4 & Fig.5, we have represented the variation in the density of deformed population for different values of the uptake rate coefficients γ_1 and γ_2 (all the parameters same as eq. (4.2) except γ_1 and γ_2 respectively). Here figures are showing that when the uptake rates of toxicants increase, density of deformed population increases.

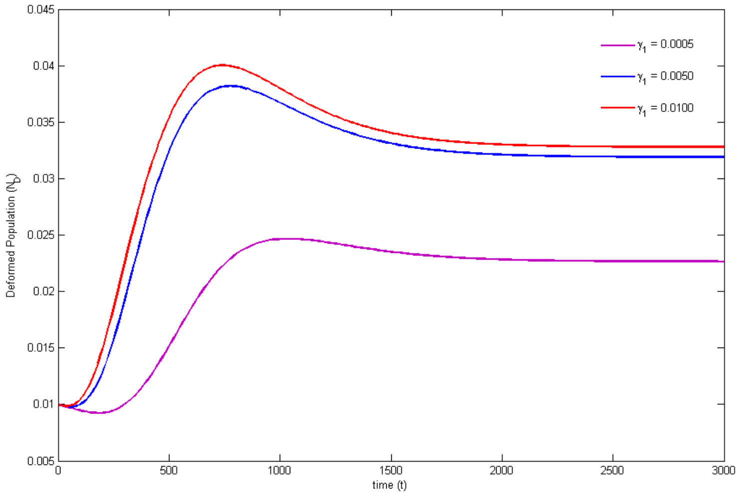

Fig.4: Variation of deformed population N_D with time for different values of γ_1

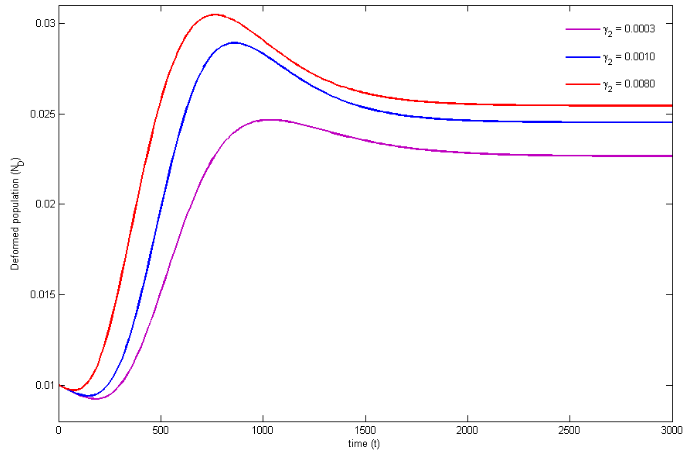

Fig.5: Variation of deformed population N_D with time for different values of γ_2

In Fig.6, we have shown the variation in density of deformed population corresponding to the decay rate of the deformed population due to high toxicity α (all the parameters same as eq. (4.2) except α). In this figure, we can see that when the decay rate of deformed population increases density of deformed population decreases.

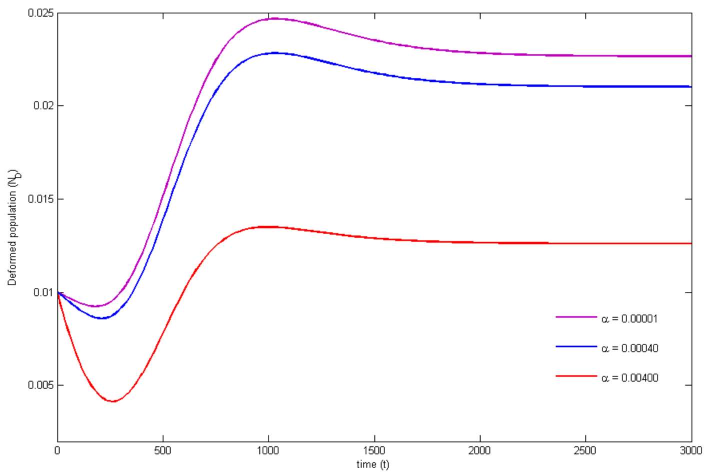

Fig.6: Variation of deformed population N_D with time for different values of α

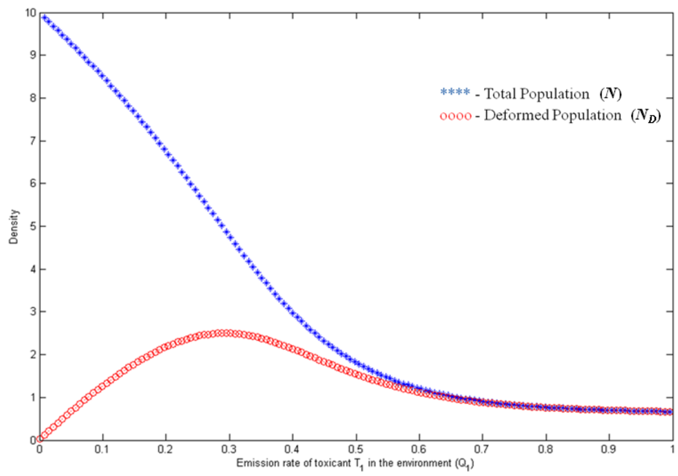

Fig.7: N and N_D for large emission rate of toxicant T_1 in the environment

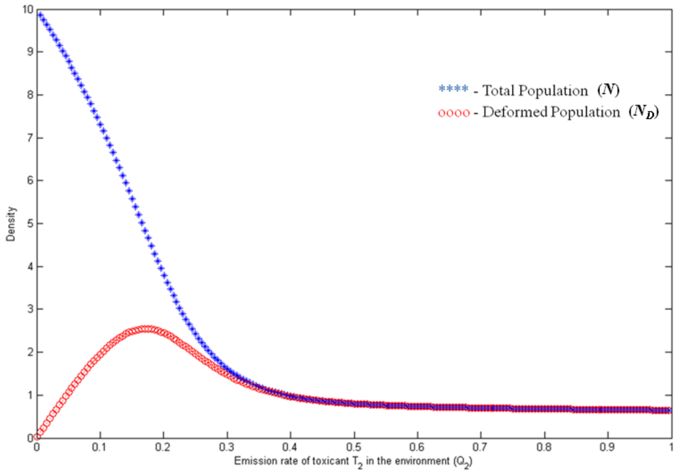

Fig.8: N and N_D for large emission rate of toxicant T_2 in the environment

In Fig.7 & Fig.8, we have represented the variation in the densities of Total population (N) and Deformed population (N_D) for large emission rate of toxicants Q_1 and Q_2. These figures show that density of total population gets severely affected and is not capable in reproduction for large emission rates.

V. Conclusion

In this paper, we have proposed and analyzed a mathematical model to study the simultaneous effect of two toxicants on a biological population, in which a subclass of biological population is severely affected and exhibits abnormal symptoms like deformity, fecundity, necrosis, etc. Here, we assume that these two toxicants are being emitted into the environment by some external sources such as industrial discharge, vehicular exhaust, waste water discharge from cities, etc. The model (2.2) has two equilibrium points E_1 and E_2 in which E_1 is saddle point and E_2 is locally and globally stable under some conditions. The qualitative behavior of model (2.2) shows that the density of total population will settle down to an equilibrium level, lower than its initial carrying capacity. It is assumed that a subclass of biological population is not capable in reproduction. Under this assumption, it is found that the density of this subclass increases as emission rates of toxicants or uptake rates of toxicants increase and when the decay rate of deformed population increases, density of deformed population decreases. For large emission rates, it may happen that the entire population gets severely affected and is not capable in reproduction and after a time period all the population may die out. So, we need to control the emission of toxicants from industries, household and vehicular discharges in the environment to protect biological species from deformity.

Appendix A. Proof of the Theorem 2.

Proof: we consider a positive definite function about E_2

$$W(N, N_D, T_1, T_2, U_1, U_2) = \frac{1}{2}(N-N^*)^2 + \frac{1}{2}(N_D-N_D^*)^2 + \frac{1}{2}(T_1-T_1^*)^2 + \frac{1}{2}(T_2-T_2^*)^2 + \frac{1}{2}(U_1-U_1^*)^2 + \frac{1}{2}(U_2-U_2^*)^2$$

Differentiating W with respect to t along the solution of (2.2), we get

$$\frac{dW}{dt} = (N-N^*)\left[rN - \frac{rN^2}{K(T_1,T_2)} - (\alpha+b)N_D\right] + (N_D-N_D^*)\left[(r_1U_1+r_2U_2)(N-N_D) - \frac{rN_DN}{K(T_1,T_2)} - (\alpha+d)N_D\right] + (T_1-T_1^*)[Q_1 - \delta_1T_1 - \gamma_1T_1N + \pi_1\nu_1NU_1] + (T_2-T_2^*)[Q_2 - \delta_2T_2 - \gamma_2T_2N + \pi_2\nu_2NU_2] + (U_1-U_1^*)[\gamma_1T_1N - \beta_1U_1 - \nu_1NU_1] + (U_2-U_2^*)[\gamma_2T_2N - \beta_2U_2 - \nu_2NU_2]$$

using (3.1-3.8), we get after some calculation

$$\frac{dW}{dt} = -\left[r\left(\frac{2N^*}{K(T_1^*,T_2^*)}-1\right)\right](N-N^*)^2 - \left[(r_1U_1^*+r_2U_2^*)\frac{N^*}{N_D^*}\right](N_D-N_D^*)^2 - (\delta_1+\gamma_1N^*)(T_1-T_1^*)^2$$
$$- (\delta_2+\gamma_2N^*)(T_2-T_2^*)^2 - (\beta_1+\nu_1N^*)(U_1-U_1^*)^2 - (\beta_2+\nu_2N^*)(U_2-U_2^*)^2$$
$$+\left[-(\alpha+b)+r_1U_1^*+r_2U_2^*-\frac{rN_D}{K(T_1^*,T_2^*)}\right](N-N^*)(N_D-N_D^*)$$
$$+[(\pi_1\nu_1U_1-\gamma_1T_1)-rN^2\eta_1(T_1,T_2)](N-N^*)(T_1-T_1^*)$$
$$+[(\pi_2\nu_2U_2-\gamma_2T_2)-rN^2\eta_2(T_1^*,T_2)](N-N^*)(T_2-T_2^*)$$
$$+(\gamma_1T_1-\nu_1U_1)(N-N^*)(U_1-U_1^*)+(\gamma_2T_2-\nu_2U_2)(N-N^*)(U_2-U_2^*)$$
$$-rNN_D\eta_1(T_1,T_2)(N_D-N_D^*)(T_1-T_1^*)-rNN_D\eta_2(T_1^*,T_2)(N_D-N_D^*)(T_2-T_2^*)$$
$$+r_1(N-N_D)(N_D-N_D^*)(U_1-U_1^*)+r_2(N-N_D)(N_D-N_D^*)(U_2-U_2^*)$$
$$+(\pi_1\nu_1N^*+\gamma_1N^*)(T_1-T_1^*)(U_1-U_1^*)+(\pi_2\nu_2N^*+\gamma_2N^*)(T_2-T_2^*)(U_2-U_2^*)$$

where,

$$\eta_1(T_1,T_2) = \begin{cases} \dfrac{\left[\frac{1}{K(T_1,T_2)}-\frac{1}{K(T_1^*,T_2)}\right]}{T_1-T_1^*}, & T_1 \neq T_1^* \\ -\dfrac{1}{K^2(T_1^*,T_2)}\dfrac{\partial K}{\partial T_1}(T_1^*,T_2), & T_1 = T_1^* \end{cases},$$

$$\eta_2(T_1^*,T_2) = \begin{cases} \dfrac{\left[\frac{1}{K(T_1^*,T_2)}-\frac{1}{K(T_1^*,T_2^*)}\right]}{T_2-T_2^*}, & T_2 \neq T_2^* \\ -\dfrac{1}{K^2(T_1^*,T_2^*)}\dfrac{\partial K}{\partial T_2}(T_1^*,T_2^*), & T_2 = T_2^* \end{cases}$$

Thus, $\frac{dw}{dt}$ can be written as sum of the quadratics,

$$\frac{dw}{dt} = \left\{-\frac{1}{2}b_{11}(N-N^*)^2+b_{12}(N-N^*)(N_D-N_D^*)-\frac{1}{2}b_{22}(N_D-N_D^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{11}(N-N^*)^2+b_{13}(N-N^*)(T_1-T_1^*)-\frac{1}{2}b_{33}(T_1-T_1^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{11}(N-N^*)^2+b_{14}(N-N^*)(T_2-T_2^*)-\frac{1}{2}b_{44}(T_2-T_2^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{11}(N-N^*)^2+b_{15}(N-N^*)(U_1-U_1^*)-\frac{1}{2}b_{55}(U_1-U_1^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{11}(N-N^*)^2+b_{16}(N-N^*)(U_2-U_2^*)-\frac{1}{2}b_{66}(U_2-U_2^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{22}(N_D-N_D^*)^2+b_{23}(N_D-N_D^*)(T_1-T_1^*)-\frac{1}{2}b_{33}(T_1-T_1^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{22}(N_D-N_D^*)^2+b_{24}(N_D-N_D^*)(T_2-T_2^*)-\frac{1}{2}b_{44}(T_2-T_2^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{22}(N_D-N_D^*)^2+b_{25}(N_D-N_D^*)(U_1-U_1^*)-\frac{1}{2}b_{55}(U_1-U_1^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{22}(N_D-N_D^*)^2+b_{26}(N_D-N_D^*)(U_2-U_2^*)-\frac{1}{2}b_{66}(U_2-U_2^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{33}(T_1-T_1^*)^2+b_{35}(T_1-T_1^*)(U_1-U_1^*)-\frac{1}{2}b_{55}(U_1-U_1^*)^2\right\}$$
$$+\left\{-\frac{1}{2}b_{44}(T_2-T_2^*)^2+b_{46}(T_2-T_2^*)(U_2-U_2^*)-\frac{1}{2}b_{66}(U_2-U_2^*)^2\right\}$$

where,

$$b_{11} = \frac{2}{5}\left[r\left(\frac{2N^*}{K(T_1^*,T_2^*)}-1\right)\right], \quad b_{22} = \frac{2}{5}\left[(r_1U_1^*+r_2U_2^*)\frac{N^*}{N_D^*}\right], \quad b_{33} = \frac{2}{3}(\delta_1+\gamma_1N^*)$$
$$b_{44} = \frac{2}{3}(\delta_2+\gamma_2N^*), \quad b_{55} = \frac{2}{3}(\beta_1+\nu_1N^*), \quad b_{66} = \frac{2}{3}(\beta_2+\nu_2N^*)$$

$$b_{12} = \left[-(\alpha + b) + r_1 U_1^* + r_2 U_2^* - \frac{rN_D}{K(T_1^*, T_2^*)}\right], \qquad b_{13} = [(\pi_1 \nu_1 U_1 - \gamma_1 T_1) - rN^2 \eta_1(T_1, T_2)]$$
$$b_{14} = [(\pi_2 \nu_2 U_2 - \gamma_2 T_2) - rN^2 \eta_2(T_1^*, T_2)], \qquad b_{15} = (\gamma_1 T_1 - \nu_1 U_1), \qquad b_{16} = (\gamma_2 T_2 - \nu_2 U_2)$$
$$b_{23} = -rNN_D \eta_1(T_1, T_2), \qquad b_{24} = -rNN_D \eta_2(T_1^*, T_2), \qquad b_{25} = r_1(N - N_D), \qquad b_{26} = r_2(N - N_D)$$
$$b_{35} = (\pi_1 \nu_1 N^* + \gamma_1 N^*), \qquad b_{46} = (\pi_2 \nu_2 N^* + \gamma_2 N^*)$$

Thus, $\frac{dW}{dt}$ will be negative definite provided

$$b_{12}^2 < b_{11} b_{22} \tag{3.33}$$
$$b_{13}^2 < b_{11} b_{33} \tag{3.34}$$
$$b_{14}^2 < b_{11} b_{44} \tag{3.35}$$
$$b_{15}^2 < b_{11} b_{55} \tag{3.36}$$
$$b_{16}^2 < b_{11} b_{66} \tag{3.37}$$
$$b_{23}^2 < b_{22} b_{33} \tag{3.38}$$
$$b_{24}^2 < b_{22} b_{44} \tag{3.39}$$
$$b_{25}^2 < b_{22} b_{55} \tag{3.40}$$
$$b_{26}^2 < b_{22} b_{66} \tag{3.41}$$
$$b_{35}^2 < b_{33} b_{55} \tag{3.42}$$
$$b_{46}^2 < b_{44} b_{66} \tag{3.43}$$

We note that (3.33-3.43) ⇒ (3.22-3.32) respectively. So, W is a Lyapunov's function with respect to the equilibrium E_2 and therefore E_2 is globally asymptotically stable under the conditions (3.22-3.32). Hence the theorem. □

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