*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 6 Ver. V (Nov. - Dec. 2015), PP 19-25*
*www.iosrjournals.org*

# Thermal Stress in a Half-Space with Mixed Boundary Conditions due to Time Dependent Heat Source

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***Abstract:*** *We consider a mixed boundary value problem in thermal stress in half space. The surface of the half space is heated by a time dependent source which produces temperature changes in the material. The resulting thermal stresses are our main focus in this study. We assume that the surface of the half space satisfies mixed boundary conditions. In the part of the boundary ($x < 0$) is stress free while in the remaining boundary ($x > 0$), the gradient of the stress vanishes. The determination of thermal stress is carried out using the Jones's modification of the so-called Wiener-Hopf technique.The solution in terms of the thermal stress in closed form is obtained.*
***Keywords:*** *Heat Conduction Equation, Thermal Stress, Mixed BoundaryProblem and Wiener-Hopf technique.*

## I. Introduction

The heat conduction problem in solids is of great interest in many engineering situations. Carslaw and Jaeger [1] has given a detailed account of linear heat flow in space, rods and plates having a homogeneous and composite structure. The thermal stress produced due to the temperature variation in a solid is an interesting situation in which stress within the body produced by thermal effects. An introductory description of thermoelectricity can be found, for example, in Hunter [2]. More analysis on thermal stress problem can be found in Yilbas[3]. In this study we are interested in the mixed boundary value problem in that, part of the boundary of the half space has varnishing stress while the other part has gradient of the stress varnishing. Such mixed boundary value problems in the context of temperature distribution has been studied by various authors. Evans [4] has considered a steady state temperature distribution within the cylinder at the point of entry into a cooling fluid. In Chakrabarti [5], the explicit solution of the sputtering temperature of a cooling cylindrical rod with an insulated core when allowed to enter into a cold fluid of large extent with a uniform speed $v$ in the positive semi-infinite range while the negative semi-infinite range is kept outside is determined, and a simple integral expression is derived for the value of the sputtering temperature of the rod at the points of entry. Georgiadis et al [6] considered infinite dissimilar materials which are joined and brought in contact over half of their common boundary and the other half insulated all along the common boundary (interface). Chakrabarti and Bera [7] studied a mixed boundary-valued problem associated with the diffusion equation which involves the physical problem of cooling of an infinite slab in a two-fluid medium. An analytical solution is derived for the temperature distribution at the quench fronts being created by two different layers of cold fluids having different cooling abilities moving on the upper surface of the slab at constant speed. Similarly, Zaman [8] studied a heat conduction problem across a semi-infinite interface in layered plates. The two layered plates are kept in contact, in which the contact between the layers takes place in one part of the interface while the outer part is perfectly insulated. Zaman and Al-Khairy [9] considered a steady state temperature distribution in a homogeneous rectangular infinite plate. They assumed that the lower part to be cooled by a fluid flowing at a constant velocity while the upper part satisfies the general mixed boundary conditions. However, in this work, we employ the modified Wiener-Hopf technique due to Jones [10] to solve our problem.

## II. Formulation of the Problem

We consider two dimensional heat conduction equation in half space ($y > 0$). The governing partial differential equation is:

$$\phi_{xx}(x,y,t) + \phi_{yy}(x,y,t) = \frac{1}{k}\,\phi_t(x,y,t) + f(x,t), \tag{1}$$

where,
$f(x,t) = I_0\,\delta(x-a)e^{-\beta t}$ is the heat source, $\delta(x-a)$ is the Dirac delta function. As for $\beta, a$ are the source parameters and $I_0$ is a positive constant.
The initial condition and boundary conditions are given by:

$$\phi(x,y,0) = 0, \qquad -\infty < x < \infty, 0 \le y < \infty. \tag{2}$$

$\lim_{y\to\infty} \phi(x,y,t) = 0, \quad -\infty < x < \infty, t \geq 0$ (3a)

$\phi_y(x,0,t) = 0, \; -\infty < x < \infty, t \geq 0$ (3b)

The plane stress coupled equation [4] is given by:

$\sigma_{xx}(x,y,t) + \sigma_{yy}(x,y,t) - \mu^2\sigma_{tt}(x,y,t) = \lambda\, \phi_{tt}(x,y,t),$ (4)

here,$\mu$ and $\lambda$ are positive constants while $\phi_{tt}$is the second derivative of the temperature obtained from the solution of Eq. (1).

The initial conditions are assumed as follows:

$\sigma(x,y,0) = 0, \quad -\infty < x < \infty, \; 0 \leq y < \infty$ (5a)

$\sigma_t(x,y,0) = 0 \;, \; -\infty < x < \infty, \; 0 \leq y < \infty$ (5b)

while the boundary conditions are:

$\sigma(x,0,t) = 0 \;, \quad -\infty < x < 0, \; t \geq 0$ (6a)

$\sigma_y(x,0,t) = 0, \;\; 0 < x < \infty, \; t \geq 0.$ (6b)

Also,

$\lim_{y\to\infty} \sigma(x,y,t) = 0, \quad -\infty < x < \infty, \; t \geq 0.$ (7)

## III. The Wiener-Hopf Equation

The Laplace transform in the time variable t and its inverse transform in s are defined by:

$L\{\sigma(x,y,t)\} = \int_0^\infty \sigma(x,y,t)e^{-st}dt = \overline{\sigma}(x,y,s)$ (8)

and

$L^{-1}\{\overline{\sigma}(x,y,s)\} = \frac{1}{2\pi i}\int_{i\infty-c}^{i\infty+c} \overline{\sigma}(x,y,s)e^{st}ds = \sigma(x,y,t).$ (9)

In the same way, we define the Fourier transform in $x$ and its corresponding inverse Fourier transform in $\alpha$ by:

$\mathcal{F}\{\sigma(x,y,t)\} = \int_{-\infty}^{\infty} \sigma(x,y,t)e^{i\alpha x}dx = \sigma^*(\alpha,y,t)$ (10)

and

$\mathcal{F}^{-1}\{\sigma^*(\alpha,y,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma^*(\alpha,y,t)e^{-i\alpha x}d\alpha = \sigma(x,y,t),$ (11)

with $\alpha = \xi + i\tau$. Moreover, we also introduce the half range Fourier transforms as

$\int_0^\infty \sigma(x,y,t)e^{i\alpha x}dx = \sigma_+^*(\alpha,y,t)$ (12)

and

$\int_{-\infty}^{0} \sigma(x,y,t)e^{i\alpha x}dx = \sigma_-^*(\alpha,y,t).$ (13)

So that,

$\sigma^*(\alpha,y,t) = \sigma_+^*(\alpha,y,t) + \sigma_-^*(\alpha,y,t).$ (14)

Where

$\sigma_+^*(\alpha,y,t) = O(e^{\tau_- x})$as$x \to \infty$ and $\sigma_-^*(\alpha,y,t) = 0(e^{\tau_+ x})$ as $x \to -\infty$.

Thus $\sigma_+^*(\alpha,y,t)$ is an analytic function of $\alpha$ in the upper half-plane $\tau > \tau_-$ , while $\sigma_-^*(\alpha,y,t)$ is an analytic function of $\alpha$ in the lower half-plane $\tau < \tau_+$ respectively. Therefore, $\sigma^*(\alpha,y,t)$defined an analytic function in the common strip $\tau_- < \tau < \tau_+$with $\tau = Im(\alpha)$.

## IV. Solution of the Heat Equation

Taking Laplace transform in t of Eq. (1) we get:

$\bar{\phi}_{xx}(x,y,s) + \bar{\phi}_{yy}(x,y,s) = \frac{s}{k}\bar{\phi}(x,y,t) - \frac{1}{k}\phi(x,y,0) + \bar{f}(x,s).$ (15)

Applying the initial condition to get:

$\bar{\phi}_{xx}(x,y,s) + \bar{\phi}_{yy}(x,y,s) = \frac{s}{k}\bar{\phi}(x,y,t) + \bar{f}(x,s).$ (16)

Taking Fourier transform in $x$ of Eq. (16) we obtain:

$-\alpha^2\bar{\phi}^*(\alpha,y,s) + \bar{\phi}^*{}_{yy}(\alpha,y,s) = \frac{s}{k}\bar{\phi}^*(\alpha,y,t) + \bar{f}^*(\alpha,s),$ (17)

which can be written as,

$\bar{\phi}^*{}_{yy}(\alpha,y,s) - \left(\alpha^2 + \frac{s}{k}\right)\bar{\phi}^*(\alpha,y,s) = \bar{f}^*(\alpha,s).$ (18)

Therefore the solution is,

$\bar{\phi}^*(\alpha,y,s) = A\, e^{-\sqrt{\alpha^2+\frac{s}{k}}\,y} + B\, e^{\sqrt{\alpha^2+\frac{s}{k}}\,y} - \frac{\bar{f}^*(\alpha,s)}{\left(\alpha^2+\frac{s}{k}\right)},$ (19)

where,

$\bar{f}^*(\alpha,s) = \frac{I_0 e^{i\,a\,\alpha}}{(s+\beta)}.$

Now, using boundary conditions in Eq. (3) we deduce that:

$B = 0$and$A = 0$

Thus, Eq. (19) becomes:

$\bar{\phi}^*(\alpha, y, s) = -\frac{I_0 e^{i\,\alpha\,\alpha}}{(s+\beta)\left(\alpha^2+\frac{s}{k}\right)}$. (20)

By Partial fraction decomposition, the inverse Laplace transform of Eq. (20) is obtained:

$\phi^*(\alpha, y, t) = -\frac{k\, I_0 e^{i\,\alpha\,\alpha}\left(e^{-kt\alpha^2} - e^{-t\beta}\right)}{k\alpha^2-\beta}$. (21)

So that $\phi(x, y, t)$ is given by,

$\phi(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi^*(\alpha, y, t) e^{-i\,\alpha\,x}\, d\alpha.$ (22)

## V. Solution of the Stress Equation

Taking Fourier transform in $x$ of Eq. (4) we obtain:

$-\alpha^2\sigma^*(\alpha, y, t) + \sigma^*{}_{yy}(\alpha, y, t) - \mu^2\sigma^*{}_{tt}(\alpha, y, t) = \lambda\phi^*{}_{tt}(\alpha, y, t).$ (23)

Now, taking Laplace transform in t we get:

$-\alpha^2\bar{\sigma}^*(\alpha, y, s) + \bar{\sigma}^*{}_{yy}(\alpha, y, s) - \mu^2 s^2\bar{\sigma}^*(\alpha, y, s) - \mu^2 s\, \sigma^*(\alpha, y, 0) - \mu^2\sigma^*{}_t(\alpha, y, 0) = \lambda\, s^2\bar{\phi}^*(\alpha, y, s) - \lambda\, \bar{\phi}(\alpha, y, 0) - \lambda\, \overline{\phi_t}(\alpha, y, 0).$ (24)

Applying the transformed initial conditions we get:

$-\alpha^2\bar{\sigma}^*(\alpha, y, s) + \bar{\sigma}^*{}_{yy}(\alpha, y, s) - \mu^2 s^2\bar{\sigma}^*(\alpha, y, s) = \lambda\, s^2\bar{\phi}^*(\alpha, y, t) - \lambda\, \overline{\phi_t}(\alpha, y, 0).$ (25)

$\overline{\phi_t}(\alpha, y, 0)$is computed using Eq. (21) to get:

$\overline{\phi_t}(\alpha, y, 0) = -k\, I_0 e^{i\,\alpha\,\alpha}$, so that Eq. (25) becomes,

$\bar{\sigma}^*{}_{yy}(\alpha, y, t) - (\alpha^2 + \mu^2 s^2)\bar{\sigma}^*(\alpha, y, t) = -\frac{\lambda\, s^2 I_0 e^{i\,\alpha\,\alpha}}{(s+\beta)\left(\alpha^2+\frac{s}{k}\right)} + \lambda\, k\, I_0 e^{i\,\alpha\,\alpha}.$ (26)

Therefore, the solution is

$\bar{\sigma}^*(\alpha, y, s) = C(\alpha)e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} + D(\alpha)e^{\sqrt{\alpha^2+\mu^2 s^2}\; y} + \frac{\lambda\, s^2 I_0 e^{i\,\alpha\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,\alpha\,\alpha}}{(\alpha^2+\mu^2 s^2)}.$ (27)

Using the boundary condition in Eq. (7) we deduce that $D(\alpha) = 0$.

Hence,

$\bar{\sigma}^*(\alpha, y, s) = C(\alpha)\, e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} + \frac{\lambda\, s^2 I_0 e^{i\,\alpha\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,\alpha\,\alpha}}{(\alpha^2+\mu^2 s^2)}.$ (28)

We now transform the boundaries in Eq. (6) as follows:

$\sigma(x, 0, t) = 0, \quad -\infty < x < 0, t \geq 0 \qquad \Rightarrow \bar{\sigma}^*{}_-(\alpha, 0, s) = 0$ (29)

$\sigma_y(x, 0, t) = 0, \quad 0 < x < \infty, t \geq 0 \qquad \Rightarrow \overline{\sigma_-}^{*\prime}(\alpha, 0, s) = 0$ (30)

where$\{\,'\,\}$ denotes the derivative of $\bar{\sigma}^*$ with respect to$y$.

So that,

$\bar{\sigma}^*{}_+(\alpha, 0, s) = C(\alpha) + \frac{\lambda\, s^2 I_0 e^{i\,\alpha\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,\alpha\,\alpha}}{(\alpha^2+\mu^2 s^2)}$ (31)

and

$\bar{\sigma}^{*\prime}{}_-(\alpha, 0, s) = -C(\alpha)\sqrt{\alpha^2 + \mu^2 s^2}.$ (32)

Then, Eq. (31) and Eq. (32) give:

$\bar{\sigma}^*{}_+(\alpha, 0, s) = -\frac{\overline{\sigma_-}^{*\prime}(\alpha,0,s)}{\sqrt{\alpha^2+\mu^2 s^2}} + \frac{\lambda\, s^2 I_0 e^{i\,\alpha\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,\alpha\,\alpha}}{(\alpha^2+\mu^2 s^2)}.$ (33)

This equation (33) which holds in the strip $\tau_- < \tau < \tau_+$ is the Wiener-Hopf equation. However, the unknown functions $\bar{\sigma}^*{}_+$ and $\overline{\sigma_-}^{*\prime}$ satisfying Eq. (33) are analytic in the upper $(\tau_- < \tau)$ and lower $(\tau < \tau_+)$ half planes respectively. The solution of this equation (33) is presented in the next section.

## VI. Solution of the Wiener-Hopf Equation

The goal here, is to have the terms in Eq. (33) to be either analytic in the upper half plane or lower half plane. This goal is achieved by decomposing or factoring the mixed terms in that equation using the theorems given in [11].

Now, let

$$M(\alpha) = \frac{1}{\sqrt{\alpha^2+\mu^2 s^2}},$$

then$M(\alpha)$ can be factorized as follows,

$$M(\alpha) = M_+(\alpha)M_-(\alpha).$$

However, by choosing a suitable branches for the square roots in such a way that $\frac{1}{\sqrt{\alpha+i\mu s}}$ is analytic in the upper half plane $(\tau > \tau_-)$, with $\tau_- > Im(-i\mu s)$. Similarly,$\frac{1}{\sqrt{\alpha-i\mu s}}$ is analytic in the lower half plane $(\tau < \tau_+)$, with$\tau_+ < Im(i\mu s)$. We then deduce that:

$$\begin{cases} M_+(\alpha) = \dfrac{1}{\sqrt{\alpha + i\mu s}} \ , \qquad \text{analytic in}(\tau > \tau_-) \\ \\ M_-(\alpha) = \dfrac{1}{\sqrt{\alpha - i\mu s}} \ . \ \text{analytic in}(\tau < \tau_+). \end{cases}$$

In the same way, we put

$$N(\alpha) = \frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}.$$

Then Eq. (32) becomes:

$$\overline{\sigma}^*{}_+(\alpha,0,s) = -\overline{\sigma_-}^{*'}(\alpha,0,s)M_+(\alpha)M_-(\alpha) + N(\alpha). \quad (34)$$

Now, dividing Eq. (33) by $M_+(\alpha)$ to get,

$$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} = -\overline{\sigma_-}^{*'}(\alpha,0,s)M_-(\alpha) + \frac{N(\alpha)}{M_+(\alpha)}. \quad (35)$$

In Eq. (35), we only have $\frac{N(\alpha)}{M_+(\alpha)}$ as a mixed term. We further decompose this mixed term using the decomposition theorem (see Noble [11], page 13) to get:

$$P(\alpha) = \frac{N(\alpha)}{M_+(\alpha)} = P_+(\alpha) + P_-(\alpha),$$

where$P_+(\alpha)$and$P_-(\alpha)$ are given to be :

$$P_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\left(\frac{N(z)}{M_+(z)}\right)\frac{1}{z-\alpha}\,dz \quad (36)$$

and

$$P_+(\alpha) = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id}\left(\frac{N(z)}{M_+(z)}\right)\frac{1}{z-\alpha}\,dz, \quad (37)$$

respectively, where $c$ and $d$ are chosen to be within the strip or analyticity. That is,$\tau_- < c < \tau < d < \tau_+$. However, $P_+(\alpha)$ is given in the Appendix I.

Finally, Eq. (35) becomes:

$$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*'}(\alpha,0,s)M_-(\alpha) + P_-(\alpha). \quad (38)$$

We now define $E(\alpha)$ as,

$$E(\alpha) = \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*'}(\alpha,0,s)M_-(\alpha) + P_-(\alpha). \quad (39)$$

Then, Eq. (39) defines $E(\alpha)$ only in the strip$\tau_- < \tau < \tau_+$ .The second part of the equation is defined and analytic in $\tau > \tau_-$ , and the third part is defined and analytic in $\tau < \tau_+$. Hence by analytic continuation we can define $E(\alpha)$ over the whole $\alpha$-plane and we write:

$$E(\alpha) = \begin{cases} \dfrac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) \ , & \tau > \tau_- \\ -\overline{\sigma_-}^{*'}(\alpha,0,s)M_-(\alpha) + P_-(\alpha) \ , & \tau < \tau_+ \\ \dfrac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*'}(\alpha,0,s)M_-(\alpha) + P_-(\alpha), & \tau_- < \tau < \tau_+. \end{cases} \quad (40)$$

Now, Eq. (40) defines an entire function $E(\alpha)$ in the whole $\alpha$-plane. Moreover, it can be shown from the asymptotic behavior of $E(\alpha)$ which vanishes when $|\alpha| \to \infty$, that is $E(\alpha)$ is bounded. Hence, we deduce that by the extended form of Liouville's theorem that $E(\alpha)$ is zero.

Thus, from Eq. (36) we get,

$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = 0$, which gives,

$$\overline{\sigma}^*{}_+(\alpha,0,s) = M_+(\alpha)P_+(\alpha). \quad (41)$$

Eq. (31) and (32) give:

$$C(\alpha) = \overline{\sigma}^*{}_+(\alpha,0,s) - \frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)},$$

and thus,

$$C(\alpha) = M_+(\alpha)P_+(\alpha) - \frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}, \quad (42)$$

Therefore, the thermal stress in Eq. (28) is given by

$$\overline{\sigma}^*(\alpha,y,s)=\left(M_+(\alpha)P_+(\alpha)-\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}+\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}\right)e^{-\sqrt{\alpha^2+\mu^2 s^2}\;y}+\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}-\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}. \quad (43)$$

## VII. Closed form of the thermal-stress

From Eq. (43) we have:

$$\overline{\sigma}^*(\alpha,y,s)=\left(M_+(\alpha)P_+(\alpha)-\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}+\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}\right)e^{-\sqrt{\alpha^2+\mu^2 s^2}\;y}+\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}-\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}.$$

This solution can be used to determine the overall stress effect of the body in the transformed domain.
The inverse Laplace transform and the inversion Fourier transform can then be used to obtain the thermal stress $\sigma(x,y,t)$ at given$(x,y,t)$.

Thus,

$$\sigma(x,y,t)=\frac{1}{4\pi^2 i}\int_{-i\infty+c}^{i\infty+c}\int_{-\infty}^{\infty}\left(\left(M_+(\alpha)P_+(\alpha)-\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}+\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}\right)e^{-\sqrt{\alpha^2+\mu^2 s^2}\;y}+\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}-\right.$$

*λ k I0e a αα2+μ2s2e−iaxestdα ds.*

(44)

These integrals in Eq. (44) give the closed form solution of the thermal stress. Evaluation of these integrals analytically is not easy task. This is due to the multiple valued functions resulting from the square roots in the integrand. Moreover, the singularity at infinity is not isolated, and hence we can't use the residue at infinity to evaluate such integrals. However, the contribution of the poles is investigated in the Appendix II.

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**Appendices**

**Appendix I:Decomposition of $P_+(\alpha)$**

Recall,

$$\frac{K(\alpha)}{M_+(\alpha)}=\frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)\sqrt{\alpha+i\mu s}\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)}-\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{\sqrt{\alpha+i\mu s}\,(\alpha^2+\mu^2 s^2)}.$$

From Eq. (36) we have,

$$P_+(\alpha)=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\left(\frac{\lambda\, s^2 I_0 e^{i\,a\,z}}{(s+\beta)\sqrt{z+i\mu s}\,(z^2+\mu^2 s^2)\left(z^2+\frac{s}{k}\right)}-\frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{\sqrt{z+i\mu s}\,(z^2+\mu^2 s^2)}\right)\frac{1}{z-\alpha}\,dz. \quad (A.1)$$

For this, let's consider the following closed contour
$\gamma=(-\infty+ic\,,\infty+ic\,)\cup C_R^+$, where $C_R^+$ is the semi-circle in the upper half plane since$a>0$.
Now, let

$$G(z) = \left(\frac{\lambda\, s^2 I_0 e^{i\,a\,z}}{(s+\beta)\sqrt{z+i\mu s}\,(z^2+\mu^2 s^2)\left(z^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{\sqrt{z+i\mu s}\,(z^2+\mu^2 s^2)}\right)\frac{1}{z-\alpha},$$

then, $G$ has the following simple poles which lie inside $\gamma$,

$z = \alpha$ , $z = i\mu s$ and $z = i\sqrt{\frac{s}{k}}$ . Therefore,

$\int_\gamma G(z) = \int_{-\infty+ic}^{\infty+ic} G(z) + \int_{C_R^+} G(z) = \sum_{k=1}^{3} Res[G(z), z_k]$ ,

The integral over $C_R^+$ vanishes due to Jordan Lemma.

Hence,

$P_+(\alpha) = \sum_{k=1}^{3} Res[G(z), z_k]$which is given by,

$$P_+(\alpha) = \frac{i e^{-a\sqrt{\frac{s}{k}}} I_0 k^2 \sqrt{\frac{s}{k}}\lambda^2 \sqrt{i\left(\sqrt{\frac{s}{k}}+s\mu\right)}}{2\left(\sqrt{\frac{s}{k}}+i\alpha\right)(s+\beta)\left(\sqrt{\frac{s}{k}}+s\mu\right)(-1+ks\mu^2)} - \frac{e^{-as\mu} I_0\,\lambda\,\sqrt{is\mu}\,(-s-\beta+ks\lambda+ks^2\mu^2+ks\beta\mu^2)}{2\sqrt{2}s^2(s+\beta)\mu^2(-\alpha+is\mu)(-1+ks\mu^2)} + \frac{e^{ia\alpha} I_0\,\lambda\,(s^2+ks\alpha^2+s\beta+k\alpha^2\beta-ks^2\lambda)}{(s+k\alpha^2)(s+\beta)\sqrt{\alpha+is\mu}\,(\alpha^2+s^2\mu^2)}$$

(A.2)

**Appendix II:Thermal-stress due to the poles contributions**

The integrals in Eq. (44) can be treated individually as follows:

Let

$$G(\alpha) = \left(\left(\begin{array}{c} M_+(\alpha)P_+(\alpha) - \\ \frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{array}\right) e^{-\sqrt{\alpha^2+\mu^2 s^2}\;y} + \atop \frac{\lambda\, s^2 I_0 e^{i\,a\,\alpha}}{(s+\beta)(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0 e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}\right),$$

then$G(\alpha)$ has two simple poles at:

$\alpha = i\sqrt{\frac{s}{k}}$and $\alpha = -i\sqrt{\frac{s}{k}}$. Thus, we then have,

$$\overline{\sigma}(x,y,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)\, e^{-i\alpha x} = \begin{cases} Res\left[G(\alpha), -i\sqrt{\frac{s}{k}}\right], & x > 0 \\ Res\left[G(\alpha), i\sqrt{\frac{s}{k}}\right]. & x < 0 \end{cases}$$

Finding, we get:

$$\overline{\sigma}(x,y,s) = \frac{\left(e^{a\sqrt{\frac{s}{k}}-\sqrt{\frac{s}{k}}x-y\sqrt{\frac{s(-1+ks\mu^2)}{k}}} I_0 s^2\lambda^2 \left(\begin{array}{c} 1+i\sqrt{\frac{s}{k}}-i e^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\sqrt{\frac{s}{k}}+ \\ k\sqrt{\frac{s}{k}}\mu - iks\sqrt{\frac{s}{k}}\mu^2 + i e^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}} ks\sqrt{\frac{s}{k}}\mu^2 \end{array}\right)\right)}{2(s+\beta)\left(-\sqrt{\frac{s}{k}}+s\mu\right)^2\left(\sqrt{\frac{s}{k}}+s\mu\right)^2}, \text{ if} x > 0$$

(A.3)

and

$$\overline{\sigma}(x,y,s) = \frac{\left(\begin{array}{c} i e^{-a\sqrt{\frac{s}{k}}+\sqrt{\frac{s}{k}}x-y\sqrt{\frac{s(-1+ks\mu^2)}{k}}} \times \\ \left(-1+e^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\right) I_0 s^2\sqrt{\frac{s}{k}}\lambda^2 \left(\begin{array}{c} \sqrt{\frac{s}{k}}+3s\mu+ \\ 3ks\sqrt{\frac{s}{k}}\mu^2+ks^2\mu^3 \end{array}\right) \end{array}\right)}{2(s+\beta)\left(\sqrt{\frac{s}{k}}-s\mu\right)\left(\sqrt{\frac{s}{k}}+s\mu\right)^4}. \qquad \text{if} x < 0$$

(A.4)

Eq. (A.3) and (A.4) are in the Laplace domain, and the only pole that these equations have in $s$ domain is at $s = -\beta$. Therefore, we get:

$$\sigma(x,y,t)=\frac{\left(e^{-t\beta+a\sqrt{-\frac{\beta}{k}}-x\sqrt{-\frac{\beta}{k}}-y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}I_0\,\beta^2\lambda^2\left(\begin{array}{c}1+i\sqrt{-\frac{\beta}{k}}-ie^{y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}\sqrt{-\frac{\beta}{k}}+\\ k\sqrt{-\frac{\beta}{k}}\mu+ik\beta\sqrt{-\frac{\beta}{k}}\mu^2-\\ ie^{y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}k\beta\sqrt{-\frac{\beta}{k}}\mu^2\end{array}\right)\right)}{2\left(-\sqrt{-\frac{\beta}{k}}+\beta\mu\right)^2\left(\sqrt{-\frac{\beta}{k}}+\beta\mu\right)^2}\text{if}x>0$$

(A.5)

and

$$\sigma(x,y,t)=-\frac{\left(\begin{array}{c}ie^{-t\beta-a\sqrt{-\frac{\beta}{k}}+x\sqrt{-\frac{\beta}{k}}-y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}\times\\ \left(-1+e^{y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}\right)i\beta^2\sqrt{-\frac{\beta}{k}}\lambda^2\left(\begin{array}{c}-\sqrt{-\frac{\beta}{k}}+3\beta\mu+\\ 3k\beta\sqrt{-\frac{\beta}{k}}\mu^2-k\beta^2\mu^3\end{array}\right)\end{array}\right)}{2\left(\sqrt{-\frac{\beta}{k}}-\beta\mu\right)^4\left(\sqrt{-\frac{\beta}{k}}+\beta\mu\right)}\text{if } x<0.$$

(A.6)