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On Modifying Eigenfunction-Expansions For Unsteady Couette Flow Of A Second Grade Fluid In A Layer Of Porous Medium

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Abstract : *A modified separation of variables (Eigenfunction-Expansions) suggested by Ivan [1] was investigated for unsteady Couette flow of a second grade fluid in a layer of porous medium. We find that it produces the correct solution shown to agree identically with the Laplace Transform obtained for a class of unsteady flows of generalized second grade fluid for the same problem.*

Keywords: *second grade fluid, Eigenfunction expansions, unsteady flow, Laplace transform*

I. Introduction

The interest in flows of non-Newtonian fluids through a porous medium has grown considerably because of their applications in engineering [2]. Recently , Ivan[1] explained that inmost mathematics and the process of the steady state of a linear parabolic partial differential equation engineering textbooks describe as a technique for obtaining a boundary-value problem with homogeneous boundary conditions that can be solved by separation of variables (i.e .,eigenfunction expansions). While this method produces the correct solution for the start-up of the flow of a Newtonian fluid between parallel plates, it can lead to erroneous solutions to the corresponding problem for a class of non-Newtonian fluids. He showed that the reason for this is the non-rigorous enforcement of the start-up conditions in the textbook approach, which lead to a violation of the principle of causality. Nevertheless, he said these boundary-value problems can be solved correctly using Modifying eigenfunction expansions, and he presented mathematically-correct way by writing u (0, t) = H (t) by using a Heavaside unit step function. Thus, it is important that the start-up condition is always written explicitly (though the appropriate H(t) prefactor) to obtain the physical solution to the start-up problem and, moreover, to not alter the state of rest prior to start-up.The purpose of the present article is to show that is not merely a semantic distinction of no consequence, and that it fundamentally affects the method of solution. Specifically, (in essence, an application of Duhamel's principle) [3] .We examine the suggestion of Ivan's [1] paper for the generalized second grade fluid with the Laplace transform in the paper of Hayat [2] and we focus on special case of governing problem :

$$(1+\beta^2 l^2)\,\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial y^2}-l^2\,\frac{\partial^3 u}{\partial y^2\partial t}+\beta^2 u=0 \tag{1}$$

Special case when $\beta = 0$

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}}-\frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2}-l^2\,\frac{\partial^3 \mathrm{u}}{\partial \mathrm{y}^2\,\partial \mathrm{t}}=0 \tag{2}$$

The paper is organized as follows: In section (III), we gave a summary of the solutions of generalized second grade fluid by Laplace transform which were given by Hayat[2].In Section (IV), the solutions are found by the textbook eigenfunction expansion technique. In Section (V), the solutions are derived by using the eigenfunction expansion technique modified by Ivan [1], In Section (VI), we gives a critical discussion of the present work.

II. Basic Equations

We consider the flow of a second grade non-Newtonian fluid between two horizontal parallel impermeable plates. The distance between two plates is d. The general constitutive equation given by Rivlin and Ericksen [4] can be written in the following form:

$$\tau = -\rho I + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \tag{3}$$

Where p is the pressure, I is the unit tensor, μ is the dynamic viscosity α_1, α_2 , are the normal stress moduli, A_1 , A_2 are first two Rivlin-Ericksen kinematic tensors Where:

$$A_1 = \nabla V + (\nabla V)^T \tag{4}$$

$$A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + A_2(\nabla V)^T \tag{5}$$

Where (dA_1/dt) denotes the material time derivative, V is the velocity field and grad is the gradient operator, T is the matrix transpose.

The considered plates are rigid and infinite.Under these assumptions, the flow velocity at a given point in the porous layer
depends only on its y-coordinate and time t and thus the velocity is

$$V\ (\mathrm{u}\ (\mathrm{y},\mathrm{t})\ ,0,0) \quad (6)$$

In which u is the x-component of the velocity. Since the flow is unsteady, the interaction terms depend upon the drag and virtual mass effect. The relation between the pressure drop and velocity for a second grade fluid in porous media is

$$\nabla p = -\frac{\emptyset}{k}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)V \quad (7)$$

where $(k > 0)$ and $\emptyset(0 < \emptyset < 1)$ are the (constant) permeability and porosity, respectively. Note that Eq. (7) ignores the boundary effects on the flow and cannot be directly used to analyze flow problems in a porous space.Thus modified Darcy's law based on a local volume averaging technique [5,6] will be considered in a porous layer. Under consideration of the balance of forces acting on a volume element of fluid, the local volume average balance of linear momentum is given by [5,6]

$$\rho\frac{\mathrm{dV}}{\mathrm{dt}} = \mathrm{divT} + \mathrm{r} \quad (8)$$

In which ρ is the fluid density and r is the Darcy resistance for a second grad fluid in the porous space. Due to the volume averaging process, some information is lost, thus requiring supplementary empirical relation for the Darcy resistance [5] to be known as a measure of the resistance to the flow in the bulk of the porous space and r is also a measure of the flow resistance offered by the solid matrix; then r satisfies the following equation [5]:

$$r = -\frac{\emptyset}{k}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)V \quad (9)$$

Using (9) into (7) we have

$$\rho\frac{\mathrm{dV}}{\mathrm{dt}} = \mathrm{div\ T} - \frac{\emptyset}{k}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)V \quad (10)$$

Substituting (3) into above equation, one obtains

$$\rho\frac{\mathrm{dV}}{\mathrm{dt}} = -\nabla p + \mathrm{divS} - \frac{\emptyset}{k}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)V \quad (11)$$

where S is the extra stress tensor which for second grade fluid is

$$S = \mu A_1 + \alpha 1 A_2 + \alpha_2 A_2 \quad (12)$$

It is noted that if the terms dV/dt and divS are ignored then $Eq(11)$ reduces to(7). Now from $Eqs(4)$, (10) and (11) we can write:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} - \mathrm{v}\,\frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2} - d^2\,\frac{\partial^3 \mathrm{u}}{\partial \mathrm{y}^2\,\partial \mathrm{t}} + \frac{\emptyset}{k}\left(V + d^2\,\frac{\partial}{\partial t}\right)\mathrm{u} = 0 \quad (13)$$

in which $\nu = \mu/\rho$ is the kinematic viscosity, $\rho d^2 = \alpha_1(d \geq 0)$, the elastic coefficient, has the unit of length [2]) and pressure gradient in the $x - direction$ is neglected which is reasonable when there is no applied pressure gradient. We are interested here in initial-boundary value problems of Couette flow with sudden motion of bottom plate, in special case when $\beta = 0$.

III. Solution By Laplace Transform For Unsteady Couette Flow Of A Second Grade Fluid In A Layer Of Porous Medium[2]

Hayat [2], used Laplace transform to solve the governing equation for the flow of the second grade fluid between two parallel plates. This section deals with the solution of a second grade fluid in a porous layer in absence of the pressure gradient. The flow is induced due to motion of the pressure gradient. The flow is induced due to motion of the lower plate i.e., for $\mathrm{t} > 0$, the plate at $\mathrm{y} = 0$ starts to slide in its own plane with a constant speed U_0, i.e. the velocity of the plate is given by $(\mathrm{U}_0, 0, 0)$. The plate at $\mathrm{y} = \mathrm{h}$ is kept fixed. under this situation, the governing equation (1) with the boundary - initial conditions are given by :

$$(1 + \beta^2 l^2)\,\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} - l^2\,\frac{\partial^3 \mathrm{u}}{\partial \mathrm{y}^2\,\partial \mathrm{t}} + \beta^2 \mathrm{u} = 0$$

$$\mathrm{u}\ (0,\mathrm{t}) = \mathrm{U}_0\mathrm{H}(\mathrm{t}) \quad (14)$$

$$\mathrm{u}\ (\mathrm{h},\mathrm{t}) = 0 \quad (15)$$

$$\mathrm{u}\ (\mathrm{y},0) = 0, \qquad (\mathrm{y} > 0) \quad (16)$$

H(t) denotes the Heaviside unit step function, Defining the dimensionless quantities

$$\acute{u}=\frac{u}{U_0}\ ,\acute{y}=\frac{y}{h}\ ,t^{\prime}=\frac{vt}{h^2}\ ,\omega^{\prime}=\frac{h^2\,\omega}{v},l=\frac{d}{h}\ ,\beta=h\sqrt{\emptyset/k}$$

the governing problem is :

$$\left(1+\beta^2 l^2\right)\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial y^2}-l^2\frac{\partial^3 u}{\partial y^2\,\partial t}+\beta^2 u=0$$

$$u\,(0,t)=H\,(t) \tag{17}$$

$$u\,(1,t)=0 \tag{18}$$

$$u\,(y,0)=0\ (y>0) \tag{19}$$

where the primes have been suppressed for simplicity.

For the governing problem subject to $(17-19)$, we define

$$\bar{u}\,(y,s)=L[u\,(y,t)]=\int_0^\infty e^{-st}u\,(y,t)dt$$

as the Laplace transform of $u\,(y,t)$ (where s is a Laplace transform parameter). Taking Laplace transform of (1) and , we arrive at

$$\frac{\partial^2\bar{u}}{\partial y^2}-\left(\frac{\beta^2+s\left(1+\beta^2 l^2\right)}{1+sl^2}\right)\bar{u}=0 \tag{20}$$

$$\bar{u}\,(0,s)=\frac{1}{s}\ ,\qquad \bar{u}(1,s)=0 \tag{21}$$

Solving the above problem we have

$$\bar{u}\,(y,s)=\frac{\sinh q(1-y)}{s\sinh q} \tag{22}$$

$$q=\left[\frac{\beta^2+s\left(1+\beta^2 l^2\right)}{1+sl^2}\right]^{\frac{1}{2}} \tag{23}$$

Taking inverse Laplace transform we obtain

$$u\,(y,t)=U\theta\,(t)\frac{1}{2\pi i}\int_0^\infty\frac{\sinh q(1-y)e^{-st}}{s\sinh q}\,ds. \tag{24}$$

In order to obtain the solution, we have to solve the integral in (24). For that we use the complex variable theory. It is seen that $s=0$ is a simple pole.

Therefore, the residue at $s=0$ is :

$$Res\,(0)=\frac{\sinh\beta(1-y)}{s\,sinh\,\beta} \tag{25}$$

The other singular points are the zeros of

$$sinh\,q=0. \tag{26}$$

Setting $q=i\lambda$, we find

$$sin\,\lambda=0. \tag{27}$$

If $\lambda_n=n\pi, n=1,2,3,\ldots,\infty$ are the zeros of (27) then

$$s_n=-\left[\frac{\beta^2+n^2\pi^2}{1+(\beta^2+n^2\pi^2)l^2}\right],\quad n=1,2,3,\ldots,\infty \tag{28}$$

are the poles. The residue at all these poles is obtained as

$$Res\,(s_n)=\frac{2(-1)^n n\,\pi e^{s_n t}}{(\beta^2+n^2\pi^2)\,[1+(\beta^2+n^2\pi^2)l^2\,]} \tag{29}$$

Res (0) and Res(s_n) , a complete solution is given by

$$u\,(y,t)=\theta(t)\left[\frac{\sinh\beta(1-y)}{sinh\,\beta}+2\pi\sum_{n=0}^{\infty}\frac{(-1)^n n\,e^{s_n t}}{(\beta^2+n^2\pi^2)\,[1+(\beta^2+n^2\pi^2)l^2\,]}\sin n\pi\,(1-y)\right] \tag{30}$$

In special case when $\beta=0$ we find that:

$$u\,(y,t)=H(t)\left[(1-y)+2\pi\sum_{n=0}^{\infty}\frac{(-1)^n n\,e^{s_n t}}{(n^2\pi^2)(1+n^2\pi^2 l^2)}\sin n\pi\,(1-y)\right] \tag{31}$$

$$s_n=-\left[\frac{n^2\pi^2}{1+(n^2\pi^2)l^2}\right] \tag{32}$$

IV. Solutions By The Textbook Eigenfunction Expansion Technique.

The governing differential equation in Special case when $\beta = 0$ is :

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} - l^2 \frac{\partial^3 u}{\partial y^2 \partial t} = 0 \quad (33)$$

With boundary conditions:

$$u(0,t) = U_0 H(t) \quad (34)$$
$$u(h,t) = 0 \quad (35)$$
$$u(y,0) = 0 \quad for \quad 0 \quad (36)$$

Now, we introduce a transformation functionUsing this observation, the textbook approach[Carslaw and Jaeger, 1959; Batchelor, 1967; Leal, 2007; Bruus, 2008] is to now make the change of dependent variable

$$u(y,t) = v(y,t) + U_{ss} \quad (37)$$

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}, \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^3 u}{\partial y^2 \partial t} = \frac{\partial^3 v}{\partial y^2 \partial t} \quad (38)$$

we find that $v(y,t)$satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial y^2} - l^2 \frac{\partial^3 v}{\partial y^2 \partial t} -= 0 \quad (39)$$

under conditions :

$$v(0,t) = U_0 H(t) \quad (40)$$
$$v(1,t) = 0 \quad (41)$$
$$v(y,0) = -U_{ss} \quad (42)$$

We seek a solution by the form

$$v(y,t) = \sum_{n=1}^{\infty} T_n(t) . \Psi_n(y), \quad (43)$$

It is well known (Titchmarsh, 1962)[11] that the eigenvalue problem:

$$\Psi''_n(y) = -\lambda \Psi_n(y) \quad (44)$$

Substitution into the Eq. (38) we have:

$$\sum_{n=1}^{\infty} \frac{dT_n}{dt} \Psi_n(y) - \left(\sum_{n=1}^{\infty} T_n(t) \Psi''_k(y)\right) - l^2 \left(\sum_{n=1}^{\infty} \frac{dT_n}{dt}(t) \Psi''_k(y)\right) = 0 \quad (45)$$

Multiplying by $\Psi_m(y)$, and integrating from $0 \to 1$ we find that

$$\sum_{n=1}^{\infty} \frac{dT_n}{dt} \Psi_n(y) - \left(\sum_{n=1}^{\infty} T_n(t) \Psi''_k(y)\right) - l^2 \left(\sum_{n=1}^{\infty} \frac{dT_n}{dt}(t) \Psi''_k(y)\right) = 0 \quad (46)$$

$$(1 + l^2 \lambda) \left(\frac{dT_n}{dt}\right) + \lambda_n T_n(t) = 0 \quad (47)$$

$$T_0 = \frac{-2}{n\pi}$$

$$\frac{dT_n}{dt} + \frac{\lambda_n}{(1 + l^2 \lambda)} T_n(t) = 0 \quad (48)$$

Let: $\beta = \frac{\lambda_n}{(1+l^2\lambda)}$

$$T_n(t) = \frac{-2}{n\pi} \exp(-\beta t) \quad (49)$$

The solution is:

$$u(y,t) = (1-y) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\exp(-\beta\lambda t) \sin(n\pi y)}{n} \quad (50)$$

V. Solution Using The Modifiyng Eigenfunction Expansion Suggested By Ivan's [1]

The governing differential equation as in section (IV) is :

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} - l^2 \frac{\partial^3 u}{\partial y^2 \partial t} -= 0$$

With the same boundary conditions:

$$u(0,t) = H(t) U_0$$
$$u(1,t) = 0$$
$$u(y,0) = 0, (y > 0)$$

Now, we introduce a transformation function suggested by Ivan is :

$$u(y,t) = v(y,t) + U_{ss}\,H(t)\,, \qquad U_{ss} = 1 - y \tag{51}$$

So we have:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = \frac{\partial \mathrm{v}}{\partial \mathrm{t}} + U_{ss}(y)\delta(\mathrm{t})\ , \qquad \frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2} = \frac{\partial^2 \mathrm{v}}{\partial \mathrm{y}^2}\ ,\ \frac{\partial^3 \mathrm{u}}{\partial \mathrm{y}^2\,\partial \mathrm{t}} = \frac{\partial^3 \mathrm{v}}{\partial \mathrm{y}^2\,\partial \mathrm{t}}$$

we find $\mathrm{v(y,t)}$satisfies :

$$\left(\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + U_{ss}(y)\delta(\mathrm{t})\right) - \frac{\partial^2 \mathrm{v}}{\partial \mathrm{y}^2} - l^2\,\frac{\partial^3 \mathrm{v}}{\partial \mathrm{y}^2\,\partial \mathrm{t}} -= 0 \tag{52}$$

underconditions :

$$v(0,t) = 0 \tag{53}$$

$$v(1,t) = 0 \tag{54}$$

$$v(y,0) = 0 \tag{55}$$

This interpretation is a demonstration of Duhamel's principle [3](Duhamel, 1833; namely, that a time-varying boundary condition can be "exchanged" for a homogeneous boundary condition at the "cost" of adding a time-varying source term to the linear BVP. Notice that the textbook approach exchanges the inhomogeneous boundary conditions for a homogeneous boundary condition at the cost of an inhomogeneous initial condition. Philosophically, this is already problematic because the cumulative effects of the boundary condition from t = 0 up to t = ∞ , have been "condensed" into an initial condition and imposed $t = 0$, an act that readily violates the principle of causality, namely "no output before the input" (Toll) [12].

The method of separation of variables suggests that the ansatz substituting and using the orthogonality relation from Eq. (44), we seek a solution of the form

$$(\mathrm{y,t}) = \sum_{\mathrm{n}=1}^{\infty} \mathrm{T_n(t)}\Psi_{\mathrm{n}}\,(\mathrm{y}),$$

Substitution into the Eq. (52) we have :

$$\left(\sum_{\mathrm{n}=1}^{\infty}\frac{\mathrm{dT_n}}{\mathrm{dt}}\,.\Psi_{\mathrm{n}}(\mathrm{y}) + U_{ss}(y)\delta(\mathrm{t})\right) + \lambda\sum_{\mathrm{n}=1}^{\infty}\mathrm{T_n(t)}.\Psi_n(y) + \lambda l^2\sum_{\mathrm{n}=1}^{\infty}\frac{\mathrm{dT_n}}{\mathrm{dt}}.\Psi_n(y) = 0 \tag{56}$$

Substituting into Eq.(56) and using the orthogonality relation from Eq.(44), Multiplying by $\Psi_{\mathrm{m}}(\mathrm{y})$, and integrating from $0 \to 1$, we find that :

$$(1 + \lambda l^2)\frac{dT_n}{dt} + \lambda T_n(t) = \frac{-2}{n\pi}\,\delta(\mathrm{t}) \tag{57}$$

$$\mathrm{T}(0) = 0$$

So by Laplace transform:

$$(1 + \lambda l^2)sT(s) - T(0) + \lambda T(s) + T(s) = -\frac{2}{n\pi} \tag{58}$$

$$\mathrm{T(s)} = -\frac{2\mathrm{U}}{\mathrm{n}\pi}.\left(\frac{1}{(1+\lambda l^2)s+\lambda}\right) \tag{59}$$

$$\mathrm{T(t)} = -\frac{2\mathrm{U}}{\mathrm{n}\pi}\,\mathrm{L}^{-1}.\left(\frac{1}{(1+\lambda l^2)s+\lambda}\right) \tag{60}$$

$$\mathrm{T(t)} = -\frac{2}{\mathrm{n}\pi}\frac{1}{(1+\lambda l^2)+\lambda}.\exp\left(-\frac{\lambda \mathrm{t}}{(1+\lambda l^2)}\right) \tag{61}$$

The solution is:

$$u\,(y,t) = H(t)\left[(1-y) + 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n n\,e^{s_n t}}{(n^2\pi^2)(1+n^2\pi^2 l^2)}\,\sin n\pi\,(1-y)\right] \tag{62}$$

VI: Discussion And Conclusion:

Ivan Christov explained that in [1] most mathematics and engineering textbooks describe the process of steady state of a linear parabolic partial differential equation as a technique for obtaining a boundary-value problem with homogeneous boundary conditions that can be solved by separation of variables (i.e ., eigenfunction- expansions), these boundary-value problems can be solved correctly using eigenfunction expansions, and he presented the formulation that makes this possible So in this paper we examined the suggestion of Ivan's for the generalized second grade fluid with the Laplace Transform in the paper of Hayat [2] in speical case of the Eq. (1) when , $\beta = 0$. We found that the solution of the paper of Hayat with Laplace Transform in Eq. (31) agrees exactly with the solution of modifying separation of variables (i.e., eigenfunction expansions) in section (V) in Eq .(62) , In section (IV) the solution in Eq.(50) are not identical with the same solutions found in sections (III) , (V) for the same IBVP, so the modifying separation of variables (i.e., eigenfunction expansions) is more accurate than the textbook eigenfunction expansion technique and more flexible than Laplace Transform on describing the properties of a viscoelastic fluid.

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