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Extending Baire Measures To Regular Borel Measures

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Abstract: *It should be noted that a Borel measure may not be determined by its values on compact G_δ sets. But if some of the conditions are imposed on a Borel measure so that it can be determined by its values on compact G_δ sets. The answer to this question is Regularity. We can discus as in this paper by proving the result that a Borel measure is determined by its values on compact G_δ sets and further that every Baire measure has a unique extension to a regular Borel measure.*

Definition: Let X be any set, be $\mathcal{S}$ any σ-ring on X, $\mathcal{C}$ and $\mathcal{U}$ be any subclasses of $\mathcal{S}$.

(1) μ be any measure on $\mathcal{S}$ i.e. (X, $\mathcal{S}, \mu$) is a measure space then we say (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom I.
(2) $If\ \mathcal{C}$ is closed for finite unions, countable intersections, $\phi \in \mathcal{C}$ and $\mu(c) < \infty$ $\forall$ c $\in \mathcal{C}$, then we say that axiom II is satisfied. i.e. we say (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom II.
(3) $If\ \mathcal{U}$ is clsed for countable unions, finite intersections and for every E $\in \mathcal{S}$ there exist U $\in \mathcal{U}$ s.t. E $\subset$ U,then we say that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom III.

Definition: Suppose that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfy axiom I, II and III. Let E $\in \mathcal{S}$

(1) If $\mu(E)$ = inf{ $\mu(U)/$ E $\subset$ U $\in \mathcal{U}$}, then E is said to be Outer regular.
(2) If $\mu(E)$ = sup{ $\mu(C)/$ E $\supset$ C $\in \mathcal{C}$ }, then E is said to be Inner regular.
(3) The set E is said to be Regular if it is Outer regular as well as Inner regular.
(4) The measure μ is called Regular if every measurable set E in $\mathcal{S}$ is Regular.

Preposition: Suppose that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom I, II and III. Let E $\in \mathcal{S}$ then

(1) E is Outer regular iff for every $\varepsilon > 0$ there exist U $\in \mathcal{U}$ s.t. E $\subset$ U and $\mu(U) \leq \mu(E) + \varepsilon$.
(2) If for every $\varepsilon > 0$ there exist C $\in \mathcal{C}$ s.t. C $\subset$ E and $\mu(E) \leq \mu(C) + \varepsilon$, then E is called Inner regular.
(3) If E is Inner regular and $\mu(E) < \infty$ then for each $\varepsilon > 0$ there exist C $\in \mathcal{C}$, s.t. C $\subset$ E and $\mu(E) \leq \mu(C) + \varepsilon$.

Proof: (1) Suppose E is Outer regular and $\varepsilon > 0$.
Let $\mu(E) = \infty$, By axiom III there exist U $\in \mathcal{U}$ s.t. E $\subset$ U $\Rightarrow \mu(U) \geq \mu(E) = \infty$
$\Rightarrow \mu(U) = \infty \Rightarrow \mu(U) \leq \mu(E) + \varepsilon$.
Now suppose that $\mu(E) < \infty$, then we have $\mu(E) \leq \mu(E) + \varepsilon$ and
$\mu(E)$ = inf{ $\mu(U)/$ E $\subset$ U $\in \mathcal{U}$} then by definition of infimum there exist U $\in \mathcal{U}$ s.t. E $\subset$ U and $\mu(U) \leq \mu(E) + \varepsilon$, shows that the condition is necessary.
Conversely: Assume that the condition is satisfied.
To show that E is Outer regular, Let n be any natural number, By the condition taking $\varepsilon = \frac{1}{n}$ we have $U_n \in \mathcal{U}\ s.t.$ E $\subset U_n$ and $\mu(U_n) \leq \mu(E) + \frac{1}{n}$.
Let $V_n = \bigcap_{i=1}^{n} U_i$ then $V_n \in \mathcal{U}$, (V_n) is a decreasing sequence and
$\mu(E) + \frac{1}{n} \geq \mu(U_n) \geq \mu(V_n) \quad \forall$ n
Therefore $\lim_{n\to\infty}\{ \mu(E) + \frac{1}{n} \} \geq \lim_{n\to\infty} \mu(U_n)$
$\Rightarrow \mu(E) \geq$ inf{ $\mu(U_n)$} $\geq$ inf { $\mu(V_n)/$ E $\subset$ V, V $\in \mathcal{U}$ } ……….(*)
On the other hand $\mu(E) \leq \mu(V)$ for all V s.t. E $\subset$ V, V $\in \mathcal{U}$.
$\Rightarrow \mu(E) \leq$ inf{ μ(V)/E $\subset$ V, V $\in \mathcal{U}$ } …………(**)
From (*) and (**) we have $\mu(E)$ = inf { μ(V)/E $\subset$ V, V $\in \mathcal{U}$ }, shows that E is Outer regular.
(2) And (3) can be proved similarly by using the definition of Supremum.

Preposition: (1) If $\mu(E) = \infty$ then E is Outer regular.
(2) Every member of $\mathcal{U}$ is Outer regular.
(3) If V = $\bigcap_{n=1}^{\infty} U_n$, $U_n \in \mathcal{U}$, $\mu(U) < \infty$ then V is Outer regular.
Proof: (1) Let $\mu(E) = \infty$ and U $\in \mathcal{U}$ s.t. E $\subset$ U, then $\mu(U) = \infty$

$\Rightarrow \inf\{\mu(\mathrm{U})/\mathrm{E} \subset \mathrm{U},\ \mathrm{U} \in \mathcal{U}\} = \infty \Rightarrow \mu(E) = \inf\{\mu(\mathrm{U})/\mathrm{E} \subset \mathrm{U},\ \mathrm{U} \in \mathcal{U}\}$. Then E is Outer regular.
(2) Let $\mathrm{W} \in \mathcal{U}$ then $\mu(W) \geq \inf\{\mu(\mathrm{U})/\mathrm{W} \subset \mathrm{U},\ \mathrm{U} \in \mathcal{U}\}$
Let $\mathrm{U} \in \mathcal{U}$ s.t. $\mathrm{W} \subset \mathrm{U}$ then $(W) \leq \mu(\mathrm{U}) \Rightarrow \mu(W) \leq \inf\{\mu(\mathrm{U})/\mathrm{W} \subset \mathrm{U},\ \mathrm{U} \in \mathcal{U}\}$
$\Rightarrow \mu(W) = \inf\{\mu(\mathrm{U})/\mathrm{W} \subset \mathrm{U},\ \mathrm{U} \in \mathcal{U}\}$, shows that W is Outer regular.
(3) Let (U_{n}) be any sequence of members of $\mathcal{U}$ s.t. $\mu(U_1) < \infty$
Let $\mathrm{V} = \bigcap_{\mathrm{n}=1}^{\infty} U_{\mathrm{n}}$, Defne $V_{\mathrm{n}} = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} U_{\mathrm{i}}$ then $V_{\mathrm{n}} \in \mathcal{U}$ and $(V_{\mathrm{n}}) \downarrow \bigcap_{\mathrm{n}=1}^{\infty} V_{\mathrm{n}} = \bigcap_{\mathrm{n}=1}^{\infty} U_{\mathrm{n}} = \mathrm{V}$
$\Rightarrow \mu(\mathrm{V}) = \lim_{n\to\infty} \mu(V_{\mathrm{n}}) = \inf\{\mu(V_{\mathrm{n}})\} \geq \inf\{\mu(\mathrm{U})/\ \mathrm{V} \subset \mathrm{U} \in \mathcal{U}\}$
$\Rightarrow \mu(\mathrm{V}) \geq \inf\{\mu(\mathrm{U})/\ \mathrm{V} \subset \mathrm{U} \in \mathcal{U}\}$ ……….(*)
Let U be any member of $\mathcal{U}$ s.t. $\mathrm{V} \subset \mathrm{U}$ then $\mu(\mathrm{V}) \leq \mu(\mathrm{U})$
$\Rightarrow \mu(\mathrm{V}) \leq \inf\{\mu(\mathrm{U})/\ \mathrm{V} \subset \mathrm{U} \in \mathcal{U}\}$ …………..(**)
From (*) and (**) we get, $\mu(\mathrm{V}) = \inf\{\mu(\mathrm{U})/\ \mathrm{V} \subset \mathrm{U} \in \mathcal{U}\}$
Shows that V is Outer regular.

Preposition: (1) If $\mu(\mathrm{E}) = 0$ then E is Inner regular.
(2) Every member of $\mathcal{C}$ is Inner regular.
(3) Countable unions of members of $\mathcal{C}$ is Inner regular.
Proof: (1) Let $\mu(\mathrm{E}) = 0$ and $\mathrm{C} \in \mathcal{C}$ and $\mathrm{C} \subset \mathrm{E}$ then $\mu(\mathrm{C}) \leq \mu(\mathrm{E}) \Rightarrow \mu(\mathrm{C}) = 0$
$\Rightarrow \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{E}, \mathrm{C} \in \mathcal{C}\} = 0 \Rightarrow \mu(\mathrm{E}) = \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{E}, \mathrm{C} \in \mathcal{C}\}$.
Hence E is Inner regular.
(2) Let $\mathrm{D} \in \mathcal{C}$ then $\mathrm{D} \subseteq \mathrm{D}$, and $\mathrm{D} \in \mathcal{C}$
Therefore $\mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\} \geq \mu(\mathrm{D})$
Also $\mathrm{C} \subset \mathrm{D} \Rightarrow \mu(\mathrm{C}) \leq \mu(\mathrm{D}) \Rightarrow \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\} \leq \mu(\mathrm{D})$
Therefore $\mu(\mathrm{D}) = \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\} \Rightarrow$ D is Inner regular.
(3) Let (D_{n}) be a sequence of members of $\mathcal{C}$ and $\mathrm{D} = \bigcup_{\mathrm{n}=1}^{\infty} D_{\mathrm{n}}$, Define $C_{\mathrm{n}} = \bigcup_{\mathrm{j}=1}^{\mathrm{n}} D_{\mathrm{j}}$ then $C_{\mathrm{n}} \in \mathcal{C}$ for all n and (C_{n}) is monotone increasing sequence with $\bigcup_{\mathrm{n}=1}^{\infty} \mathrm{C} = \bigcup_{\mathrm{n}=1}^{\infty} D_{\mathrm{n}} = \mathrm{D}$
$\Rightarrow (C_{\mathrm{n}}) \uparrow \mathrm{D} \Rightarrow \mu(C_{\mathrm{n}}) \to \mu(\mathrm{D})$
$\Rightarrow \mu(\mathrm{D}) = \lim_{n\to\infty} \mu(\mathrm{C_n}) = \mathrm{Sup}\{\mu(\mathrm{C_n})\} \leq \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\}$
$\Rightarrow \mu(\mathrm{D}) \leq \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\}$ On the other hand if $\mathrm{C} \subset \mathrm{D}$,
$\mathrm{C} \in \mathcal{C}$ then $\mu(\mathrm{C}) \leq \mu(\mathrm{D})$
$\Rightarrow \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\} \leq \mu(\mathrm{D}) \Rightarrow \mu(\mathrm{D}) = \mathrm{Sup}\{\mu(\mathrm{C})/\ \mathrm{C} \subset \mathrm{D}, \mathrm{C} \in \mathcal{C}\}$
$\Rightarrow$ D is Inner regular.
Hence the proof.

Theorem: Countable union of outer regular sets is outer regular.
Proof: Let $(\mathrm{E_n})$ be any sequence of outer regular sets and $\mathrm{E} = \bigcup_{\mathrm{n}=1}^{\infty} E_{\mathrm{n}}$
If $\mu(\mathrm{E}) = \infty$, then E is outer regular as proved earlier.
Now suppose that $\mu(\mathrm{E}) < \infty$, Let $\varepsilon > 0$, since $\mathrm{E_n}$ are outer regular we can find a set $\mathrm{U_n} \in \mathcal{U}$ s.t. $\mathrm{E_n} \subset \mathrm{U_n}$, and $\mu(\mathrm{U_n}) \leq \mu(\mathrm{E_n}) + \frac{\varepsilon}{2^n}$
Let $\mathrm{U} = \bigcup_{\mathrm{n}=1}^{\infty} U_{\mathrm{n}}$, then $\mathrm{U} \in \mathcal{U}$ and $\mathrm{E} \subset \mathrm{U}$ then
$\mu(\mathrm{U} - \mathrm{E}) = \mu\left((\bigcup_{\mathrm{n}=1}^{\infty} U_{\mathrm{n}}) - (\bigcup_{\mathrm{n}=1}^{\infty} E_{\mathrm{n}})\right)$
$\leq \mu\left(\bigcup_{\mathrm{n}=1}^{\infty} (U_{\mathrm{n}} - E_{\mathrm{n}})\right) \leq \sum_{n=1}^{\infty} \mu(U_{\mathrm{n}} - E_{\mathrm{n}}) = \sum_{n=1}^{\infty} [\mu(U_{\mathrm{n}}) - \mu(E_{\mathrm{n}})]$ [Because $\mu(\mathrm{E_n}) < \infty$]
$\leq \sum_{n=1}^{\infty} [\frac{\varepsilon}{2^n}] = \varepsilon$
$\Rightarrow \mu(\mathrm{U} - \mathrm{E}) \leq \varepsilon \Rightarrow \mu(\mathrm{U}) \leq \mu(\mathrm{E}) + \varepsilon \Rightarrow$ E is outer regular.

Theorem: Finite union of outer regular sets is outer regular.
Proof: Let $E_1, E_2, \ldots\ldots, E_{\mathrm{k}}$ be k outer regular sets and let $\mathrm{E} = \bigcup_{\mathrm{i}=1}^{\mathrm{k}} E_{\mathrm{i}}$,
Define $E_{\mathrm{n}} = E_{\mathrm{k}}$ for n > k. Then (E_{n}) is a sequence of outer regular sets and $\bigcup_{\mathrm{n}=1}^{\infty} E_{\mathrm{n}} = \bigcup_{\mathrm{n}=1}^{\mathrm{k}} E_{\mathrm{n}} = \mathrm{E}$.
Hence E is outer regular set by the proceeding theorem.

Theorem: Finite intersection of outer regular sets of finite measure is outer regular.
Proof: Suppose E and F are outer regular sets and $\mu(E) < \infty$ *and* $\mu(F) < \infty$.
Let $\varepsilon > 0$ *be* given, by the outer regularity of E, we can find a set $U \in \mathcal{U}$ s.t. $E \subset U$ and $\mu(U) \leq \mu(E) + \frac{\varepsilon}{2}$
Similarly we can find a set $V \in \mathcal{U}$ s.t. $F \subset V$ and $\mu(V) \leq \mu(F) + \frac{\varepsilon}{2}$
Then $U \cap V \in \mathcal{U}$, $E \cap F \subset U \cap V$ and $\mu[(U \cap V) - (E \cap F)] \leq \mu[(U - E) \cup (V - F)]$
$\leq \mu(U - E) + \mu(V - F) = [\mu(U) - \mu(E)] + [\mu(V) - \mu(F)] = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
$\Rightarrow \mu[(U \cap V) - (E \cap F)] \leq \varepsilon \Rightarrow E \cap F$ is outer regular.

Theorem: The countable intersection of outer regular sets of finite measure is outer regular.
Proof: Let (E_n) be any sequence of outer regular sets of finite measure and $E = \bigcap_{n=1}^{\infty} E_n$.
To show that E is outer regular.
Let $\varepsilon > 0$, Define $F_n = \bigcap_{j=1}^{n} E_j$, then (F_n) is a decreasing sequence of outer regular sets and $\lim_{n\to\infty}(F_n) = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} E_n = E$. Thus $(F_n) \downarrow E$ and $\mu(F_i) < \infty$ for all i.
By continuity of measure for decreasing sequences we obtain that
$\mu(E) = \lim_{n\to\infty} \mu(F_n)$ i.e. $\mu(F_n) \to \mu(E)$
$\Rightarrow$ There exist k s.t. $\mu(F_k) \leq \mu(E) + \frac{\varepsilon}{2}$, Since F_k is outer regular and $\mu(F_k) < \infty$, we can find
$U \in \mathcal{U}$ s.t. $F_k \subset U$ and $\mu(U) \leq \mu(F_k) + \frac{\varepsilon}{2}$,
Thus $E \subset U$ and $\mu(U - E) = \mu[(U - F_k) \cup (F_k - E)]$
$\leq \mu(U - F_k) + \mu(F_k - E) \leq \mu(U) - \mu(F_k) + \mu(F_k) - \mu(E) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
$\Rightarrow \mu(U - E) < \varepsilon$, Shows that E is outer regular.

Theorem: Finite union of inner regular sets is inner regular.
Proof: Let E and F be two inner regular sets. To show that $E \cup F$ is also inner regular.
(1) Let $\mu(E) = \infty$, then $\text{Sup}\{\mu(C) / E \supset C \in \mathcal{C}\} = \infty$
$\Rightarrow \text{Sup}\{\mu(C) / C \subset E \cup F, C \in \mathcal{C}\} = \infty$
The fact $\mu(E) = \infty$ gives that $\mu(E \cup F) = \infty \Rightarrow \mu(E \cup F) = \text{Sup}\{\mu(C) / C \subset E \cup F, C \in \mathcal{C}\}$
$\Rightarrow E \cup F$ is inner regular.
(2) Let $\mu(F) = \infty$, then the argument is same as above.
(3) Finally suppose that $\mu(E) < \infty$, $\mu(F) < \infty$, Consider any $\varepsilon > 0$, as E is inner regular and $\mu(E) < \infty$, therefore there exist $C \in \mathcal{C}$ s.t. $C \subset E$ and $\mu(E) < \mu(C) + \frac{\varepsilon}{2}$
By the same argument there exist D, $D \subset F$ and $\mu(F) < \mu(D) + \frac{\varepsilon}{2}$
Now $C \cup D \in \mathcal{C}$, $C \cup D \subset E \cup F$ and $\mu[(E \cup F) - (C \cup D)] \leq \mu[(E - C)] + \mu[(F - D)]$
$\leq \mu[(E - C)] + \mu[(F - D)] = \mu(E) - \mu(C) + \mu(F) - \mu(D) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
$\Rightarrow \mu[(E \cup F) - (C \cup D)] \leq \varepsilon \Rightarrow \mu(E \cup F) \leq \mu(C \cup D) + \varepsilon \Rightarrow E \cup F$ is inner regular.

Theorem: The countable union of Inner regular sets is Inner regular.
Proof: Let (E_n) be any sequence of inner regular sets and $E = \bigcup_{n=1}^{\infty} E_n$,
Let $F_n = \bigcup_{i=1}^{n} E_i$, then in view of the above theorem F_n is inner regular for all n. Also F_n is monotonic increasing sequence and $\lim_{n\to\infty}(F_n) = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n = E$
i.e. $(F_n) \uparrow E \Rightarrow \mu(F_n) \to \mu(E)$ ……………….(*)
Case (1) Suppose $\mu(E) = \infty$. Let n be any natural number.
Since $\mu(F_n) \to \infty$, we can find k s.t. $\mu(F_k) > n$,
As F_k is inner regular, we have $\mu(F_k) = \text{Sup}\{\mu(C) / C \subset F_k, C \in \mathcal{C}\}$ and $C \subset F_k \subset E$
$\Rightarrow C \in \mathcal{C}$, $C \subset E$ and $\mu(C) > n \Rightarrow \text{Sup}\{\mu(C) / E \supset C \in \mathcal{C}\} = \infty$
$\Rightarrow \mu(E) = \text{Sup}\{\mu(C) / E \supset C \in \mathcal{C}\} \Rightarrow$ E is inner regular.
Case(2) *Let* $\mu(E) < \infty$, Then *take* $\varepsilon > 0$, as $\mu(E) < \infty$ and $\mu(F_k) \to \mu(E)$, we can find k s.t.
$\mu(F_k) \leq \mu(E) + \frac{\varepsilon}{2}$, for inner regularity of F_k, we can find $D \in \mathcal{U}$ s.t. $D \subset F_k$
And $\mu(F_k) < \mu(D_k) + \frac{\varepsilon}{2}$,

$Then\ \mu(E-D) = \mu[(E-F_k) \cup (F_k - D)] = \mu(E) - \mu(F_k) + \mu(F_k) - \mu(D) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
$\Rightarrow \mu(E-D) < \varepsilon \Rightarrow \mu(E) - \mu(D) < \varepsilon \Rightarrow \mu(E) < \mu(D) + \varepsilon$, Proves that E is inner regular.

Theorem: Countable intersection of inner regular sets of finite measure is inner regular.
Proof: Let (E_n) be any sequence of inner regular sets s.t. $\mu(E_n) < \infty, \forall$ n.
Let E = $\bigcap_{n=1}^{\infty} E_n$, Let $\varepsilon > 0$, Since E_n is inner regular and $\mu(E_n) < \infty$, we can find a set $C_n \in \mathcal{C}$ s.t. $C_n \subset E_n$ and $\mu(E_n) < \mu(C_n) + \frac{\varepsilon}{2^n}$, Define C = $\bigcap_{n=1}^{\infty} C_n$, Then C $\in \mathcal{C}$, C $\subset$ E
and $\mu(E-C) = \mu[\left(\bigcap_{n=1}^{\infty} E_n\right) - \left(\bigcap_{n=1}^{\infty} C_n\right)] \leq \mu\left(\bigcup_{n=1}^{\infty} (E_n - C_n)\right) \leq \sum_{n=1}^{\infty} \mu(E_n - C_n)$
$\leq \sum_{1}^{\infty} \mu(E_n) - \sum_{1}^{\infty} \mu(C_n) \leq \sum_{1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$
$\Rightarrow \mu(E) - \mu(C) < \varepsilon \Rightarrow \mu(E) < \mu(C) + \varepsilon \Rightarrow$ E is inner regular.

Theorem: Finite intersection of inner regular sets of finite measure is inner regular.
Proof: Let $E_1, E_2, \ldots\ldots, E_k$ be finitely many inner regular sets with $\mu(E_i) < \infty$ for $1 \leq i \leq k$
Define $E_n = E_k$ for n > k.
Then (E_n) is a sequence of inner regular sets with $\mu(E_n) < \infty$ for all n.
By the proceeding theorem $\bigcap_{n=1}^{\infty} E_n$ is inner regular.
But $\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{k} E_n \Rightarrow \bigcap_{n=1}^{k} E_n$ is inner regular.
Note: From the above said theorems we can say that
(1) Countable union of regular sets is regular.
(2) Finite union of regular sets is regular.
(3) Countable intersection of regular sets of finite measure is regular.
(4) Finite intersection of regular sets of finite measure is regular.

Properties of Baire Measure
Remark: Let ν be any Baire measure on L.C.H. space X, Ω be the σ-ring of Baire sets, Let $\mathcal{C}$ denote the class of compact G_δ sets and $\mathcal{U}$ be the class of open Baire sets. Then axioms I,II,III are satisfied.
Proof: (1) Axiom (I) is obvious.
(2) $\phi \in \mathcal{C}$, Let A,B $\in \mathcal{C}$, then A and B are compact G_δ sets $\Rightarrow$ A$\cap$ B is compact G_δ set, A $\cup$ B is also compact G_δ set $\Rightarrow$ A$\cap$ B, A $\cup$ B $\in \mathcal{C}$.
By $\nu(C) < \infty\ \forall$ C $\in \mathcal{C}$, By definition of Baire measure.
Let (C_n) be any sequence of members of $\mathcal{C}$ and C = $\bigcap_{n=1}^{\infty} C_n$, Since each C_n is compact, C is closed and C $\subset C_n \Rightarrow$ C is compact. As countable intersection of G_δ sets is G_δ set,
Hence C $\in \mathcal{C}$. Therefore axiom II is satisfied.
(3) Let A, B $\in \mathcal{U}$, then A and B are open Baire sets $\Rightarrow$ A$\cap$B is open Baire set$\Rightarrow$ A$\cap$B $\in \mathcal{U}$.
Let (A_n) be any sequence of members of $\mathcal{U}$ and A= $\bigcup_{n=1}^{\infty} A_n$, Since A_n are open for all n and union of open sets is open, it follows that A is open. $A_n \in \Omega \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Omega \Rightarrow$ A $\in \Omega$,
Thus A is a open Baire set $\Rightarrow$ A $\in \mathcal{U}$. Let E$\in \Omega$. Since $\Omega = \mathfrak{S}(\mathcal{C})$, we have E $\subset \bigcup_{n=1}^{\infty} K_n$,where $K_n \in \mathcal{C}\ \forall$n $\Rightarrow K_n \subset$ X $\forall$ n.
$\Rightarrow$There exist an open Baire set V_n s.t. $K_n \subset V_n$ [By Sandwich Theorem]
Let V = $\bigcup_{n=1}^{\infty} V_n$, then V is an open Baire set and hence V $\in \mathcal{U}$. From E $\subset \bigcup_{n=1}^{\infty}$ $K_n \Rightarrow$ E $\subset$ V and V $\in \mathcal{U}$. Shows that axiom III is also satisfied.

Definition: Let ν be any Baire measure on X, $\mathcal{C}$ be the class of compact G_δ sets and $\mathcal{U}$ be the class of open Baire sets. If ν be inner regular w.r.t. $\mathcal{C}$ and outer regular w.r.t. $\mathcal{U}$ then ν is called a **Regular Baire Measure** i.e. ν is called regular if it is regular w.r.t. (X, $\Omega, \nu, \mathcal{C}, \mathcal{U}$)where $\mathcal{C}$ is the class of G_δ sets and $\mathcal{U}$ be the class of open Baire sets.

Theorem: Let C be any compact G_δ set then C is regular.
Proof: As proved earlier that every member of $\mathcal{C}$ is inner regular. Therefore it is enough to show that C is outer regular.

As C is a G_δ set, therefore there exist a sequence (U_n) of open sets s.t. C = $\bigcap_{n=1}^{\infty} U_n$
⇒ C ⊂ U_n, C is compact, U_n is open for all n.
Hence by Baire Sandwich Theorem there exists Baire sets V_n and D_n s.t. C ⊂ $V_n \subset D_n \subset U_n$.
V_n is open , D_n is compact G_δ set, Obviously C = $\bigcap_{n=1}^{\infty} V_n$, V_n is open Baire set ⇒ $V_n \in \mathcal{U}, \forall$ n.
⇒ V_n is outer regular for all n.
As $V_n \subset D_n$ and $v(D_n) < \infty$ ∀ n, It follows that $\bigcap_{n=1}^{\infty} V_n$ is outer regular, Shows that C is outer regular. Follows that C is regular.

Theorem: Let C and D be compact G_δ sets, then C-D is regular.
Proof:
Case (1): Assume that C ⊃ D, since D is a G_δ set there exist a sequence (U_n) of open sets s.t. D = $\bigcap_{n=1}^{\infty} U_n$ ⇒ D ⊂ U_n for all n ⇒ There exist open Baire set V_n s.t. D ⊂ $V_n \subset U_n$ for all n and V_n is a countable union of compact G_δ sets. [By Baire Sandwich Theorem]
Obviously D = $\bigcap_{n=1}^{\infty} V_n$, Hence C-D = C- $(\bigcap_{n=1}^{\infty} V_n)$= C ∩ $\left(\bigcap_{n=1}^{\infty} V_n\right)^c$ = C ∩ $(\bigcup_{n=1}^{\infty} V_n^c)$
= $\bigcup_{n=1}^{\infty}(C \cap V_n^c) = \bigcup_{n=1}^{\infty}(C - V_n)$
⇒ C-D = $\bigcup_{n=1}^{\infty}(C - V_n)$ …………(1)
As V_n is open is open and countable union of compact G_δ sets, it follows that V_n^c is closed G_δ set.
⇒X - V_n is closed G_δ set ⇒C ∩ (X - V_n) is compact G_δ set ⇒ C - V_n is compact G_δ set
⇒ C - V_n is regular ⇒ $\bigcup_{n=1}^{\infty}$ (C - V_n) is regular ⇒ C-D is regular.
Case (2): Let C and D be any compact G_δ sets, Define E = C ∩ D then C-D = C-E ⇒ C ∩ D is compact G_δ set, then C ⊃ E and C and E are compact G_δ sets, therefore from case (1) C-E is regular
⇒ C-E = C-D is regular.

Theorem: Every Baire measure is regular.
Proof: Let v be any Baire measure Ω be the σ-ring of Baire sets, $\mathcal{C}$ be the class of compact G_δ sets, $\mathcal{U}$ be the class of open Baire sets.
Let $\mathcal{R}$ be the ring generated by $\mathcal{C}$, then every member of $\mathcal{R}$ is of the type $\bigcup_{i=1}^{n}(A_i - B_i)$ where A_i *and* B_i are members of $\mathcal{C}$ i.e. compact G_δ sets and $A_i \supset B_i$ and $A_i - B_i$ are disjoint for
1 ≤ i ≤ n.
Since difference of compact G_δ sets is regular ⇒ $A_i - B_i$ is regular for each i
⇒ $\bigcup_{i=1}^{n}(A_i - B_i)$ is regular.
Thus every member of $\mathcal{R}$ is regular …………(1)
Let C be any compact G_δ set. Define $\mathcal{M}$ = {E ∈ Ω/C ∩ E is regular}
Let A ∈ $\mathcal{R}$, then A is regular from (1), Also C is regular [Because compact G_δ set]
Hence C ∩ A is regular ⇒ A ∈ $\mathcal{M}$
Shows that $\mathcal{R} \subset \mathcal{M}$ ……………..(2)
Let (E_n) be any increasing sequence of members of $\mathcal{M}$ then C∩ E_n is regular for all n.
⇒ $\bigcup_{n=1}^{\infty}(C \cap E_n)$ is regular ⇒ C∩$(\bigcup_{n=1}^{\infty} E_n)$ is regular ⇒ $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.
Let (F_n) be any decreasing sequence of members of $\mathcal{M}$ then C∩ F_n is regular for all n and $v(C \cap F_n) \leq v(C) < \infty$
⇒ $\bigcap_{n=1}^{\infty}(C \cap F_n)$ is regular ⇒ C∩$(\bigcap_{n=1}^{\infty} F_n)$ is regular ⇒ $\bigcap_{n=1}^{\infty} F_n \in \mathcal{M}$.
Shows that $\mathcal{M}$ is closed for monotone limits, Hence $\mathcal{M}$ is the monotone class. From (2) we have $\mathcal{R} \subset \mathcal{M}$.
Therefore by lemma on monotone classes we have $\mathfrak{S}(\mathcal{R}) \subset \mathcal{M}$ ⇒ $\Omega \subset \mathcal{M}$ ⇒ C ∩ E is regular ∀ E ∈ Ω ………..(3)
Let E be any Baire set. Then E ∈ Ω and $\Omega = \mathfrak{S}(\mathcal{C})$ ⇒ E = $\bigcup_{n=1}^{\infty} K_n$ where $K_n \in \mathcal{C}$ ∀ n.
i.e. K_n is a compact G_δ set ∀ n. Thus E = E∩ $(\bigcup_{n=1}^{\infty} K_n)$ = $\bigcup_{n=1}^{\infty}$ (E ∩ K_n)
From (3) we have E ∩ K_n is regular ∀ n ⇒ $\bigcup_{n=1}^{\infty}$ (E ∩ K_n) is regular ⇒ E is regular ∀ E ∈ Ω

Prove that v is regular.

Properties of Borel Measures:

Definition: Let X be L.C.H. space, $\wedge$ be the σ-ring of Borel sets, μ be the Borel measure, $\mathcal{C}$ be the class of compact sets and $\mathcal{U}$ be the class of open Borel sets. If μ is regular w.r.t. the system $(X, \wedge, \mu, \mathcal{C}, \mathcal{U})$. Then μ is regular Borel Measure.

Preposition: Let X be L.C.H. space, $\wedge$ be the σ-ring of Borel sets, μ be the Borel measure, $\mathcal{C}$ be the class of compact sets and $\mathcal{U}$ be the class of open Borel sets. Then $(X, \wedge, \mu, \mathcal{C}, \mathcal{U})$ satisfies axiom I,II,III.

Proof: Axiom I is obviously satisfied.

Axiom II: $\in \mathcal{C}$, as ϕ is compact.

Let A , B $\in \mathcal{C}$ then A and B are compact sets. $\Rightarrow$ A$\cup$ B is compact $\Rightarrow$ A$\cup$ B $\in \mathcal{C}$.

Let (A_n) be any sequence of members of $\mathcal{C}$. Let A = $\bigcap_{n=1}^{\infty} A_n$, since A_n is closed set. As A$\subset A_n$, A_n is compact, It follows that A is compact$\Rightarrow$ A $\in \mathcal{C}$. The measure μ is finite on $\mathcal{C}$. Hence by definition Axiom II is also satisfied.

Axiom III: Let A, B $\in \mathcal{U}$ then A and B are open Borel sets. $\Rightarrow$ A$\cap$ B is open Borel set $\Rightarrow$ A$\cap$ B $\in \mathcal{U}$.

Let (A_n) be any sequence of members of $\mathcal{U}$. Let E = $\bigcup_{n=1}^{\infty} A_n$, since A_n are open Borel sets. $\Rightarrow \bigcup_{n=1}^{\infty} A_n$ is open Borel set $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{U}$.

Finally consider any Borel set S, then S $\in \wedge$ and $\wedge = \mathfrak{S}(\mathcal{C}) \Rightarrow S \subset \bigcup_{n=1}^{\infty} K_n$ where $K_n \in \mathcal{C}$ for all n.

By Baire Sandwich Theorem there exist an open Baire set V_n such that $K_n \subset V_n \subset$ X for all n.

Let $V_n = \bigcup_{n=1}^{\infty} V_n$, then V is an open Baire set and S $\subset \bigcup_{n=1}^{\infty} K_n \subset \bigcup_{n=1}^{\infty} V_n$ i.e. S $\subset$ V.

Since every Baire set is a Borel set, we have S $\subset$ V and V $\in \mathcal{U}$.

Proves that axiom III is satisfied.

Theorem: Let μ be any regular Borel measure and C be any compact set, then there exist a compact G_δ set D s.t. C $\subset$ D and $\mu(C) = \mu(D)$.

Proof: Since μ is regular, the set C is regular, hence C is outer regular, therefore for any natural number n there exist an open Borel set U_n s.t. C $\subset U_n$ and $\mu(U_n) \leq \mu(C)+\frac{1}{n}$

$\Rightarrow \mu(C) \leq \mu(U_n) \leq \mu(C)+\frac{1}{n}$

$\Rightarrow \mu(C) \leq \text{Inf}\{\mu(U_n)\} \leq \text{Inf}\{\mu(C)+\frac{1}{n}\} \Rightarrow \mu(C) \leq \text{Inf}\mu(U_n) \leq \mu(C)$

$\Rightarrow \mu(C) = \text{Inf}\,\mu(U_n)$ ....(*)

Since C is compact contained in U_n then by Baire Sandwich Theorem there exist a compact G_δ set D_n s.t. C $\subset D_n \subset U_n$.

Let D = $\bigcap_{n=1}^{\infty} D_n$ then D is compact G_δ set and C $\subset$ D.

From C $\subset$ D$\subset U_n$ for all n, we get $\mu(C) \leq \mu(D) \leq \mu(U_n)$ for all n

$\Rightarrow \mu(C) \leq \mu(D) \leq Inf\mu(U_n)$ for all n

$\Rightarrow \mu(C) \leq \mu(D) \leq \mu(C) \Rightarrow \mu(C) = \mu(D)$, Proved.

Theorem: Let μ_1, μ_2 be two Borel measures and $\mu_1(C) = \mu_2(C)$ for every compact set C then $\mu_1 = \mu_2$.

Proof: Let $\mathcal{C}$ be the class of compact subsets of X, $\mathcal{R}$ be the ring generated by $\mathcal{C}$ then every member of $\mathcal{R}$ is of the form $\bigcup_{i=1}^{n}(A_i - B_i)$ where $A_i, B_i \in \mathcal{C}, A_i \supset B_i$ and $A_i - B_i$ are disjoint.

Let A$\in \mathcal{R}$ and A = $\bigcup_{i=1}^{n}(A_i - B_i)$, then $\mu_1(A) = \sum_{i=1}^{n} \mu_1(A_i - B_i)$

$= \sum_{i=1}^{n}[\mu_1(A_i) - \mu_1(B_i)]$ [Because μ_1 is finite on $\mathcal{C}$]

$= \sum_{i=1}^{n}[\mu_2(A_i) - \mu_2(B_i)]$ [Because $\mu_1 = \mu_2$ *on* $\mathcal{C}$]

$=\sum_{i=1}^{n} \mu_2(A_i - B_i) = \mu_2[\bigcup_{i=1}^{n}(A_i - B_i)] = \mu_2(A)$, shows that $\mu_1 = \mu_2$ on $\mathcal{R}$.

By Caratheodory's Extension Theorem $\mu_1 = \mu_2$ on $\mathfrak{S}(\mathcal{R}) \Rightarrow \mu_1 = \mu_2$ on $\mathfrak{S}(\mathcal{C})$

i.e. $\mu_1 = \mu_2$ on σ-ring of Borel sets.

Remark: By the above theorem we show that a Borel measure is uniquely determined by its values on compact sets. It should be noted that a Borel measure may not be uniquely fixed by its values on compact G_δ sets. However the result holds in case of regular Borel measures.

Theorem: Let μ_1, μ_2 be two Regular Borel Measures and $\mu_1(D) = \mu_2(\text{D})$ for every compact G_δ set D then $\mu_1(\text{E}) = \mu_2(\text{E})$ for every Borel set E.

Proof: To prove that $\mu_1(\text{E}) = \mu_2(\text{E})$ for every Borel set E it is enough to prove that $\mu_1(\text{C}) = \mu_2(\text{C})$ for every compact set C.

Let C be any compact set, since μ_1 is regular Borel measure we can find a compact G_δ set D_1 such that C $\subset D_1$ and $\mu_1(\text{C}) = \mu_2(D_1)$ ………………(1)

By the same argument we can find a compact G_δ set D_2 such that C $\subset D_2$ and

$\mu_2(\text{C}) = \mu_2(D_2)$ ………………(2)

Define D = $D_1 \cap D_2$, then D is a compact G_δ set and C $\subset D_1$ and C $\subset D_2$ $\Rightarrow$C $\subset$D.

$\Rightarrow \mu_1(\text{C}) \leq \mu_1(D_1) \leq \mu_1(\text{D}) = \mu_2(\text{D}) \leq \mu_2(D_2) = \mu_2(\text{C}) \Rightarrow \mu_1(\text{C}) \leq \mu_2(\text{C})$

By the same argument we get $\mu_2(\text{C}) \leq \mu_1(\text{C})$.

Hence $\mu_1(\text{C}) = \mu_2(\text{C})$, this completes the proof.

Theorem: Let μ_1, μ_2 be two Regular Borel Measures then the following statements are equivalent

(a) $\mu_1(\text{E}) = \mu_2(\text{E})$ for every Borel set E.

(a') $\mu_1(\text{C}) = \mu_2(\text{C})$ for every compact set C.

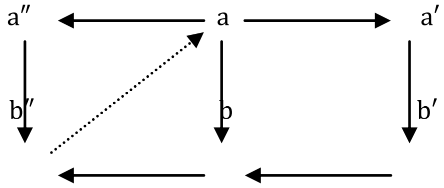
(a'') $\mu_1(\text{U}) = \mu_2(\text{U})$ for every open bounded set U.

(b) $\mu_1(\text{F}) = \mu_2(\text{F})$ for every Baire set F.

(b') $\mu_1(\text{D}) = \mu_2(\text{D})$ for every compact G_δ set D.

(b'') $\mu_1(\text{F}) = \mu_2(\text{F})$ for every open bounded Baire set F.

Proof: For the proof we draw a picture here



From the picture it is obvious that we simply need to prove that $(b'') \rightarrow$ (a)

Suppose that (b'') holds, to show that $\mu_1 = \mu_2$

For this it is enough to prove that $\mu_1(\text{D}) = \mu_2(\text{D})$ for every compact G_δ set D.

As μ_1 *and* μ_2 are regular, so consider any compact G_δ set D, Let D = $\bigcap_{n=1}^{\infty} U_n$, where U_n are open sets.

By Baire sandwich theorem, for every n there exist an open Baire set V_n and a compact G_δ set D_n such that D $\subset V_n \subset \text{D}_n \subset U_n \Rightarrow$ D = $\bigcap_{n=1}^{\infty} V_n$, V_n is open bounded Baire set because $V_n \subset \text{D}_n$ and D_n are compact.

Define $W_n = \bigcap_{i=1}^{n} V_i$, then (W_n) is a decreasing sequence of open bounded and Baire sets and

$\bigcap_{n=1}^{\infty} W_n = \bigcap_{n=1}^{\infty} V_n$ =D $\Rightarrow (W_n) \downarrow$ D $\Rightarrow \mu_1(\text{D}) = \lim_{n\to\infty} \mu_1(W_n)$ and $\mu_2(\text{D}) = \lim_{n\to\infty} \mu_2(W_n)$

But $\mu_1(W_n) = \mu_2(W_n)$ for all n from the supposition.

Hence $\lim_{n\to\infty} \mu_1(W_n) = \lim_{n\to\infty} \mu_2(W_n) \Rightarrow \mu_1(\text{D}) = \mu_2(\text{D})$ for all compact G_δ set D.

$\Rightarrow \mu_1(\text{E}) = \mu_2(\text{E})$ for every Baire set E. Proved.

Note: The above theorem, in addition brings out the fact that a Baire measure can have at the most one extension to a Regular Borel Measure. It will be proved here that every Baire measure can be extended to a regular Borel measure. This will settle the question that every Baire measure possess a unique extension to a regular Borel measure.

Definition: If U- C $\in \mathcal{U}$ for every U $\in \mathcal{U}$ and C $\in \mathcal{C}$, then we say that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$)satisfy axiom IV.

Theorem: Suppose (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom I to IV and every member of $\mathcal{C}$ is outer regular then C-D is outer regular foe every C and D $\in \mathcal{C}$.

Proof: Assume first that C $\supset$D, then from the hypothesis of the theorem C is outer regular, Let $\varepsilon > 0$, we can find V $\in \mathcal{U}$ s.t. C $\subset$ V and $\mu(U) \leq \mu(C) + \varepsilon$.

As axiom IV be satisfied, it follows that U-D $\in \mathcal{U}$.

Also $\mu[\,(U-D)-(C-D)] = \mu[\,(U-C)] = \mu(U) - \mu(C) \leq \varepsilon$. [Because $\mu(C) < \infty$]

$\Rightarrow \mu[\,(U-D)-(C-D)] \leq \varepsilon \Rightarrow \mu[\,(U-D)] \leq \mu[\,(C-D)] + \varepsilon$

Shows that C-D is outer regular.

Now suppose that C and D be any members of $\mathcal{C}$.

Define E = C $\cap$ D then C-D = C- (C $\cap$ D) = C-E, Since C $\in \mathcal{C}$, E = C $\cap$ D $\in \mathcal{C}$ and C $\supset$ E.

It follows from above that C-E is outer regular ⇒ C-D is outer rgular.

Definition: Let A ⊂X, Suppose there exist C ∈ $\mathcal{C}$s.t. A ⊂ C, then A is said to be bounded.

Definition: Suppose every member of $\mathcal{C}$ is conained in a bounded member of $\mathcal{U}$, i.e. for every C ∈ $\mathcal{C}$ there exist U ∈ $\mathcal{U}$, D ∈ $\mathcal{C}$ s.t. C ⊂ U⊂ D, Then we say that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom V.

Theorem: Suppose (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$)satisfies axioms I to V, assume that every bounded member of $\mathcal{U}$ is inner regular, then C-D is inner regular for all C and D ∈ $\mathcal{C}$.

Proof: Let C and D ∈ $\mathcal{C}$, assume that C ⊃ D. Let $\varepsilon > 0$, by axiom V there exist a bounded member U ∈ $\mathcal{U}$ s.t. C ⊂ U. By axiom IV, U-D ∈ $\mathcal{U}$, also U-D is bounded.

By hypothesis of the theorem U-D is inner regular.

Let $\varepsilon > 0$, Let E ∈ $\mathcal{C}$s.t. E ⊂ U-D

And $\mu[\,(U - D)] \leq \mu(E) + \varepsilon$, C ∩E ⊂ C-D and C ∩E ∈ $\mathcal{C}$ also (C-D)- (C ∩E) ⊂ U- D-E

⇒ μ[(C-D) - (C∩E)] ≤ $\mu[\,(U - D) - E]$

= $\mu[\,(U - D)] - \mu[(E)] \leq \varepsilon$ [Because μ(C) $< \infty, E \in \mathcal{C}$]

⇒ μ (C-D) ≤ μ (C ∩E) + ε, shows that C-D is inner regular.

Now suppose that C and D are any members of $\mathcal{C}$, Let E = C ∩D then E ∈ $\mathcal{C}$, C ⊃ E therefore C-E is inner regular , But C – E = C-D, Hence C-D is inner regular.

Definition: Suppose C-V ∈ $\mathcal{C}$ for all C ∈ $\mathcal{C}$, U ∈ $\mathcal{U}$, then we say that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom VI.

Theorem: Suppose axiom I to VI are satisfied, then the following statements are equivalent.

(a) Every member of $\mathcal{C}$ is outer regular.

(b) Every bounded member of $\mathcal{U}$ is inner regular.

Proof: Suppose (a) is true. Let U be any bounded member of $\mathcal{U}$. Let C ∈ $\mathcal{C}$ s.t. U ⊂ C, then by axiom VI, C-U ∈ $\mathcal{C}$. By supposition C- U is outer regular.

Let $\varepsilon > 0$, then there exist V ∈ $\mathcal{U}$ s.t. C-U ⊂ V and μ [V-(C-U)] ≤ ε ……..(1)

From U ⊂ C and C-U ⊂ V we get U –(C-V) ⊂ V – (C-U)

Therefore μ [U-(C-V)] ≤ μ [V-(C-U)] ≤ ε [From (1)]

⇒ μ (U) ≤ μ (C-V) + ε and C −V ∈ $\mathcal{C}$, C - V ⊂ U, Shows that U is inner regular , Hence (a) ⇒ (b).

Further Let (b) holds, to show that (a) is true, let C ∈ $\mathcal{C}$. By the axiom V, there exist a bounded member U of $\mathcal{U}$ s.t. C ⊂ U , But by axiom IV we get U-C ∈ $\mathcal{U}$,

Also U- C is bounded and by supposition U-C is inner regular.

Let $\varepsilon > 0$, by inner regularity we can find D ∈ $\mathcal{C}$ s.t. D ⊂ U- C and μ [(U-C)-D] ≤ ε …..(2)

By axiom IV, U-D ∈ $\mathcal{U}$, From C ⊂ U, D ⊂ U- C, we get C ⊂ U- D and

μ [(U-D)-C] = μ [(U-C)-D] ≤ ε [From (2)]

⇒ μ (U –D) ≤ μ (C) + ε, V-D ∈ $\mathcal{U}$ and C ⊂ U- D, Shows that C is outer regular.

Hence (b) ⇒ (a).

Definition: Suppose $\mathcal{S} = \mathfrak{S}(\mathcal{C})$, then we say that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom VII.

Theorem: Suppose (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axioms I to VII, then the following statements are equivalent.

(a) μ is regular.

(b) Every member of $\mathcal{C}$ is outer regular.

(c) Every bounded member of $\mathcal{U}$ is inner regular.

Proof: In proceeding theorems we proved that (b) ⇔ (c), Thus obviously (a) ⇒ (b) and (a) ⇒(c), therefore it is enough to prove that (c) ⇒ (a).

Suppose that (c) holds, then (b) is also true, to show that (a) holds.

Let $\mathcal{R}$ be the ring generated by $\mathcal{C}$. Let S ∈ $\mathcal{R}$ then S = $\bigcup_{i=1}^{n}(A_i - B_i)$ where $A_i, B_i \in \mathcal{C}$, It is earlier proved that C-D is regular for all C and D ∈ $\mathcal{C}$. Hence $A_i - B_i$ is regular for each i.

⇒ $\bigcup_{i=1}^{n}(A_i - B_i)$ is regular ⇒ S is regular for every S ∈ $\mathcal{R}$…………..(1)

Consider any C ∈ $\mathcal{C}$ define $\mathcal{M}$ = { E ∈ $\mathcal{S}$ /C∩ E is regular }, If S ∈ $\mathcal{R}$ then S is regular, As C ∈ $\mathcal{C}$ and every member of $\mathcal{C}$ is outer regular hence regular, it follows that C is regular. This implies that C∩ S is regular⇒ S ∈ $\mathcal{M}$. Shows that $\mathcal{R} \subset \mathcal{M}$………….(2)

Let (E_n) be any increasing sequence of members of $\mathcal{M}$.

Then C ∩ E_n is regular for all n. Therefore $\bigcup_{n=1}^{\infty}$(C ∩ E_n) is regular. This implies that

C ∩ ($\bigcup_{n=1}^{\infty} E_n$) is regular ⇒ $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M} \Rightarrow \lim_{n\to\infty}(E_n) \in \mathcal{M}$…………(*)

Let (F_n) be any decreasing sequence of of members of $\mathcal{M}$ then C $\cap F_n$ is regular for all n. Therefore $\bigcap_{n=1}^{\infty}$(C $\cap F_n$) is regular. This implies that C $\cap$ ($\bigcap_{n=1}^{\infty} F_n$) is regular.
$\Rightarrow \bigcap_{n=1}^{\infty} F_n \in \mathcal{M} \Rightarrow \lim_{n\to\infty}(F_n) \in \mathcal{M}$. …………(**)
From (*) and (**) we note that $\mathcal{M}$ is closed for monotone limits of its members, therefore $\mathcal{M}$ must be a monotone class.
From (2) we have $\mathcal{R} \subset \mathcal{M} \Rightarrow \mathfrak{S}(\mathcal{R}) \subset \mathcal{M}$ {By L.M.C.} $\Rightarrow \mathfrak{S}(\mathcal{C}) \subset \mathcal{M}$
$\Rightarrow \mathcal{S} \subset \mathcal{M}$ {Axiom VII]
$\Rightarrow$ C$\cap$ E is regular for every E $\in \mathcal{S}$ …………(3)
Let A $\in \mathcal{S}$, as $\mathcal{S} = \mathfrak{S}(\mathcal{C})$, every member of $\mathcal{S}$ is contained in a countable union of members of $\mathcal{C}$. Therefore A $\subset \bigcup_{n=1}^{\infty} K_n$, where $K_n \in \mathcal{C}$ $\forall$ n.
This gives A = A $\cap \left(\bigcup_{n=1}^{\infty} K_n\right) = \bigcup_{n=1}^{\infty}$ (A $\cap K_n$)
From (3) we see that A $\cap K_n$ is regular for every n. This gives that $\bigcup_{n=1}^{\infty}$ (A $\cap K_n$) is regular.
$\Rightarrow$ A is regular $\forall$ A $\in \mathcal{S}$, that means μ is regular and this completes the proof.

Theorem: Let μ be a Borel measure on X, then the following statements are equivalent.
(a) μ is regular.
(b) Every compact set is outer regular.
(c) Every bounded open Borel set is inner regular.
Proof: Let $\wedge$ be the σ-ring of Borel sets, $\mathcal{C}$ be the class of compact sets, $\mathcal{U}$ be the class of open Borel sets of X. It is proved earlier that (X, $\wedge, \mu, \mathcal{C}, \mathcal{U}$) satisfies axioms I, II, III.
Now we have

Axiom IV: Let U $\in \mathcal{U}$ and C $\in \mathcal{C}$, then C is compact $\Rightarrow$ C is closed $\Rightarrow$ C^c is open.
Since U is open we get U$\cap$ C^c is open $\Rightarrow$ U-C is open. Also U-C is Borel set as U and C are Borel sets. Shows that U-C $\in \mathcal{U}$. Therefore Axiom IV is satisfied.

Axiom V: Let C $\in \mathcal{C}$, then C is compact.
By Baire sandwich theorem there exist Baire sets V and D s.t. C $\subset$ V $\subset$ D and V is open, D is compact G_δ set. Shows that axiom V is satisfied.

Axiom VI: Let C$\in \mathcal{C}$ and U $\in \mathcal{U}$ then U is open $\Rightarrow$ U^c is closed and C is also closed
$\Rightarrow$ C$\cap$ U^c is closed $\Rightarrow$ C-U is closed subset of a compact set C, hence C-U is a compact set
$\Rightarrow$ C-U $\in \mathcal{C}$, shows that axiom VI is satisfied.

Axiom VII: As $\mathcal{S} = \mathfrak{S}(\mathcal{C})$, then we say that (X, $\mathcal{S}, \mu, \mathcal{C}, \mathcal{U}$) satisfies axiom VII.
Definition: Let X be a topological space then Baire $\sigma - algebra$ $\mathfrak{B}_0$(X) on X is the smallest $\sigma - algebra$ containing the pre-images of all continuous functions f: X→ X. And if there exist a measure μ on $\mathfrak{B}_0$(X) s.t. $\mu(X) < \infty$. Then μ is called a finite Baire measure on X. Further if $\mathfrak{B}(X)$ is the Borel $\sigma - algebra$ on X (i.e. the smallest $\sigma - algebra$ containing the open sets of X) then $\mathfrak{B}_0$(X)$\subset$ $\mathfrak{B}(X)$.
Definition: Borel sets are those sets of X belonging to the smallest $\sigma - algebra$ that contains all closed subsets of X. Clearly a Baire set is always a Borel set. But in many familiar spaces including all metric spaces the classes of all Baire sets and Borel sets are coincide.
Remark: If X be the metric space then $\mathfrak{B}_0$(X)= $\mathfrak{B}(X)$.

Regular Borel Measure: Let μ be a Borel measure on a space X and let E $\in \mathfrak{B}$. We say that the measure μ is outer regular on E if $\mu(E) = inf\{\mu(U): E \subset U, U \text{ is open}\}$ and we say that measure μ is inner regular on E if $\mu(E) = sup\{\mu(K) : K \subset U, K \text{ is compact} \}$. If μ is both inner and outer regular on E then we say that μ is regular on E. Further μ is called Regular Borel measure if it is regular on every Borel set. For example a Radon measure is a Borel measure which is
a. Finite on every compact set.
b. Outer regular on every Borel set.
c. Inner regular on every open set.

Preposition: Let μ be a Borel measure which is finite on compact sets. Then the following statements are equivalent.

1. μ is outer regular on σ-bounded sets.
2. μ is inner regular on σ-bounded sets.

Proof: (1) ⇒ (2) Suppose that E is a bounded Borel set and E $\subset L$. Where L is compact. Assume that$\varepsilon > 0$. We have to prove that there is a compact set K $\subseteq E$ with μ (K) $\geq \mu$ (E)-ε. As the relative compliment L/E is bounded by outer regularity there is an open set O$\supseteq L/E$ such that μ (O) $\leq \mu$ (L/E) +ε. It follows that K = L/O = L$\cap O^c$ is a compact set of E satisfying μ (K) =μ (L) - μ (L$\cap O$) $\geq \mu$ (L)- μ (O) $\geq \mu$(L/E)-ε, as required .

In general let $E = E_1 \cup E_2 \cup E_3 \cup ....$ is a countable union of bounded Borel sets E_i. We may assume that the sets E_i are disjoint. If some of the E_i has finite measure, then by above we have Sup{μ(K): K $\subseteq$ *Ei*, K∈𝓀 = *μEi*=+∞ , where 𝓀 is the family of compact sets. Then Sup*μ*K: K ⊆*E*, K∈𝓀 = *μE*=+∞ Proved.

But on the other hand if $\mu(E_i) < \infty$ for each i then for any $\varepsilon > 0$ we can find a sequence of compact sets $K_i \subseteq E_i$ with the property that $\mu(E_i) \leq \mu(K_i) + \frac{\varepsilon}{2^i}$.

Taking $L_n = K_1 \cup K_2 \cup ...... \cup K_n$, it is clear that L_n is a compact subset of E for which $\mu(L_n) = \sum_{i=}^{n} \mu(K_i) \geq \sum_{i=}^{n} \mu(K_i) - \frac{\varepsilon}{2^i} \geq \sum_{i=}^{n} \mu(K_i) - \varepsilon$. Taking supremum over n we get $Sup\, \mu(L_n) \geq \mu(E) - \varepsilon$. Which shows that μ is Inner regular on σ-bounded sets.

(2) ⇒ (1) Let E be abounded Borel set. Then closure of E is $\bar{E}$ and is compact set and by single covering there exist a bounded open set U s.t. $\bar{E} \subseteq U$. Let $\varepsilon > 0$ then L/E is a bounded Borel set, then by Inner regularity there exist a bounded compact set K ⊆ L/E *with* the property that μ(K) $\geq \mu$(L/E) $- \varepsilon$. Let V = U/K = U$\cap K^c$ then V is a bounded and open which contains E and μ(V) = μ(U $\cap K^c$) $\leq \mu$(L $\cap K^c$) = μ(L)- μ(K) $\leq \left(\mu\left(\frac{L}{E}\right) - \varepsilon\right)$= μ(E) $- \varepsilon$. As ε is arbitrary positive and this proves that μ is outer regular on bounded sets.

Further let E = $\bigcup_n E_n$ where each E_n is a bounded Borel set and each E_i is disjoint and $\mu(E_n) < \infty$ for all n. From the above we have a sequence of open sets $O_n \supseteq E_n$ such that $(O_n) \leq \mu(E_n) + \frac{\varepsilon}{2^n}$. Therefore the set E is contained in the union of O = $\bigcup_n O_n$ and we get $\mu(O) \leq \sum_{i=}^{n} \mu(O_i) \leq \sum_{i=}^{n} \mu(E_i) + \varepsilon = \mu(E) + \varepsilon$. Hence the proof.

Content: A real valued function defined on a $\sigma - algebra$ $\mathcal{A}$ of sub sets of a space X is said to be a content on X if 1) $\mu(A) > 0$ for all A∈ $\mathcal{A}$.
2) $\mu(\emptyset) = 0$ and 3) $\mu(A_1 \cup A_2)$= $\mu(A_1) + \mu(A_2)$ for all $A_1, A_2 \in \mathcal{A}$.

Preposition: Let λ be a content on X and μ be a regular Borel measure induced by λ, then the following statements are equivalent.
1. λ is regular content.
2. μ is an extension of λ.

Proof: Suppose that (1) holds i.e. λ is a regular content. Let C be any compact set, and > 0 , By the regularity of λ , we can find a compact set D s.t.
C⊂D and $\lambda(D) \leq \lambda$(C) + ε ....(1)
Let λ^* be the outer measure induced by λ. Then let U= D then C⊂U ⊂D and U is an open bounded Borel set, we have $\lambda^*(C) \leq \lambda^*(U)$ {*Because* λ^* *is monotone*}
$\leq \lambda(D)$) $\leq \lambda$(C) + ε {*By* (1)}
Hence $\lambda^*(C) \leq \lambda$(C) But $\mu(C) = \lambda^*(c)$ Therefore $\lambda^*(C) = \lambda$(C) i.e. $\mu(C) = \lambda$(C) which shows that μ is an extension of λ.
Now suppose that (2) holds, that means μ is an extension of λ. Hence λ is restriction of μ *and* μ is regular Borel measure. Hence by case (1) λ is a regular content.

Remark: Let C and D be two disjoint sets of X, and then there exist open bounded Baire sets U and V s.t. C ⊂U and D ⊂V.
Proof: Let $x \in C$ then $x \notin D$ we can find disjoint open sets U_x and V_x s.t. .$x \in U_x$, D⊂ V_x . It is clear that $\{V_x / x \in C\}$ is an open cover for C and C is compact, hence there exist a finite sub cover x_1, x_2......... x_n s.t. C⊂ $\bigcup_{i=1}^{n} U_{x_i}$. Let $U^* = \bigcup_{i=1}^{n} U_{x_i}$ *and* $V^* = \bigcap_{i=1}^{n} V_{x_i}$ then U^* *and* V^* are disjoint open sets and C ⊂ U^* and D ⊂ V^*, By Baire sandwich theorem there exist open bounded Baire sets U and V s.t. C ⊂U ⊂ U^* and D ⊂V D ⊂ V^* and obviously U and V are disjoint.

Main Result: Every Baire measure has a unique extension to a regular Borel measure.

Lemma: Let υ be any Baire measure on X. Define for compact set C $\lambda(C) = Inf\{\upsilon(C)/ C \subset U, U\ is\ open\ Baire\ set\}$. Then λ is a regular content and $\lambda(D) = \mu(D)$ for every compact G_δ set D.

Proof of the Lemma: (1) Since $\nu \geq 0$ then obviously $\lambda \geq 0$.
Let C be any compact set, By Baire sandwich theorem we can find an open Baire set U and a compact G_δ set D s.t. C ⊂U ⊂ D. Which gives that $\lambda(C) \leq \nu(U) \leq \nu(D) < \infty$ which shows that is λ a real valued.
(2) Suppose C and D are two compact sets and C⊂ D. Let U be any open Borel set s.t. D ⊂U then C $\subset U \Rightarrow \lambda(C) \leq \nu(U) \Rightarrow \lambda(C) = Inf\{\upsilon(U)/\ U\ is\ open\ Baire\ set\ and\ \text{D} \subset \text{U}\}$.
$\Rightarrow \lambda(C) \leq \lambda(D) \Rightarrow \lambda$ is monotone.
(3) Let C and D be compact sets and U be any open Baire set s.t. C $\subset U$ and V any open Baire set s.t. D $\subset V$ then C∪D ⊂ U∪V $\Rightarrow \lambda(\text{C} \cup \text{D}) \leq \nu(\text{U} \cup \text{V}) \leq \nu(U) + \nu(V)$
$\Rightarrow \lambda(\text{C} \cup \text{D}) \leq \inf\{\nu(U)\} + \inf\{\nu(V)\} \Rightarrow \lambda(\text{C} \cup \text{D}) \leq \lambda(C) + \lambda(D) \Rightarrow \lambda$ is sub additive.
(4) Let C and D be any two disjoint compact sets then by above remark we can find disjoint open bounded Baire sets U and V s.t. C ⊂U and D ⊂V. Ler W be an open Baire set s.t. C∪D ⊂W then C ⊂ U∩ W and D ⊂ V∩ W.
$Since\ W \supset$ (U∩ W) ∪ (V∩ W), we get $\nu(W) \geq \nu[(\text{U}\cap W) \cup (\text{V}\cap W)] = \nu(\text{U}\cap W) + \nu(\text{V}\cap W) \geq \lambda(C) + \lambda(D)$. ⇒Inf $\{\nu(W)\} \geq \lambda(C) + \lambda(D)$ then from (3) we get $\lambda(\text{C} \cup \text{D}) \leq \lambda(C) + \lambda(D)$. Hence we get $\lambda(\text{C} \cup \text{D}) = \lambda(C) + \lambda(D)$. This proves that λ is content.
(5) **Regularity**: Let C be any compact set, $\varepsilon > 0$ then by definition of λ we can find an open Baire set U such that C⊂ $U$and $\nu(U) \leq \lambda(C) + \varepsilon$. By Baire sandwich theorem we can find an open Baire set V and a compact G_δ set D s.t. C⊂ $V \subset$ D ⊂ U. Then C≼ D and $\lambda(D) \leq \nu(U) \leq \lambda(C) + \varepsilon$. $\Rightarrow \lambda(C) = Inf\{\lambda(D)/\text{C} \preccurlyeq D$,D is compact. This proves that λ is regular content.
(6) **Finally:** Let D be any compact G_δ set. Let (U_n) be any sequence of open sets s.t.
D $= \bigcap_{i=1}^{n}(\text{U}_\text{n})$ for each n, D ⊂ U_n, By Baire sandwich theorem we can find an open Baire set V_n and compact G_δ set D_n s.t. D ⊂ V_n ⊂ D_n ⊂ U_n ⇒ D$=\bigcap_{i=1}^{n}(\text{V}_\text{n})$.
$Define$ $\text{W}_\text{n} = \bigcap_{i=1}^{n}(\text{V}_\text{i})$. Then ($\text{W}_\text{n}$) is a monotone decreasing sequence of open bounded Baire sets s.t. $(\text{W}_\text{n}) \rightarrow$ D $\Rightarrow \nu(\text{W}_\text{n})\ ) \rightarrow \nu(\text{D})$. as D⊂ W_n for all n $\Rightarrow \lambda(D) \leq \lim_{n\to\infty} \nu(\text{W}_\text{n}) \Rightarrow \lambda(D) \leq \nu(D)$ ....(*)
$If\ V\ be\ any\ Open\ Baire\ set\ s.t.$ D⊂ Vthen $\nu(D) \leq \nu(V) \Rightarrow \nu(D) \leq \text{Inf}\{\nu(W)\} \Rightarrow \nu(D) \leq \lambda(D)$ ....(**)
$From\ (*) and (**)$ we get $\lambda(D) = \nu(D)$.
Proof of the Main Theorem: Let ν be any Baire measure on X.
Define for compact set C, $\lambda(C) = Inf\{\nu(U)/\text{C} \subset U$, U is Baire set}. Then λ is a regular content. Let μ be the regular Borel measure induced by λ. Then Let μ is an extension of λ.Let $\nu^{/}$ be the Baire restriction of μ . Let D be any compact G_δ set,
then $\nu(D) = \lambda(D) = \nu^{/}(D)$ [By above lemma]
$\Rightarrow \nu(E) = \nu^{/}(E)$ For all Baire set E. $\Rightarrow \nu(E) = \mu(E)$ For every Baire set E.
i.e. μ is an extension of ν. Thus Baire measure ν has been extended to a regular Borel measure μ.

Uniqueness: Let μ_1 and μ_2 be two regular Borel measures such that $\mu_1(D) = \mu_2(\text{D})$ for every compact G_δ set D.
To prove above we have to show that $\mu_1(E) = \mu_2(\text{E})$ for every Borel set E. For this it is suffices to prove that $\mu_1(C) = \mu_2(\text{C})$ for every compact set C.
Let C be any compact set. Since $\mu_1$is regular Borel measure we can find a compact G_δ set D_1 such that C $\subset D_1$ and $\mu_1(C) = \mu_1(D_1)$ ....(1)
By the same argument we can find a compact G_δ set D_2 such that C $\subset D_2$ and $\mu_2(C) = \mu_2(D_2)$ ....(2)
Define D$=D_1 \cap D_2$, then D is a compact G_δ set and C $\subset D_1$and C $\subset D_2 \Rightarrow C \subset D$ shows that $\mu_1(C) = \mu_1(D_1) \leq \mu_1(D) = \mu_2(\text{D}) \leq \mu_2(D_2) = \mu_2(\text{C}) \Rightarrow \mu_1(C)) \leq \mu_2(\text{C})$
By the same argument $\mu_2(C) \leq \mu_1(\text{C})$.
Hence proved that $\mu_1(C) = \mu_2(\text{C})$.

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