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Existence and Uniqueness result for Boundary value problems involving capillarity problems

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[2010]47H05, 47H09 Monotone operator; generalized p-Laplacian operator; non-linear boundary value problem; capillarity problem.

Abstract: *In this paper, we study a nonlinear boundary value problem (bvp) which generalizes capillarity problem. An existence and uniqueness result is obtained using the knowledge of range for nonlinear operator. Ours extends the result in [12].*

I. Introduction

A research on the existence and uniqueness result for certain nonlinear boundary value problems of capillarity problem has a close relationship with practical problems. Some significant work has been done on this, see Wei et al. [**Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**]. In 1995, Wei and He [**Error! Reference source not found.**] used a perturbation result of ranges for m-accretive mappings in Calvert and Gupta [**Error! Reference source not found.**] to obtain a sufficient condition so that the zero boundary value problem,

$$-\nabla_p u+g(x,u(x))=f(x),\text{a.e in } \Omega,$$

EQ – \F(∂u,∂n)=0\,a.e in Γ\,

has solutions in $L^p(\Omega)$, where $2\le p<+\infty$. In 2008, as a summary of the work done in[**Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**], Wei et al used some new technique to work for the following problem with so-called generalized p-Laplacian operator:

$$-\text{div}[(c(x)+|\nabla u|^2)^{(p-2)/2}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x),\text{a.e in } \Omega$$

EQ –〈v\,(c(x)+|∇u|\s\up5(2))\s\up5(\F(,))(p–2)2∇u〉∈β\s\do5(x)(u(x))\, a.e in Γ

where $0\le c(x)\in L^p(\Omega)$, ε is a non-negative constant and v denotes the exterior normal derivatives of Γ. It was shown in [7] that (1.2) has solutions in $L^p(\Omega)$ under some conditions where $2N/(N+1)<p\le s<+\infty$, $1\le q<+\infty$ if $p\ge N$, and $1\le q\le N_p/(N-p)$ if $p<N$, for $N\ge 1$. In Chen lup[8], the authors studied the eigenvalue problem for the following generalized capillarity equations:

$$-\text{div}[(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{(p-2)}\nabla u]=\lambda(|u|^{q-2}u+|u|^{r-2}u), \text{ in } \Omega,$$

EQ u=0\,a.e. on ∂Ω.

In their paper [10], Wei et al, borrowed the main ideas dealing with the nonlinear elliptic boundary value problem with the generalized p-Laplacian operator to study the nonlinear generalized Capillarity equations with Neumann boundary conditions. They used the perturbation results of ranges for m-accretive mappings in [**Error! Reference source not found.**] again to study

$$-\langle v,(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{(p-2)}\nabla u\rangle\in\beta_x(u(x)), \text{ a.e on } \Gamma$$

Motivated by [10,12],we study the following boundary value problem:

$$-\mathrm{div}[(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{(p-2)}\nabla u]+\lambda(|u|^{q_1-2}u+|u|^{q_2-2}u+\cdots|u|^{q_m-2}u)$$

EQ + g(x\,u(x)\,∇u(x))=f(x)\,a.e. inΩ\,

$$-\langle v,(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{(p-2)}\nabla u\rangle\in\beta_x(u(x)),\ \text{a.e in } \Gamma$$

This equation generalized the Capillarity problem considered in [10]. We replaced the nonlinear term $g(x,u(x))$ by the term $g(x,u(x),\nabla u(x))$ which is rather general. In this paper, we will use some perturbation results of the ranges for maximal monotone operators by Pascali and Shurlan [10] to prove that (**Error! Reference source not found.**) has a unique solution in $W^{1,p}(\Omega)$ and later show that this unique solution is the zero of a suitably defined maximal monotone operator.

II. Preliminaries

We now list some basic knowledge we need. Let X be a real Banach space with a strictly convex dual space X^*.Using "↪" and "w-lim" to denote strong and weak convergence respectively. For any subset G of X, let intG denote its interior and $\bar{G}$ its closure. Let " X↪↪Y" denote that space X is embedded compactly in space Y and " X↪Y" denote that space X is embedded continuously in space Y. A mapping, $T:D(T)=X\to X^*$ is said to be hemicontinuous on X if $w-\lim_{t\to 0}T(x+ty)=Tx$, for any $x,y\in X$ Let J denote the duality mapping from X into 2^X, defined by

$$f(x)=f\in x^*:(x,f)=||x||.||f||,||f||=||x||,x\in X$$

where (.,.) denotes the generalized duality pairing between X and X^* Let $A:X\to 2^X$ be a given multi-valued mapping, A is boundedly-inversely compact if for any pair of bounded subsets G and G' of X, the subset $GA^{-1}(G')$ is relatively compact in X.

The mapping $A:X\to 2^X$ is said to be accretive if $((v_1-v_2),J(u_1-u_2))\geq 0$, for any $u_i\in D(A)$ and $v_i\in Au_i$; i=1,2.

The accretive mapping A is said to be m-accretive if $R(I+\mu A)=X$, for some $\mu>0$.

Let $B:X\to 2^{X^*}$ be a given multi-valued mapping, the graph of B, G(B) is defined by $G(B)=\{[u,w]\mid u\in D(B),w\in Bu\}$. $B:X\to 2^{X^*}$ is said to be monotone [11] if G(B) is a monotone subset of $X\times X^*$ in the sense that

$$(u_1-u_2,w_1-w_2)\geq 0,\text{for any } [u_i,w_i]\in G(B);i=1,2.$$

The monotone operator B is said to be maximal monotone if G(B) is maximal among all monotone subsets of $X\times X^*$ in the sense of inclusion the mapping B is said to be strictly monotone if the equality in (**Error! Reference source not found.**) implies that $u_1=u_2$. The mapping B is said to be coercive if $\lim_{n\to+\infty}((x_n,x_n^*)/\|x_n\|)=\infty$ for all $[x_n,x_n^*]\in G(B)$ such that $\lim_{n\to+\infty}\|x_n\|=+\infty$.

Definition 1 The duality mapping $J:X\to 2^{X^*}$ is said to be satisfying condition (I) if there exists a function $\eta:X\to[0,+\infty)$ such that

$$\|Ju-J\upsilon\|\leq\eta(u-\upsilon),\ \text{for all } u,\upsilon\in X.$$

Definition 2 Let $A:X\to 2^X$ be an accretive mapping and $J:X\to X^*$ be a duality mapping. We say that A satisfies condition (*) if, for any $f\in R(A)$ and $a\in D(A)$ and $a\in D(A)$, there exists a constant C(a,F) such that

$$(\upsilon-f,J(u-a))\geq C(a,f),\ \text{for any } u\in D(A),\ \upsilon\in Au.$$

Lemma 3 (Li and Guo) Let Ω be a bounded conical domain in R^N. Then we have the following results;

1. If $mp>N$ then $W^{m,p}(\Omega)\hookrightarrow C_B(\Omega)$; if $mp<N$ and $q=Np/(N-mp)$, then $W^{m,p}(\Omega)\hookrightarrow L^q(\Omega)$; if $mp=N$, and $p>1$, then for $1\le q<+\infty, W^{m,p}(\Omega)\to L^q(\Omega)$
2. If $mp>N$ then $W^{m,p}(\Omega)\hookrightarrow\hookrightarrow C_B(\Omega)$; if $0<mp\le N$ and $q_0=Np/(N-mp)$, then $W^{m,p}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega), 1\le q<q_0$;

Lemma 4 (Pascali and Sburlan[11]) If $B:X\to 2^{X^*}$ is an everywhere defined, monotone and hemicontinuous operator, then B is maximal monotone.

Lemma 5 (Pascali and Sburlan[11]). If $B:X\to 2^{X^*}$ is maximal monotone and coercive, then $R(B)=X^*$
Lemma 6 (Pascali and Sburlan[11]). If $\Phi:X\to(-\infty,+\infty]$ is a proper, convex and lower semicontinuous function, then $\partial\Phi$ is maximal monotone from X to X^*.
Lemma 7 [**Error! Reference source not found.**]. If B_1 and B_2 are two maximal monotone operators in X such that $(\operatorname{int}D(B_1))D(B_2)\neq\emptyset$, then B_1+B_2 is maximal monotone.

Lemma 8 (Calvert and Gupta[1]). Let $X=L^p(\Omega)$ and Ω be a bounded domain in $\Re^N$. For $2\le p<+\infty$, the duality mapping $J_p:L^p(\Omega)\to L^{p'}(\Omega)$ defined by $J_p u=|u|^{p-1}\operatorname{sgn} u\|u\|_p^{2-p}$, for $u\in L^P(\Omega)$, satisfies condition (2); for $2N/(N+1)<p\le 2$ and $N\ge 1$, the duality mapping $J_p:L^p(\Omega)\to L^{p'}(\Omega)$ defined by $J_p u=|u|^{p-1}\operatorname{sgn} u$, for $u\in L^P(\Omega)$, satisfies condition (2), where $(1/p)+(1/p')=1$

III. Main Result

3.1 Notations and Assumptions of (Error! Reference source not found.**)**
We assume, in this paper, that $2N/(N+1)<p<+\infty, 1\le q_1,q_2,\cdots,q_m<+\infty$ if $p\ge N$, and $1\le q_1,q_2,\cdots,q_m\le Np/(N-p)$ if $p<N$, where $N\ge 1$. We use $\|\cdot\|_p, \|\cdot\|_{q_1}, \|\cdot\|_{q_2}, \cdots, \|\cdot\|_{q_m}$, and $\|\cdot\|_{1,p,\Omega}$ to denote the norms in $L^p(\Omega), L^{q_1}(\Omega), L^{q_2}(\Omega), \cdots, L^{q_m}(\Omega)$ and $W^{1,p}(\Omega)$ respectively. Let $(1/p)+(1/p')=1$, $(1/q_1)+(1/q_1')=1$, $(1/q_2)+(1/q_2')=1,\cdots$, $(1/q_m)+(1/q_m')=1$

In (**Error! Reference source not found.**), Ω is a bounded conical domain of a Euclidean space $\Re^N$ with its boundary $\Gamma\in C^1$,(c.f.[4]).
Let $|.|$ denote the Euclidean norm in $\Re^N$, $\langle .,. \rangle$ the Euclidean inner-product and υ the exterior normal derivative of Γ. λ is a nonnegative constant.
Lemma 1 Define the mapping $B_{p,q_1,q_2,\cdots,q_m}:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ by

$$(v,B_{p,q_1,q_2,\cdots,q_m}u) = \int_\Omega \langle (1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u,\nabla v\rangle dx + \lambda\int_\Omega |u(x)|^{q_1-2}u(x)v(x)dx+\lambda\int_\Omega |u(x)|^{q_2-2}u(x)v(x)dx + \cdots+\lambda\int_\Omega |u(x)|^{q_m-2}u(x)v(x)dx$$

for any $u,v\in W^{1,p}(\Omega)$ .Then $B_{p,q_1,q_2,\cdots,q_m}$ is everywhere defined, strictly monotone, hemicontinuous and coercive.
The proof of the above lemma will be done in four steps:
[Sorry. Ignored \begin{proof} ... \end{proof}]

Definition 2 Define a mapping $A_p:L^p(\Omega)\to 2^{L^p(\Omega)}$ as follows:

$D(A_p)=\{u \in L^p(\Omega) \mid$ there exist an $f \in L^p(\Omega)$, such that $f \in B_{p,q_1,q_2,\cdots,q_m}u+\partial\Phi_p(u)\}$

EQ for u ∈ D(A\s\do5(p))\, let A\s\do5(p)u={f ∈ L\s\up5(p)(Ω)\, such that f ∈ B\s\do5(p\,q\s\do4(1)\,q\s\do4(2)\,⋯\,q\s\do4(m))u+∂Φ\s\do5(p)(u)}

Definition 3 : The mapping

$A_p:L^p(\Omega)\to 2^{L^p(\Omega)}$ is m–accretive.

[Sorry. Ignored \begin{proof} ... \end{proof}]

Then χ_n is monotone, Lipschitz with $\chi_n(0)=0$ and χ'_n is continuous except at finitely many points on R. so $(\chi_n(u_1-u_2),\partial\Phi_p(u_1)-\partial\Phi_p(u_2))\geq 0$.

Then, for $u_i\in D(A_p)$ and $v_i\in A_p u_i, i=1,2,$ we have

$$(v_1-v_2,J_p(u_1-u_2)) = (|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2),B_{p,q_1,q_2,\cdots,q_m}u_1-B_{p,q_1,q_2,\cdots,q_m}u_2) + (|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2),\partial\Phi_p(u_1)-\partial\Phi_p(u_2))$$
$$= (|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2),B_{p,q_1,q_2,\cdots,q_m}u_1-B_{p,q_1,q_2,\cdots,q_m}u_2) + \lim_{n\to\infty}(\chi_n(u_1-u_2),\partial\Phi_p(u_1)-\partial\Phi_p(u_2))\geq o$$

Step 2 $R(I+\mu A_p)=L^P(\Omega)$, for every $\mu>0$.

We first define the mapping $I_p:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ by $I_p u=u$ and $(v,I_p u)(W^{1,p}(\Omega))^*\times W^{1,p}(\Omega)-(v,u)v(\Omega)$ for $u,v\in W^{1,p}(\Omega)$, where $\langle .,. \rangle_{L^2(\Omega)}$ denotes the inner product of $L^p(\Omega)$. Then I_p is maximal monotone [7].

Secondly, for any $\mu>0$, let the mapping $T_\mu:W^{1,p}(\Omega)\to 2^{(W^{1,p}(\Omega))^*}$ be defined by

$$T\mu u=I_p u+\mu B_{p,q_1,q_2,\cdots,q_m}u\mu\partial\Phi_p(u)$$

,for $u\in W^{1,p}(\Omega)$. Then similar to that in [7], by lemmas 2.4, 2.6, 2.7 and 2.5 we see that T_μ is maximal monotone and coercive, so that $R(T\mu)=(W^{1,p}(\Omega))^*$, for any $\mu>0$

Therefore, for any $f\in L^P(\Omega)$, there exists $u\in W^{1,p}(\Omega)$, such that

$$f=T_\mu u=u+\mu B_{p,q_1,q_2,\cdots,q_m}u\mu\partial\Phi_p(u) \tag{2}$$

From the definition of A_p, it follows that $R(I+\mu A_p)=L^p(\Omega)$, for all $\mu>0$. This completes the proof.

Lemma 4 The mapping $A_p:L^p(\Omega)\to 2L^p(\Omega)$, has a compact resolvent for $2N/(N+1)<p<2$ and $N\geq 1$.

[Sorry. Ignored \begin{proof} ... \end{proof}]

Remark 5 Since $\Phi_p(u+\alpha)=\Phi_p(u)$, for any $u\in W^{1,p}(\Omega)$ and $\alpha\in C_0^\infty(\Omega)$, we have $f\in A_p u$ implies that $f=B_{p,q_1,q_2,\cdots,q_m}$ in the sense of distributions.

Proposition 6 For $f\in L^p(\Omega)$,if there exists $u\in L^p(\Omega)$ such that $f\in A_p u$, then u is the unique solution of (1.7).

[Sorry. Ignored \begin{proof} ... \end{proof}]

Remark 7 If $\beta_x\equiv 0$ for any $x\in\Gamma$ then $\partial\Phi_p(u)\equiv 0$, for all $u\in W^{1,p}(\Omega)$.

Proposition 8 If $\beta_x \equiv 0$ for any $x \in \Gamma$ then $\{f \in L^P(\Omega) | \int_\Omega f dx = 0\} \subset R(A_p)$.

[Sorry. Ignored \begin{proof} ... \end{proof}]

Definition 9 (see[1,7]). For $t \in R_t, x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq 0$ and $\beta_x^0(t) = \pm\infty$, where $t>0$ or $t<0$ respectively, in case $\beta_x(t) = \varnothing$. Finally, let $\beta_x(t) = \lim_{t \to \pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$.$\beta_x(t)$ define measurable functions on Γ, in view of our assumptions on β_x.

Proposition 10 Let $f \in L^p(\Omega)$ such that

$$\int_\Gamma \beta\_(x) d\Gamma(x) < \int_\Omega f dx < \int_\Gamma \beta_+(x) d\Gamma(x)$$

Then $f \in \mathrm{Int}\, R(A_p)$.

[Sorry. Ignored \begin{proof} ... \end{proof}]

This completes the proof.

Proposition 11 $A_p + B_1 : L^p(\Omega) \to L^p$(Omega) is m-accretive and has a compact resolvent.

[Sorry. Ignored \begin{proof} ... \end{proof}]

Theorem: Let $f \in L^p(\Omega)$ be such that

$$\int_\Gamma \beta\_(x) d\Gamma(x) + \int_\Omega g\_(x) dx < \int_\Omega f(x) dx \quad < \quad \int_\Gamma \beta_+(x) d\Gamma(x) + \int_\Omega g_+(x) dx$$

then(1.4) has a unique solution in $L^p(\Omega)$, where $2N/(N+1) < p < +\infty$ and $N \geq 1$

[Sorry. Ignored \begin{proof} ... \end{proof}]

Remark: Compared to the work done in [1-7], not only the existence of the solution of (1.4) is obtained but also the uniqueness of the solution is obtained. Furthermore, our work extended the work of [12]

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