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Harmonic Analysis Associated With The Dunkl-Bessel-Struve Transform

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Abstract:*In this paper we consider the Dunkl-Struve Laplace operator $\Delta_{k,\alpha}$ on $\mathbb{R}^{d+1}$, we define the Dunkl-Bessel-Struve intertwining operator $\mathcal{X}_{k,\alpha}$ which turn out to be transmutation operator between $\Delta_{k,\alpha}$ and the Laplacian Δ_{d+1}, next we construct ${}^t\mathcal{X}_{k,\alpha}$ the dual of the Dunkl-Bessel-Struve intertwining operator. We exploit these operators to develop a new harmonic analysis corresponding to $\Delta_{k,\alpha}$.*

1. Introduction

In this paper we consider the Dunkl-Bessel-Struve Laplace operator defined by

(1)
$$\Delta_{k,\alpha} = \Delta_{k,x'} + l_{\alpha,x_{d+1}}, \quad x' \in \mathbb{R}^d, \ x_{d+1} \in \mathbb{R},$$

where Δ_k is the Dunkl-Laplacian operator on $\mathbb{R}^d$ (see[1]), l_α is the Bessel-Struve operator on $\mathbb{R}$ given by

(2)
$$l_\alpha u(x) = \frac{d^2}{dx_{d+1}^2}u(x', x_{d+1}) + \frac{2\alpha+1}{x_{d+1}}\left(\frac{du(x', x_{d+1})}{dx_{d+1}} - \frac{du(x', 0)}{dx_{d+1}}\right), \quad \alpha > \frac{-1}{2}.$$

Through this paper, we provide a new harmonic analysis on $\mathbb{R}^{d+1}$ corresponding to the Dunkl-Struve Laplace operator $\Delta_{k,\alpha}$.
The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Dunkl transform and the Bessel-Struve transform.
In section 3, we construct the Dunkl-Bessel-Struve intertwining operator $\mathcal{X}_{k,\alpha}$ and its dual ${}^t\mathcal{X}_{k,\alpha}$, next we exploit these operators to build a new harmonic analysis on $\mathbb{R}^{d+1}$ corresponding to operator $\Delta_{k,\alpha}$.

2. Preliminaries

Throughout this paper, we denote by

- $a_\alpha = \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$ where $\alpha > \dfrac{-1}{2}$.
- $x = (x_1, ..., x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}^{d+1}$.
- $\lambda = (\lambda_1, ..., \lambda_{d+1}) = (\lambda', \lambda_{d+1}) \in \mathbb{C}^{d+1}$.
- $E(\mathbb{R}^{d+1})$ (resp. $D(\mathbb{R}^{d+1})$) the space of C^∞ functions on $\mathbb{R}^{d+1}$, even with respect to the last variable (resp. with compact support).
- $\mathcal{R}$ the root system in $\mathbb{R}^d\backslash\{0\}$, $\mathcal{R}_+$ is a fixed positive subsystem and $k \in \mathcal{R} \to]0, \infty[$ a multiplicity function.

- T_j the Dunkl operator defined for $j = 1, ..., d$, on $\mathbb{R}^d$ and $f \in E(\mathbb{R}^d)$ by
$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j \frac{(f(x) - f(\sigma_\alpha(x))}{< \alpha, x >}.$$
where $<,>$ is the usual scalar product, σ_α is the orthogonal reflection in the hyperplane orthogonal to α and the multiplicity function k is invariant by the finite reflection group W generated by the reflection σ_α $(\alpha \in \mathcal{R})$.
- Δ_k the Dunkl-Laplace operator defined by
$$\Delta_k f(x) = \sum_{j=0}^{d} T_j^2 f(x).$$
- w_k the weight function defined by
$$w_k(x') = \prod_{\alpha \in \mathcal{R}_+} | < \alpha, x' > |^{2k(\alpha)}, \; x' \in \mathbb{R}^d.$$
- $L_k^p(\mathbb{R}^d), \;\; 1 \le p \le +\infty$ the space of measurable functions on $\mathbb{R}^d$ such that

(3) $$\|f\|_{k,p} = \left(\int_{\mathbb{R}^d} |f(x')|^p w_k(x') dx' \right)^{\frac{1}{p}} < +\infty, \;\; if \; 1 \le p < +\infty,$$

(4) $$\|f\|_{k,\infty} = ess \sup_{x' \in \mathbb{R}^d} |f(x')| < +\infty, \;\; if \; p = \infty.$$

- $L_\alpha^p(\mathbb{R}), \;\; 1 \le p \le +\infty$ the space of measurable functions on $\mathbb{R}$ such that

(5) $$\|f\|_{\alpha,p} = \left(\int_{\mathbb{R}} |f(t)|^p |t|^{2\alpha+1} dt \right)^{\frac{1}{p}} < +\infty, \;\; if \; 1 \le p < +\infty,$$

(6) $$\|f\|_{\alpha,\infty} = ess \sup_{t \in \mathbb{R}} |f(t)| < +\infty, \;\; if \; p = \infty.$$

In this section we recall some facts about harmonic analysis related to the Dunkl-Laplace operator Δ_k and harmonic analysis associated with the generalized Bessel-Struve operator $l_{\alpha,n}$. We cite here, as briefly as possible, only some properties. For more details we refer to [1, 2, 3].

Definition 1. *For all $x \in \mathbb{R}^{d+1}$ we define the measure $\xi_x^{k,\alpha}$ on $\mathbb{R}^{d+1}$ by*

(7) $$d\xi_x^{k,\alpha}(y) = a_\alpha x_{d+1}^{-2\alpha} (x_{d+1}^2 - y_{d+1}^2)^{\alpha - \frac{1}{2}} 1_{]0,x_{d+1}[}(y_{d+1}) d\mu_{x'}(y') dy_{d+1},$$

where $\mu_{x'}$ is a probability measure on $\mathbb{R}^d$, with support in the closed ball $B(o, \|x\|)$ of center o and radius $\|x\|$. $1_{]0,x_{d+1}[}$ is the characteristic function of the interval $]0, x_{d+1}[$.

The Dunkl intertwining operator $\mathcal{V}_k$ is defined on $C(\mathbb{R}^d)$ by

(8) $$\forall x \in \mathbb{R}^d \quad \mathcal{V}_k f(x') = \int_{\mathbb{R}^d} f(x') d\mu_{x'}(y').$$

The dual of the Dunkl intertwining is defined on $D(\mathbb{R}^d)$ by

(9) $$^t\mathcal{V}_k(f)(y') = \int_{\mathbb{R}^d} f(x') d\nu_{y'}(x'),$$

where $\nu_{y'}$ is a positive measure on $\mathbb{R}^d$ with support in the set $\left\{x' \in \mathbb{R}^d,\ \|x'\| \geq \|y'\|\right\}$.
$\mathcal{V}_k$ and ${}^t\mathcal{V}_k$ are related with the following formula

$$\int_{\mathbb{R}} \mathcal{V}_k(f)(x')g(x)w_k(x')dx = \int_{\mathbb{R}} f(y)^t\mathcal{V}_k g(y')dy'. \tag{10}$$

For each $y' \in \mathbb{R}^d$, the system

The Dunkl kernel defined by

$$\begin{cases} T_j u(x', y') = y'_j u(x', y'),\ j = 1, ...d, \\ u(0, y') = 1 \end{cases}$$

admits a unique analytic solution $K(x', y'),\ x' \in \mathbb{R}^d$, called the Dunkl kernel. This kernel possesses the following properties

$$K(x', -i\lambda') = \int_{\mathbb{R}^d} e^{-i<y',\lambda'>} d\mu_{x'}(y'). \tag{11}$$

This kernel possesses the following properties

$$K(x', \lambda') = \mathcal{V}_k(e^{<.,\lambda'>})(x). \tag{12}$$

Proposition 1. i) *$\mathcal{V}_k$ is a topological isomorphism from $E(\mathbb{R}^d)$ onto itself satisfying the following transmutation relation*

$$\Delta_k(\mathcal{V}_k f) = \mathcal{V}_k(\Delta_d f),\ \forall f \in E(\mathbb{R}^d),$$

where $\Delta_d = \sum_{j=1}^{d} \frac{d^2}{dx_j^2}$ *is the Laplacian on* $\mathbb{R}^d$.

ii) ${}^t\mathcal{V}_k$ *is a topological isomorphism from* $D(\mathbb{R}^d)$ *onto itself.*

Proposition 2. *The Dunkl-Laplace operator Δ_k and the function K are related by the following relation*

$$\Delta_k(K(x', \lambda')) = -\|\lambda'\|^2 K(x', \lambda'). \tag{13}$$

Definition 2. *The Dunkl transform of a function f in $D(\mathbb{R}^d)$ is given by*

$$\forall y' \in \mathbb{R}^d,\ \mathcal{F}_k(f)(y') = \int_{\mathbb{R}^d} f(x')K(x', -iy')w_k(x')dx'.$$

Proposition 3. i) *If $f \in L^1_k(\mathbb{R})$ then $\|\mathcal{F}^k(f)\|_{k,\infty} \leq \|f\|_{k,1}$.*

ii) *For all $f \in D(\mathbb{R}^d)$ we have*

$$\mathcal{F}_k(f) = \mathcal{F} \circ^t V_k(f),$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathbb{R}^d$ defined by

$$\mathcal{F}(f)(\lambda') = \int_{\mathbb{R}^d} f(x')e^{-i<x',\lambda'>}dx'.$$

In the next we recall some facts about harmonic analysis associated with the Bessel-Struve operator l_α.

For $\lambda_{d+1} \in \mathbb{C}$, the differential equation:

$$\begin{cases} l_\alpha u(z) = \lambda_{d+1}^2 u(z) \\ u(0) = 1,\ u'(0) = \frac{\lambda_{d+1}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})} \end{cases} \tag{14}$$

possesses a unique solution denoted $\Phi_\alpha(\lambda_{d+1}z)$. This eigenfunction, called the Bessel-Struve kernel, is given by:

$$\Phi_\alpha(\lambda_{d+1}z) = j_\alpha(i\lambda_{d+1}z) - ih_\alpha(i\lambda_{d+1}z), \quad z \in \mathbb{R}.$$

j_α and h_α are respectively the normalized Bessel and Struve functions of index α .These kernels are given as follows

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{m=0}^{+\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m}}{m!\Gamma(m+\alpha+1)}$$

and

$$h_\alpha(z) = \Gamma(\alpha+1)\sum_{m=0}^{+\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\alpha+\frac{3}{2}\right)}.$$

The kernel Φ_α possesses the following integral representation:

$$\Phi_\alpha(\lambda_{d+1}z) = a_\alpha\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}e^{\lambda_{d+1}zt}dt, \quad \forall z\in\mathbb{R}, \quad \forall\lambda_{d+1}\in\mathbb{C}. \tag{15}$$

The Bessel-Struve intertwining operator on $\mathbb{R}$ denoted $\mathcal{X}_\alpha$ introduced by L. Kamoun and M. Sifi in [3], is defined by:

$$\mathcal{X}_\alpha(f)(z) = a_\alpha\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}f(zt)dt\ , f\in E(\mathbb{R}),\ z\in\mathbb{R}. \tag{16}$$

By change of variable $\mathcal{X}_\alpha$ can be also written in the form

$$\mathcal{X}_\alpha(f)(z) = a_\alpha z^{-2\alpha}\int_0^z(z^2-t^2)^{\alpha-\frac{1}{2}}f(t)dt\ , f\in E(\mathbb{R}),\ z\in\mathbb{R}. \tag{17}$$

The Bessel-Struve kernel Φ_α is related to the exponential function by

$$\forall z\in\mathbb{R}, \quad \forall\lambda_{d+1}\in\mathbb{C}, \quad \Phi_\alpha(\lambda_{d+1}z) = \mathcal{X}_\alpha(e^{\lambda_{d+1}\cdot})(z). \tag{18}$$

$\mathcal{X}_\alpha$ is a transmutation operator from l_α into $\frac{d^2}{dz^2}$ and verifies

$$l_\alpha\circ\mathcal{X}_\alpha = \mathcal{X}_\alpha\circ\frac{d^2}{dz^2}. \tag{19}$$

Theorem 1. *The operator $\mathcal{X}_\alpha$, $\alpha > \frac{-1}{2}$ is topological isomorphism from $E(\mathbb{R})$ onto itself. The inverse operator $\mathcal{X}_\alpha^{-1}$ is given for all $f\in E(\mathbb{R})$ by*

(i) *if $\alpha = r+m$, $m\in\mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$*

$$\mathcal{X}_\alpha^{-1}(f)(z) = \frac{2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)}z(\frac{d}{dz^2})^{m+1}\left[\int_0^z(z^2-t^2)^{-r-\frac{1}{2}}f(t)|t|^{2\alpha+1}dt\right]. \tag{20}$$

(ii) *if $\alpha = \frac{1}{2}+m$, $m\in\mathbb{N}$*

$$\mathcal{X}_\alpha^{-1}(f)(z) = \frac{2^{2m+1}m!}{(2m+1)!}z(\frac{d}{dz^2})^{m+1}(z^{2m+1}f(z)),\ z\in\mathbb{R}. \tag{21}$$

Definition 3. *The Bessel-Struve transform is defined on $L^1_\alpha(\mathbb{R})$ by*

$$\forall\lambda_{d+1}\in\mathbb{R}, \quad \mathcal{F}^\alpha_{B,S}(f)(\lambda_{d+1}) = \int_{\mathbb{R}}f(z)\Phi_\alpha(-i\lambda_{d+1}z)|z|^{2\alpha+1}dz. \tag{22}$$

Proposition 4. *If $f\in L^1_\alpha(\mathbb{R})$ then $\|\mathcal{F}^\alpha_{B,S}(f)\|_\infty \leq \|f\|_{1,\alpha}$.*

Definition 4. *For $f\in L^1_\alpha(\mathbb{R})$ with bounded support, the integral transform W_α, given by*

$$W_\alpha(f)(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{|z|}^{+\infty}(t^2-z^2)^{\alpha-\frac{1}{2}}tf(sgn(z)t)dt, \quad z\in\mathbb{R}\backslash\{0\} \tag{23}$$

is called Weyl integral transform associated with Bessel-Struve operator.

Remark 1. *If we denote $dw_z(t) = a_\alpha 1_{]|z|,+\infty[}(t)\left(t^2-z^2\right)^{\alpha-\frac{1}{2}}tdt$ we can write*

$$W_\alpha f(z) = \int_{\mathbb{R}}f(sgn(z)t)\,dw_z(t).$$

Proposition 5. (i) *W_α is a bounded operator from $L^1_\alpha(\mathbb{R})$ to $L^1(\mathbb{R})$, where $L^1(\mathbb{R})$ is the space of lebesgue-integrable functions.*

(ii) *Let f be a function in $E(\mathbb{R})$ and g a function in $L_\alpha(\mathbb{R})$ with bounded support, the operators $\mathcal{X}_\alpha$ and W_α are related by the following relation*

$$\int_{\mathbb{R}} \mathcal{X}_\alpha(f)(z)g(z)|z|^{2\alpha+1}dz = \int_{\mathbb{R}} f(z)W_\alpha(g)(z)dz. \tag{24}$$

(iii) *$\forall f \in L^1_\alpha(\mathbb{R})$, $\mathcal{F}^\alpha_{B,S} = \mathcal{F} \circ W_\alpha(f)$ where $\mathcal{F}$ is the classical Fourier transform defined on $L^1(\mathbb{R})$ by*

$$\mathcal{F}(g)(\lambda_{d+1}) = \int_{\mathbb{R}} g(z)e^{-i\lambda_{d+1}z}dz.$$

We designate by $\mathcal{K}_0$ the space of functions f infinitely differentiable on $\mathbb{R}^*$ with bounded support verifying for all $n \in \mathbb{N}$,

$$\lim_{y\to 0^-} t^n f^{(n)}(t) \quad and \quad \lim_{t\to 0^+} t^n f^{(n)}(t)$$

exist.

Definition 5. *We define the operator V_α on $\mathcal{K}_0$ as follows*

- *If $\alpha = m + \frac{1}{2}$, $m \in \mathbb{N}$*

$$V_\alpha(f)(z) = (-1)^{m+1}\frac{2^{2m+1}m!}{(2m+1)!}(\frac{d}{dz^2})^{m+1}(f(z)), \quad x \in \mathbb{R}^*.$$

- *If $\alpha = m + r$, $m \in \mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$ and $f \in \mathcal{K}_0$*

$$V_\alpha(f)(z) = \frac{(-1)^{m+1}2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)}\left[\int_{|z|}^{\infty}(t^2-z^2)^{-r-\frac{1}{2}}(\frac{d}{dt^2})^{m+1}f(sgn(z)t)tdt\right], \ z \in \mathbb{R}^*.$$

Proposition 6. *Let $f \in \mathcal{K}_0$ and $g \in E(\mathbb{R})$, the operators V_α and $\mathcal{X}^{-1}_\alpha$ are related by the following relation*

$$\int_{\mathbb{R}} V_\alpha(f)(z)g(z)|z|^{2\alpha+1}dz = \int_{\mathbb{R}} f(z)\mathcal{X}^{-1}_\alpha(g)(z)dz. \tag{25}$$

3. DUNKL-BESSEL-STRUVE TRANSFORM

Definition 6. *The Dunkl-Bessel-Struve intertwining operator is the operator $\mathcal{X}_{k,\alpha}$ defined on $C(\mathbb{R}^{d+1})$ by*

$$\mathcal{X}_{k,\alpha}f(x',x_{d+1}) = a_\alpha x_{d+1}^{-2\alpha}\int_0^{x_{d+1}}(x^2_{d+1}-t^2)^{\alpha-\frac{1}{2}}\mathcal{V}_k f(x',t)dt, \tag{26}$$

Remark 2. *$\mathcal{X}_{k,\alpha}$ can also be written in the form*

- $$\mathcal{X}_{k,\alpha} = \mathcal{V}_k \otimes \mathcal{X}_\alpha \tag{27}$$

  where $\mathcal{V}_k$ is the Dunkl intertwining operator given by (8) and $\mathcal{X}_\alpha$ is the Bessel-Struve intertwining operator given by (17).

- $$\mathcal{X}_{k,\alpha}f(x) = \int_{\mathbb{R}^{d+1}} f(y)d\xi^{k,\alpha}_x(y).$$

  where $d\xi^{k,\alpha}_x$ is given by (7).

Proposition 7. *$\mathcal{X}_{k,\alpha}$ is a topological isomorphism from $E(\mathbb{R}^{d+1})$ onto itself satisfying the following transmutation relation*

$$\Delta_{k,\alpha}(\mathcal{X}_{k,\alpha}f) = \mathcal{X}_{k,\alpha}(\Delta_{d+1}f), \ \forall f \in E(\mathbb{R}^{d+1}),$$

where $\Delta_{d+1} = \sum_{j=1}^{d+1} \frac{d^2}{dx_j^2}$ *is the Laplacian on* $\mathbb{R}^{d+1}$.

Proof. The result follows directly from (1), (19), (27) and Proposition 1

$$\begin{aligned}\Delta_{k,\alpha}(\mathcal{X}_{k,\alpha}f) &= (\Delta_k + l_\alpha)(\mathcal{V}_k \otimes \mathcal{X}_\alpha)(f) \\ &= \Delta_k(\mathcal{V}_k \otimes \mathcal{X}_\alpha)(f) + l_\alpha(\mathcal{V}_k \otimes \mathcal{X}_\alpha)(f) \\ &= \Delta_k(\mathcal{V}_k)\mathcal{X}_\alpha(f) + \mathcal{V}_k l_\alpha(\mathcal{X}_\alpha)(f) \\ &= \mathcal{V}_k(\Delta_d \mathcal{X}_\alpha)f + \mathcal{V}_k(\mathcal{X}_\alpha \circ \frac{d^2}{dx_{d+1}^2} f) \\ &= \mathcal{V}_k \otimes \mathcal{X}_\alpha(\Delta_d + \frac{d^2}{dx_{d+1}^2})f \\ &= \mathcal{X}_{k,\alpha}(\Delta_{d+1}f).\end{aligned}$$

■

Definition 7. *The dual of the Dunkl-Bessel-Struve intertwining operator* $\mathcal{X}_{k,\alpha}$ *is the operator* ${}^t\mathcal{X}_{k,\alpha}$ *defined on* $D(\mathbb{R}^{d+1})$ *by:* $\forall y = (y', y_{d+1}) \in \mathbb{R}^{d+1}$,

$$(28)\qquad {}^t\mathcal{X}_{k,\alpha}(f)(y', y_{d+1}) = a_\alpha \int_{|y_{d+1}|}^{\infty} (s^2 - y_{d+1}^2)^{\alpha-\frac{1}{2}}\ {}^t\mathcal{V}_k f(y', sgn(y_{d+1})s)sds,$$

where ${}^t\mathcal{V}_k$ *is the dual Dunkl intertwining operator given by (9).*

Remark 3. *The operator* ${}^t\mathcal{X}_{k,\alpha}$ *can also be write in the form*

$$(29)\qquad {}^t\mathcal{X}_{k,\alpha} = {}^t\mathcal{V}_k \otimes W_\alpha,$$

where ${}^t\mathcal{V}_k$ *is the dual Dunkl intertwining operator given by (9) and* W_α *is the Weyl integral given by (23).*

For all $y \in \mathbb{R}^d$, we define the measure $\varrho_y^{k,\alpha}$ on $\mathbb{R}^{d+1}$, by

$$(30)\qquad d\varrho_y^{k,\alpha}(x) = a_\alpha(x_{d+1}^2 - y_{d+1}^2)^{\alpha-\frac{1}{2}} x_{d+1} 1_{]|y_{d+1}|,+\infty[}(x_{d+1})d\nu_{y'}(x')dx_{d+1}.$$

From (9), (29) and Remark 1 the operator ${}^t\mathcal{X}_{k,\alpha}$ can also be written in the form

$$(31)\qquad {}^t\mathcal{X}_{k,\alpha}(f)(y) = \int_{\mathbb{R}^{d+1}} f(x', sgn(y_{d+1})x_{d+1})d\varrho_y^{k,\alpha}(x).$$

We consider the function $\Lambda_{k,\alpha}$, given for $\lambda = (\lambda', \lambda_{d+1}) \in \mathbb{C}^d \times \mathbb{C}$ by

$$(32)\qquad \Lambda_{k,\alpha}(x, \lambda) = K(x', -i\lambda')\Phi_\alpha(x_{d+1}\lambda_{d+1}).$$

Proposition 8. *The Dunkl-Struve-Laplace operator* $\Delta_{k,\alpha}$ *and the function* $\Lambda_{k,\alpha}$ *are related by the following relation*

$$(33)\qquad \Delta_{k,\alpha}(\Lambda_{k,\alpha})(x, \lambda) = -\|\lambda\|^2 \Lambda_{k,\alpha}(x, \lambda).$$

Proof. By (1), (13), (14) and (32) we obtain

$$\begin{aligned}\Delta_{k,\alpha}(\Lambda_{k,\alpha})(x, \lambda) &= (\Delta_{k,x'} + l_{\alpha,x_{d+1}})(K(x', -i\lambda')\Phi_\alpha(-i\lambda_{d+1}x_{d+1}) \\ &= \Delta_{k,x'}(K(x', -i\lambda'))\Phi_\alpha(-i\lambda_{d+1}x_{d+1}) + K(x', -i\lambda')l_{\alpha,x_{d+1}}(\Phi_\alpha(-i\lambda_{d+1}x_{d+1})) \\ &= -\|\lambda'\|^2 K(x', -i\lambda')\Phi_\alpha(-i\lambda_{d+1}x_{d+1}) + (-i\lambda_{d+1})^2 K(x', -i\lambda')\Phi_\alpha(-i\lambda_{d+1}x_{d+1}) \\ &= -\|\lambda'\|^2 K(x', -i\lambda')\Phi_\alpha(-i\lambda_{d+1}x_{d+1}) - \lambda_{d+1}^2 K(x', -i\lambda')\Phi_\alpha(-i\lambda_{d+1}x_{d+1}) \\ &= -\|\lambda\|^2 \Lambda_{k,\alpha}(x, \lambda).\end{aligned}$$

■

Proposition 9. *The Dunkl-Bessel-Struve kernel $\Lambda_{k,\alpha}$ is related to the exponential function by*

$$\forall x \in \mathbb{R}^{d+1},\ \forall \lambda \in \mathbb{C}^{d+1},\ \Lambda_{k,\alpha}(x,\lambda) = \mathcal{X}_{k,\alpha}(e^{<.,\lambda>})(x).$$

Proof. The result follows directly from (12), (18) and (27). ∎

We denote by $L^p_{k,\alpha}(\mathbb{R}^{d+1})$, $1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R}^{d+1}$ such that

(34) $$\|f\|_{k,\alpha,p} = (\int_{\mathbb{R}^{d+1}} |f(x)|^p \mathcal{A}_{k,\alpha}(x)dx)^{\frac{1}{p}} < +\infty, \quad if\ 1 \leq p < +\infty,$$

(35) $$\|f\|_{k,\alpha,\infty} = ess \sup_{\mathbb{R}^d \times [0,+\infty[} |f(x)| < +\infty, \quad if\ p = \infty$$

where

(36) $$\mathcal{A}_{k,\alpha}(x)dx = w_k(x')|x_{d+1}|^{2\alpha+1}dx'dx_{d+1}, \quad x = (x', x_{d+1}) \in \mathbb{R}^{d+1}.$$

Theorem 2. *Let $f \in L^1_{k,\alpha}(\mathbb{R}^{d+1})$ and g in $C(\mathbb{R}^{d+1})$, we have the formula*

(37) $$\int_{\mathbb{R}^{d+1}} {}^t\mathcal{X}_{k,\alpha}(f)(y)g(y)dy = \int_{\mathbb{R}^{d+1}} f(x)\mathcal{X}_{k,\alpha}(g)(x)\mathcal{A}_{k,\alpha}(x)dx.$$

Proof. An easily combination of (10), (24), (27) and (29) shows that

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}} {}^t\mathcal{X}_{k,\alpha}(f)(y)g(y)dy &= \int_{\mathbb{R}^d\times\mathbb{R}} {}^t\mathcal{V}_k \otimes W_\alpha(f)(y', y_{d+1})g(y)dy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}} {}^t\mathcal{V}_k(f)(y', y_{d+1})\mathcal{X}_\alpha g(y', y_{d+1})|y_{d+1}|^{2\alpha+1}dy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}} f(y', y_{d+1})\mathcal{V}_k \otimes \mathcal{X}_\alpha g(y', y_{d+1})\mathcal{A}_{k,\alpha}(y)dy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}} f(y', y_{d+1})\mathcal{X}_{k,\alpha} g(y', y_{d+1})\mathcal{A}_{k,\alpha}(y)dy.
\end{aligned}$$

∎

Proposition 10. *Let f be in $L^1_{k,\alpha}(\mathbb{R}^{d+1})$. Then*

$$\int_{\mathbb{R}^{d+1}} {}^t\mathcal{X}_{k,\alpha}(f)(y)dy = \int_{\mathbb{R}^{d+1}} f(x)\mathcal{A}_{k,\alpha}(x)dx.$$

Proof. Let f be in $L^1_{k,\alpha}(\mathbb{R}^{d+1})$. By taking $g \equiv 1$ in the relation (37) we deduce that

$$\int_{\mathbb{R}^{d+1}} {}^t\mathcal{X}_{k,\alpha}(f)(y)dy = \int_{\mathbb{R}^{d+1}} f(x)\mathcal{A}_{k,\alpha}(x)dx.$$

∎

Definition 8. *The operator $\mathcal{X}_{k,\alpha}$ is topological isomorphism from $D(\mathbb{R}^{d+1})$ onto itself. The inverse operator $\mathcal{X}^{-1}_{k,\alpha}$ is given for all $f \in D(\mathbb{R}^{d+1})$ by*

(i) *If $\alpha = r + m$, $m \in \mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$*

(38)
$$\mathcal{X}^{-1}_{k,\alpha}(f)(x', x_{d+1}) = \frac{2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} x_{d+1}(\frac{d}{dx^2_{d+1}})^{m+1} \left[\int_0^{x_{d+1}} (x^2_{d+1} - t^2)^{-r-\frac{1}{2}}\ {}^t\mathcal{V}_k f(x', t)|t|^{2\alpha+1}dt\right].$$

(ii) *If $\alpha = \frac{1}{2} + m$, $m \in \mathbb{N}$*

(39) $$\mathcal{X}^{-1}_{k,\alpha}(f)(x', x_{d+1}) = \frac{2^{2m+1}m!}{(2m+1)!} x_{d+1}(\frac{d}{dx^2_{d+1}})^{k+1}(x^{2m+1}_{d+1}\ {}^t\mathcal{V}_k f(x', x_{d+1})),\ x \in \mathbb{R}^{d+1}.$$

Remark 4. *The operator* $\mathcal{X}_{k,\alpha}^{-1}$ *can also be write in the form*

$$\mathcal{X}_{k,\alpha}^{-1} =^t \mathcal{V}_k \otimes \mathcal{X}_{\alpha}^{-1}. \tag{40}$$

Definition 9. *We define the operator* $V_{k,\alpha}$ *on* $D(\mathbb{R}^{d+1})$ *as follows*

- *If* $\alpha = m + \frac{1}{2}$, $m \in \mathbb{N}$

$$V_{k,\alpha}(f)(x', x_{d+1}) = (-1)^{m+1} \frac{2^{2m+1} m!}{(2m+1)!} (\frac{d}{dx_{d+1}^2})^{m+1} (\mathcal{V}_k f(x', x_{d+1})).$$

- *If* $\alpha = m + r$, $m \in \mathbb{N}$, $\frac{-1}{2} < r < \frac{1}{2}$ *and* $f \in K_0$

$$V_{k,\alpha}(f)(x', x_{d+1}) = \frac{(-1)^{m+1} 2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} \left[\int_{|x_{d+1}|}^{\infty} (y^2 - x_{d+1}^2)^{-r-\frac{1}{2}} (\frac{d}{dy^2})^{m+1} \mathcal{V}_k f(x', sgn(x_{d+1})y) y dy \right].$$

Remark 5. *The operator* $V_{k,\alpha}$ *can also be write in the form*

$$V_{k,\alpha} = \mathcal{V}_k \otimes V_{\alpha}, \tag{41}$$

where $\mathcal{V}_k$ *is the Dunkl intertwining operator given by (8) and* V_α *is the operator given in Definition 4.*

Proposition 11. *Let* f *and* g *in* $D(\mathbb{R}^{d+1})$, *the operators* $V_{k,\alpha}$ *and* $\mathcal{X}_{k,\alpha}^{-1}$ *are related by the following relation*

$$\int_{\mathbb{R}^{d+1}} V_{k,\alpha}(f)(x) g(x) \mathcal{A}_{k,\alpha}(x) dx = \int_{\mathbb{R}^{d+1}} f(x) \mathcal{X}_{k,\alpha}^{-1}(g)(x) dx.$$

Proof. From (10), (25), (40) and (41) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}} f(x) \mathcal{X}_{k,\alpha}^{-1}(g)(x) dx &= \int_{\mathbb{R}^{d+1}} f(x)\ {}^t\mathcal{V}_k \otimes \mathcal{X}_{\alpha}^{-1}(g)(x) dx \\
&= \int_{\mathbb{R}^{d+1}} V_\alpha f(x)\ {}^t\mathcal{V}_k(g)(x) |x_{d+1}|^{2\alpha+1} dx \\
&= \int_{\mathbb{R}^{d+1}} \mathcal{V}_k \otimes V_\alpha f(x)(g)(x) \mathcal{A}_{k,\alpha}(x) dx \\
&= \int_{\mathbb{R}^{d+1}} V_{k,\alpha} f(x)(g)(x) \mathcal{A}_{k,\alpha}(x) dx.
\end{aligned}$$

■

Definition 10. *The Dunkl-Bessel-Struve transform is given for* f *in* $D(\mathbb{R}^{d+1})$ *by*

$$\forall \lambda \in \mathbb{R}^{d+1}, \quad \mathcal{F}_{k,\alpha}(f)(\lambda) = \int_{\mathbb{R}^{d+1}} f(x) \Lambda_{k,\alpha}(x, \lambda) \mathcal{A}_{k,\alpha}(x) dx. \tag{42}$$

Proposition 12. *The relation (42) can also be written in the following form:*

$$\forall \lambda = (\lambda', \lambda_{d+1}) \in \mathbb{R}^d \times \mathbb{R}, \quad \mathcal{F}_{k,\alpha}(f)(\lambda) = \mathcal{F}_k \circ \mathcal{F}_{B,S}^{\alpha}(f)(\lambda), \tag{43}$$

where $\mathcal{F}_k$ *is the Dunkl transform given in Definition 2 and* $\mathcal{F}_{B,S}^{\alpha}$ *is the Bessel-Struve transform given by (22).*

Proof. Due to (22), (32), (36) and Definition 2 we have

$$\begin{aligned}
\mathcal{F}_{k,\alpha}(f)(\lambda) &= \int_{\mathbb{R}^{d+1}} f(x) \Lambda_{k,\alpha}(x, \lambda) \mathcal{A}_{k,\alpha}(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(x', x_{d+1}) \Phi_\alpha(\lambda_{d+1} x_{d+1}) K(x', -i\lambda') |x_{d+1}|^{2\alpha+1} w_k(x') dx_{d+1} dx' \\
&= \int_{\mathbb{R}^d} \mathcal{F}_{B,S}^{\alpha} f(x', \lambda_{d+1}) K(x', -i\lambda') w_k(x') dx' \\
&= \mathcal{F}_k \circ \mathcal{F}_{B,S}^{\alpha}(f)(\lambda', \lambda_{d+1}).
\end{aligned}$$

■

Proposition 13. (i) *For* $f \in L^1_{k,\alpha}(\mathbb{R}^{d+1})$, *we have*

$$\|\mathcal{F}_{k,\alpha}(f)\|_{k,\alpha,\infty} \le \|f\|_{k,\alpha,1}.$$

(ii) *For* $f \in D(\mathbb{R}^{d+1})$, *we have*

$$\mathcal{F}_{k,\alpha}(f) = \mathcal{F} \circ \, {}^t\mathcal{X}_{k,\alpha}(f),$$

where $\mathcal{F}$ *is the transform defined by* $\forall \lambda \in \mathbb{R}^{d+1}$

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}^{d+1}} f(x) e^{-i<\lambda,x>} dx.$$

Proof.

(i) By Proposition 3)-i) and Proposition 4, we can deduce that

$$\begin{aligned} \|\mathcal{F}_{k,\alpha}(f)\|_{k,\alpha,\infty} &= sup_{x\in\mathbb{R}^{d+1}} |\mathcal{F}_{k,\alpha}(f)(x)| \\ &= sup_{x_{d+1}\in\mathbb{R}} sup_{x'\in\mathbb{R}^d} |\mathcal{F}_k \circ \mathcal{F}^{\alpha}_{B,S}(f)(x)| \\ &\le sup_{x_{d+1}\in\mathbb{R}} \|\mathcal{F}^{\alpha}_{B,S}(f)(x)\|_{k,1} \\ &\le \|f\|_{k,\alpha,1}. \end{aligned}$$

(ii) From (23), (29), (43), Proposition 3)-ii) and Proposition 5-iii) we obtain

$$\begin{aligned} \mathcal{F} \circ \, {}^t\mathcal{X}_{k,\alpha}(f) &= \mathcal{F} \circ \, {}^t\mathcal{V}_k \otimes W_\alpha(f) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}} {}^t\mathcal{V}_k \otimes W_\alpha(f)(x) e^{-i<x,\lambda>} dx \\ &= a_\alpha \int_{\mathbb{R}^d} \int_{|x_{d+1}|}^{\infty} (s^2 - x_{d+1}^2)^{\alpha-\frac{1}{2}} \, {}^t\mathcal{V}_k f(x', sgn(x_{d+1})s) e^{-ix_{d+1}\lambda_{d+1}} e^{-i<x',\lambda'>} sdsdx \\ &= \int_{\mathbb{R}^d} {}^t\mathcal{V}_k(\mathcal{F}^{\alpha}_{B,S})(f)(x', \lambda_{d+1}) e^{-i<x',\lambda'>} dx' \\ &= \mathcal{F}_k \circ \mathcal{F}^{\alpha}_{B,S}(\lambda', \lambda_{d+1}) \\ &= \mathcal{F}_{k,\alpha}(\lambda). \end{aligned}$$

■

References

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