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n - Power Quasi Normal Operators on the Hilbert Space

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Abstract: *Let L (H) be the algebra of all bounded linear operators on a complex Hilbert space H. An operator $T \epsilon L (H)$ is called n power quasi normal operator if T^n commutes with T^*T that is, $T^nT^*T = T^*TT^n$ and it is denuded by [nQN]. In this paper we investigate some properties of n-power quasinormal operators. Also, the necessary and sufficient condition for a Binormal operator to be 2 power quasi normal operator is obtained.*
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I. Introduction:

Let H be a complex Hilbert space. Let L (H) be the algebra of all bounded linear operators defined in H. Let T be an operator in L(H). The operator T is called normal if it satisfies the following condition $T^*T = TT^*$, i.e., T commutes with T^*. The class of quasi – normal operators was first introduced and studied by A. brown in [1] in 1953.The operator T is quasi normal if T^*T commutes with T, i.e. $T(T^*T) = (T^*T)T$ and it is denoted by [QN]. A.A.S. Jibril [2], in 2008 introduced the class of n – power normal operators as a generalization of normal operators. The operator T is called n – power normal if T^n commutes with T^*, i.e. $T^nT^* = T^*T^n$ and is denoted by [nN]. In the year 2011, Ould Ahmed Mahmoud Sid Ahmed introduced n – power quasi normal operators [3] as a generalization of quasi normal operators. The operator T is called n power quasi normal if T^n commutes with T^*T, i.e. $T^n(T^*T) = (T^*T)T^n$ and it is denoted by [nQN]. Further we know that T is called self adjoint if $T^* = T$, unitary if $T^*T = T^*T = I$ and binormal if T^*T commutes with $T\,T^*$. In this paper we prove some new theorems based on the theorems discussed in quasi P – normal operators [4]. T is quasi P – normal if T^*T commutes with $T + T^*$, i.e. $T^*T(T + T^*) = (T + T^*)T^*T$.

In [5], Arun Bala assumed the following terminologies: Let T = R+iS, where R = Re $T = \frac{T+T^*}{2}$ and S = imT = $\frac{T-T^*}{2i}$ are the real and imaginary parts of T.

II. n – power quasi normal operators

Theorem: 1. If T is a n power quasi normal operator and $\boldsymbol{\lambda}$ is any scalar which is real, then $\boldsymbol{\lambda}$T is also a n-power quasi normal operator.

Proof:

Since T is a n power quasi normal operator we have $T^nT^*T = T^*TT^n$ -------(1)

If $\boldsymbol{\lambda}$ is any scalar, which is real then $(\lambda T)^* = \bar{\lambda}T^* = \lambda T^*$ also we have

$[(\lambda T)^*]^n = (\lambda T^*)^n = \lambda^n {T^*}^n$

Using the above results in (1)

$(\lambda T)^n(\lambda T)^*(\lambda T) = \lambda^n T^n \lambda T^* \lambda T = \lambda^{n+2} T^n T^* T$ ------ (2)

$(\lambda T)^*(\lambda T)\,(\lambda T)^n = \lambda T^*\,\lambda T \lambda^n T^n = \lambda^{n+2} T^* T T^n$ -------(3)

From (1), (2) and (3) we see that λT is also a n-power quasi normal operator.

Theorem: 2. If T is a n power quasi normal operator which is a self adjoint operator also, then T^* is also n-power quasi normal operator.

Proof:

Since T is a n power quasi normal operator, we have $T^nT^*T = T^*TT^n$ -------(1)

Since T is a self adjoint we have $T^* = T$ -------- (2)

Replace T^* by T in (1) we get, $(T^*)^n(T^*)^*T^* = (T^*)^nTT^* = T^nT^*T$----- (3)

and $(T^*)^*T^*(T^*)^n = TT^*(T^*)^n = T^*TT^n$ ------- (4)

From (1), (3) and (4) we see that T^* is alos n-power quasi normal operator.

Theorem: 3. If T is self adjoint operator, then T is n power quasi normal operator.

Proof:

Since T is a self adjoint operator, therefore $T^* = T$

Now $T^nT^*T = T^nTT = T^{n+2}$ -------(1)

$T^*TT^n = TTT^n = T^{2+n}$ ------(2)

Hence $T^nT^*T = T^*TT^n$

Therefore T is n power quasi normal operator.

Theorem: 4. Let T be any operator on a Hilbert space H. Then

(i) $(T + T^*)$ is n power quasi normal.

(ii) TT^* is n power quasi normal.

(iii) T^*T is n power quasi normal.

(iv) $I + T^*T,\ I + TT^*$ are also n power quasi normal.

Proof:

(i) Let N = $T + T^*$

Now $N^* = (T + T^*)^* = T^* + T = T + T^* = N$

$\therefore N$ is a self adjoint and from theorem 3, we know that every self adjoint operator is n power quasi normal. $\therefore N = T + T^*$ is n power quasi normal.

(ii) Similarly $(TT^*)^* = T^{**}T^* = TT^*$

(iii) $(T^*T)^* = T^*T^{**} = T^*T$

(iv) $(I + TT^*)^* = I^* + T^{**}T^* = I + TT^*$

$(I + T^*T)^* = I^* + T^*T^{**} = I + T^*T$

So all the operators mentioned above are self adjoint and therefore all these operators are n power quasi normal operators.

Remark:

Since O and I are self adjoint operators and therefore O and I are n power quasi normal operators.

Theorem: 5. Let T be a n power quasi normal operator on a Hilbert space H. Let S be self adjoint operator for which T and S commute, then ST is also n power quasi normal operator.

Proof:

Since S is self adjoint operator we have $S^* = S$. Since S and T commute we get ST = TS. Also, $(ST)^* = (TS)^*$ implies that $T^*S^* = S^*T^*$ and this implies $T^*S = ST^*$

Also, $(ST)^* = T^*S = ST^*$

Since T is a n power quasi normal operator we get $T^nT^*T = T^*TT^n$

From ST = TS and $S^* = S$, we can easily prove $(ST)^* = T^*S = ST^*$; $ST^n = T^nS$;

$$TS^n = S^nT;\ T^nS^* = S^*T^n;\ S^nT^* = T^*S^n;\ S^nS^*S = SS^*S^n\ and\ (ST)^n = S^nT^n$$

Now $(ST)^n(ST)^*(ST) = S^nT^nT^*S^*ST$

$= S^nT^nT^*S^*TS$

$= S^nT^nT^*\ TS^*S$

$= S^nT^*\ TT^nS^*S$

$= S^nT^*\ TS^*T^nS$

$= S^nT^*TS^*ST^n$

$= T^*S^nTS^*ST^n$

$= T^*TS^nS^*ST^n$

$= T^*TSS^*S^nT^n$

$= T^*STS^*S^nT^n$

$= (ST)^*(TS)(ST)^n$

Hence ST$\in [nQN]$

Theorem: 6. If T is a self adjoint operator, then T^{-1} is also a n power quasi normal operator.

Proof:

Since T is self adjoint operator we have $T^* = T$, further we have

$(T^{-1})^* = (T^*)^{-1} = T^{-1}$ [since$T^* = T$]

$(T^{-1})^* = T^{-1}$ Implies that T^{-1} is self adjoint.

But in theorem (3) we have proved that every self adjoint operator is n power quasi normal operator . T^{-1} is self adjoint operator and therefore T^{-1} is n power quasi normal operator.

Proof verification:

Since T^{-1} is self adjoint, therefore $(T^{-1})^* = T^{-1}$

Now $(T^{-1})^n(T^{-1})^*(T^{-1}) = (T^{-1})^n(T^{-1})(T^{-1}) = (T^{-1})^{n+2}$

$(T^{-1})^* (T^{-1})(T^{-1})^n = (T^{-1})(T^{-1})(T^{-1})^n = (T^{-1})^{n+2}$

Therefore, $T^{-1} \in [nQN]$

In the following theorem, we derive the condition for an invertible n power quasi normal operator to be self adjoint.

Remark: If T is isometry right invertible operator then, we can prove that $T^{-1} = T^*$.

Proof: Given T is isometry operator. Therefore, we get $T^*T = I$. Post multiply this by T^{-1} we can get the desired result.

Theorem: 7. Let T^{-1} be a n power quasi normal operator and if $T^{-1} = T^*$, then T^{-1} is self adjoint only if T is self adjoint.

Proof:

Given T^{-1} is n power quasi normal operator and therefore by definition,

$(T^{-1})^n (T^{-1})^* T^{-1} = (T^{-1})^* T^{-1} (T^{-1})^n$

$(T^{-1})^* = T^{-1}$ only if $(T^{-1})^n = (T^{-1})^*$ and $T^{-1} = (T^{-1})^n$

i.e., T^{-1} is self adjoint only if $T^{*^n} = T^{*^*}$ and $T^* = T^{*^n}$ since $T^{-1} = T^*$,

i.e., T^{-1} is self adjoint only if $T^{*^n} = T$ and $T^* = T^{*^n}$ $--(1)$

But $T = T^{*^n}$ and $T^* = T^{*^n}$ is possible only when T = T^* i.e., T is self adjoint ---(2)

Using (2) in (1), we get, T^{-1} is self adjoint only if T is self adjoint.

Theorem: 8. Let T= R + iS be an operator on a Hilbert space H for which RS = SR then T is a 2 power quasi normal operator if $SR^3 = R^3S$ and $RS^3 = S^3R$ i.e., S^3 commutes with R and R^3 commutes with S.

Proof:

Since T is an operator for which T = R+iS, so that, we get $T^* = R - iS$

Now $T^*T = (R - iS)(R + iS) = (RR + SS) + i(RS - SR)$

$= RR + SS + i.0$ [since RS = SR]

$= RR + SS = R^2 + S^2$

Also, $T^2 = (R + iS)(R + iS)$

$= (RR - SS) + i(SR + RS)$

$= (RR - SS) + i(SR + SR)$

$= (RR - SS) + 2iSR$

$T^2T^*T = [(RR - SS) + 2iSR](RR + SS)$

$= [(R^2 - S^2) + i2SR](R^2 + S^2)$

$= (R^2 - S^2)(R^2 + S^2) + i2SR(R^2 + S^2)$

$= R^4 - S^4 + R^2S^2 - S^2R^2 + i2SR(R^2 + S^2)$

$= R^4 - S^4 + R\,RS\,S - SSRR + i2SR(R^2 + S^2)$

$= R^4 - S^4 + RS\,RS - SR\,SR + i2SR(R^2 + S^2)$

$= R^4 - S^4 + SR\,SR - SR\,SR + i2SR(R^2 + S^2)$

$= R^4 - S^4 + i2SR(R^2 + S^2)$

$= R^4 - S^4 + 2i(SRR^2 + SRS^2)$

$= R^4 - S^4 + 2i(SR^3 + SRS^2)$

$= R^4 - S^4 + 2i(SR^3 + RSS^2)$

$= (R^4 - S^4) + 2i(SR^3 + RS^3)$ -------(1)

Similarly, $T^*TT^2 = (R^2 + S^2)[(R^2 - S^2) + i2SR]$

$= (R^2 + S^2)(R^2 - S^2) + i(R^2 + S^2)2SR$

$= R^4 - S^4 + S^2R^2 - R^2S^2 + 2i[R^2SR + S^2SR]$

$= R^4 - S^4 + 0 + 2i[R^2RS + S^2RS]$

$= R^4 - S^4 + 2i[R^3S + S^2RS]$

$= (R^4 - S^4) + 2i[R^3S + S^2SR]$

$= (R^4 - S^4) + 2i[R^3S + S^3R]$ --------(2)

Equations (1) and (2) are same if $R^3S = S^3R$ and $SR^3 = RS^3$

i.e., T is 2 power quasi normal if R^3 commutes with S and S^3 $commutes$ with R.

In the following theorem, we derive the necessary and sufficient condition for a binormal operator to be 2 power quasi normal operator.

Theorem: 9 A self adjoint operator on a Hilbert's space H is binormal if and only if it is 2-power quasi normal.

Proof:

Given T is self-adjoint operator. $\therefore T^* = T$.

Now suppose that T is 2 power quasi normal we have

$$T^2T^*T = T^*TT^2$$
$$\Rightarrow\ TTT^*T = T^*TTT$$
$$\Rightarrow\ TT^*T^*T = T^*T\,TT^*$$
$$\Rightarrow\ TT^*T^*T - T^*T\,TT^* = 0$$
$$\Rightarrow\ [TT^*, T^*T] = 0$$

$\therefore$ T is binormal.

Conversely, let T be binormal. Therefore by definition we get,

$$TT^*T^*T = T^*T\,TT^*$$

Since T is self adjoint $T = T^*$

$$\Rightarrow\ T\,TT^*T = T^*TTT$$
$$\Rightarrow\ T^2T^*T = T^*TT^2$$

Therefore, it is proved that $T \in 2$ power quasi normal operator.

Theorem 10. Let T be a self adjoint operator on a Hilbert space H and S be any operator on H, then S^*TS is a n power quasi normal.

Proof: Since T is self adjoint we get $T^* = T$

Consider $(S^*TS)^* = S^*T^*S = S^*TS$. Therefore S^*TS is self adjoint operator.

$\therefore$ By theorem (3), we have the desired result, i.e. if S^*TS is a self adjoint operator, then it is n power quasi normal operator. This result can be verified as follows.

Now, $(S^*TS)^n(S^*TS)^*(S^*TS) = (S^*TS)^n(S^*TS)(S^*TS) = (S^*TS)^{n+2}$ ----(1)

Similarly, $(S^*TS)^*(S^*TS)(S^*TS)^n = (S^*TS)^2(S^*TS)^n = (S^*TS)^{n+2}$ ----(2)

Equations (1) and (2) are same. $\therefore S^*TS \in$ n power quasi normal.

References

[1] A. Brown , On a class of operators, Proc. Amer. Math. Soc, 4 (1953), 723 – 728.

[2] A.A.S. Jibril, on n – power normal operators, The Journal of Science and engineering, Volume 33, Number 2A, (2008) , 247 – 251.

[3] Ould Ahmed Mahmoud Sid Ahmed, On the class of n – power quasi normal operators on the Hilbert space, Bull. Of Math.Anal. Appl., Vol 3, 2 (2011), 213 – 228.

[4] Dipshikha Bhattacharya And Narendra Prasad, Quasi-P Normal operators linear operators on Hilbert space for which T+T* And T*T commute, Ultra Scientist Vol. 24(2)A, 269-272 (2012).

[5] Arun Bala, A note on Quasi normal operators, University of Delhi, 110007, 1976.