IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 1 Ver. IV (Jan. - Feb. 2016), PP 87-94
www.iosrjournals.org

Characterizations of Radical Ideals and Ring with Nilpotent Ideals

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Abstract: *In this paper, we have mentioned some properties of radical ideals ring with nilpotent ideals. Mainly, we have focused on "the Jacobson radical of an Artinian ring R is nilpotent. In fact, $J(R)$ is the largest nilpotent (left or right or 2-sided) ideal of R and consequently, $N(R)=J(R)$". Finally, we have discussed many of theorem on nilpotent ideals.*
Keywords: *Artinian ring, Jacobson radical, Nil radical, Primary ideal, Tertiary radical.*

I. Introduction

There are several kinds of radicals in a ring. Among them, two radicals called the Nil radical and Jacobson radical. There are others like the Amitsur radical, the Brown-Mc Coy radical, the Levitzki radical, etc. Some basic properties of the first two of these radicals have been discussed. The $radR$ of R is defined to be its radical as a left module over itself: $radR := rad_R R$. The annihilator of a simple module (in other words, a primitive ideal) is evidently the intersection of the annihilator of the non-zero elements of the elements of the module; these beings all maximal left ideals, we get $radR = \bigcap$ of annihilators of simple (respectively semisimple) modules. For Artinian rings the radical is very special. The Nil radical of a ring R is defined to be the radical ideal with respect to the property that " A two – sided ideal is nil" and is denoted by $N(R)$, i. e., $N(R)$ is the largest two-sided ideal of R such that every element of $N(R)$ is nilpotent. The Jacobson radical of a ring R with 1 is defined as the radical ideal of R with respect to the property that "A two-sided ideal I is such that $1-a$ is a unit in R for all $a \in I$" and it is denoted by $J(R)$. In other words, $J(R)$ **.**is the largest two-sided ideal of R such that $1-a$ is a unit for all $a \in J(R)$. The Jacobson radical of a ring R consists of those elements in R which annihilates all simple right R-module. One important property of Jacobson radical for Artinian ring is that let R be a left (or right) Artinian ring. Then, $J(R)$ is a nilpotent ideal of R (i.e. $J(R)^n$ for some $n>0$) and is equal to the sum of all nilpotent ideals of R. Then we have talked about the ring with nilpotent ideals.

II. Discussion

In this section we have worked on radical ideals. Some properties and characterizations of Nil and Jacobson ideals have also been discussed.

1.1 Radical ideal:

A two sided ideal I in a ring R with 1 is called a radical ideal with respect to a specified property p if

1. The ideal I possesses the property p and
2. The ideal I is maximal for the property p, i. e., if J is a two sided ideal of R having the property p, then $J \subseteq I$.

2.2 Nil radical:

Now special cases have been discussed before proving the existence of the nil radical.

Examples:

1. If R has no non-trivial nilpotent elements, in particular, R an integral domain, then $N(R)=0$.
2. If R is commutative, then the set $N(R)$ of all nilpotent elements of R which is an ideal, is the nil radical of R. (If R has 1, then $N(R)$ is the intersection of all prime ideals of R .)

3. If R is a nil ring. i. e., every element of R is nilpotent, then $N(R)=R$. For instance, $R=2Z/4Z$ or $R=$ strictly upper triangular matrices over any ring.
4. $N(M_r(D))=(0)$ for any division ring D because $R=M_r(D)$ is not a nil ring and it has no two-sided ideals other than (0) and R. (Here we observe that R has nilpotent element if $r\geq 2$ but they do not form an ideal.)

2.1 Theorem:

For any ring R, the nil radical $N(R)$ exists and it is characterized by $N(R)=\{a\in R/$ the principal two-sided ideal (a) is a nil ideal$\}$.

Proof:

We have to first prove that $N=N(R)$ as above is a two-sided ideal and secondly that it is the largest for that property.

1. Since $0\in N$, $N\neq\phi$. If $a\in N$ and $x\in R$, then $(xa)\subseteq(a)$ and $(ax)\subseteq(a)$ and so both (xa) and (ax) are nil ideals and hence $ax, xa\in N$. Thus we have only to prove the following.
2. N is additive subgroup of R.

To see this, for $a,b\in N$, we have to show that $(a-b)$ is a nil ideal. Since $(a-b)\subseteq(a)+(b)$, every element $x\in(a-b)$ can be written as $x=y+z$ for some $y\in(a)$ and $z\in(b)$. Since (a) and (b) are nil ideals, both y and z are nilpotent, say $y^n=0$ and $z^n=0$ for some $n>0$. Now look at $x^n=(y+z)^n=y^n+z'=0+z'$ where z' is some of monomials in y and z in each of which z is a factor, i. e., $z'\in(z)\subseteq(b)$ and so z' is nilpotent and hence x is nilpotent, i. e., $(a-b)$ is a nil ideal, as required.

Finally, let I be any two-sided ideal of R. Then trivially, $(a)\subseteq I$, $\forall a\in I$ and hence (a) is a nil ideal, i. e., $I\subseteq N$, as required.

2.1 Corollary: We have $N\left(R/N(R)\right)=(0)$ for any ring R.

Proof:

Let $\bar{a}=a+N\in N\left(R/N\right)$ where $N=N(R)$ and $a\in R$. Then the two-sided principal ideal $(\bar{a})$ is a nil ideal in R/N, i. e., the two-sided ideal (a) in R is nil modulo N. Hence it follows that (a) is a nil ideal in R (since N is nil ideal), i. e., $a\in N$ and so $\bar{a}=0$, i. e. $a\in N$, as required.

2.3 Jacobson radical:

Before we prove the existence of the Jacobson radical, we mark the following special cases.

Examples:

1. $J(Z)=(0)$
2. $J(M_r(D))=(0)$, $\forall r\in N$ and D a division ring (since $M_r(D)$ has no 2-sided ideals other than (0) and $M_r(D)$ and the latter cannot be a candidate).
3. If R is a commutative local ring with its unique maximal ideal M, then obviously $J(R)=M$.

To prove the existence of the Jacobson radical, first we define the so called left and right Jacobson radicals $J_l(R)$ and $J_r(R)$ and show them to be equal. Secondly we show that $J_l(R)=J_r(R)=J(R)$ is the one we are looking for.

2.3.1 Left Jacobson radical:

For any ring R with 1, intersection of all maximal left ideals of R is called the left Jacobson radical or simply the left radical of R and is denoted by $J_l(R)$. (In case R is commutative, $J_l(R)$ is the intersection of all maximal left ideals of R).

Examples:

1. The left radical of a division ring is (0). More generally, the left radical of $M_n(D)$ is (0) $(\forall n \in N)$ where D is a division ring.
2. The (left) radical of Z is (0).
3. The (left) radical of a local ring is its unique maximal ideal.
4. The left radical of Z/nZ is mZ/nZ where m is the product of all distinct prime divisors of n. For instance, $J_l\left(Z/36Z\right)=6Z/36Z$, $J_l\left(Z/64Z\right)=2Z/64Z$ and $J_l\left(Z/180Z\right)=30Z/180Z$.

2.2Theorem:

For any ring R, its left radical $J_l(R)$ is the intersection of the annihilators of all simple left modules over R. In particular, $J_l(R)$ is a 2-sided ideal of R.

Proof:

1. If m is a maximal left ideal of R, then m is the annihilator of the non-zero element $\overline{1}=1+m$ is the simple R-module $S=R/m$.
2. If S is a left simple R-module and $x \in S$ is a non-zero element, then $S=Rx$ and the natural map $f_x : R \to S$, defined by $f_x(a)=ax$, $\forall a \in R$ is an endomorphism whose kernel is the annihilator of the element x. Thus we have $R/\ker(f_x) \approx Rx = S$ which is simple and hence $M_x=\ker(f_x)$ is a maximal left ideal of R. This shows that the annihilator of any non-zero element of a simple module is a maximal left ideal of R. In other words, the family of all maximal left ideals of R is the same as that of the annihilators of non-zero elements of all simple left modules over R.
3. The annihilator of any left module M is a 2-sided ideal of R and it is the intersection of the annihilators of all elements of M.
4. If M is the set of all maximal left ideals of R and L is the family of all simple left modules over R, then we have $J_l(R)=\cap_{M\in M} M$ which in turn can be written as $J_l(R)=\cap_{S\in L}(\cap_{x\in S} M_x)=\cap_{S\in L} Ann_R(S)$ (Where M_x is the annihilator of the element $x \in S$ and so $J_L(R)$ is the intersection of the family $\{Ann_R(S) | S \in L\}$ of 2sided ideals and hence 2-sided, as required.

II. Radical of Artinian Ring:

1Corollary: If R is commutative, then $N(R) \subset J(R)$.

Proof: Equality need not in the corollary. For example if $R = Z_{(p)}$ then $N(R)=0$ and $J(R)=(p)$.

1 Theorem: Let R be Artinian. Then $radR$ is the largest two-sided nilpotent ideal of R.

Proof: Any nil ideal (one sided or two-sided) is contained in the radical. It suffices to prove therefore that $radR$ is

nilpotent (we have also observed that $radR$ is a two-sided ideal, being the annihilator of all simple modules). Set $\tau := radR$. Choose n large enough so that $\tau^n = \tau^{n+1} = \cdots =: a$. It suffices to assume that $a \neq 0$ and arrive at a contradiction. Assume $a \neq 0$. Choose a minimal left ideal I with the property that $aI \neq 0$ (such an ideal exists by the Artinian hypothesis: observe that $aR = a \neq 0$, so the collection of ideals with the property is non-empty). Now, on the one hand, $a(\tau I)=(a\tau)I = aI \neq 0$, so that τI has the property; on the other, $\tau I \subseteq I$. So $\tau I - I$ by the minimality of I. We claim now that I is finitely generated. It will then follow, by Nakayama's lemma, that $I=0$, which is a contradiction, since $aI \neq 0$ by choice of I, and the proof will be over.

If I is an ideal in the Artinian ring R, then R/I is an Artinian ring.

An Artinian integral domain is a field.
If R is an Artinian ring, then every prime ideal of R is maximal. Therefore, the nil radical $N(R)$ coincides with the Jacobson radical $J(R)$.

2 Theorem: If R is Artinian, then $J(R)$ is a nilpotent ideal.

Proof:

Let $J = J(R)$; consider the descending chain of right ideals $J \supset J^2 \supset \cdots \supset J^n \supset \cdots$. Since R is Artinian there is an integer n such that $J^n = J^{n+1} = \cdots = J^{2n} = \cdots$. Hence if $xJ^{2n} = (0)$ then $xJ^n = 0$. We want to prove that $J^n = (0)$; suppose it is not. Let $W = \{x \in J \mid xJ^n = (0)\}$; W is an ideal of R. If $W \supset J^n$ then $J^n J^n = (0)$ which would yield that $(0) = J^{2n} = J^n$ the desired outcome. Suppose that $J^n \not\subset W$. Therefore in $\overline{R} = R/W$, $\overline{J}^n \neq (0)$. If $\overline{x}\overline{J}^n = (0)$ then $xJ^n \subset W$ hence $(0) = xJ^n J^n = xJ^{2n} = xJ^n$ placing x in W and so implying that $\overline{x} = 0$. That is, $\overline{x}\overline{J}^n = (0)$ forces $\overline{x} = 0$. Since $\overline{J}^n \neq (0)$ it contains a minimal right ideal $\overline{\rho} \neq (0)$ of $\overline{R}$. But in that event $\overline{\rho}$ is an irreducible $\overline{R}$-module hence is annihilated by $J(\overline{R})$. Since $\overline{J}^n \subset J(\overline{R})$ we get $_{\overline{\rho}}\overline{J}^n = (0)$. As we have seen above this forces a contradiction $\overline{\rho} = (0)$. The theorem is proved.

Now we try to prove the following theorem.

3 Theorem: The Jacobson radical of an Artinian ring R is nilpotent. In fact, $J(R)$ is the largest nilpotent (left or right or 2-sided) ideal of R and consequently, $N(R) = J(R)$.

Proof:

Since R is Artinian, the descending chain of ideals $J \supseteq J^2 \supseteq \cdots \supseteq J^n \supseteq \cdots \supseteq \cdots$ is stationary where $J = J(R)$. Say, $J^n = J^{n+1} = \cdots = J^{2n} = \cdots$ for some $n >> 0$. Write $I = J^n$. Now we have $I = I^2$ and $JI = I$. (If we know that I is finitely generated then Nakayama's lemma would have implied that $I = (0)$ which is what we are looking for. But there seems no way to ensure this crucial fact.) The following is an elementary but a subtle argument to achieve the goal. Assume, if possible, that $I \neq (0)$. Consider the family F of all left ideals K of R such that $IK \neq (0)$. Since $I^2 = I \neq (0)$, $I \in F$ and so $F \neq (0)$. Note that $(0) \notin F$. Since R is Artinian, F has a minimal member, say K. i.e., K is a left ideal of R such that $IK \neq (0)$ and K is minimal for this property. On the other hand, since $IK \neq (0)$, we can find $a \in I$ and $b \in K$ such that $ab \neq 0$ which implies that $I(Rb) \neq (0)$, i.e., $Rb \in F$. But $Rb \subseteq K$ and so $Rb = K$ by minimality of K. Thus K is a principal left ideals of R. Finally, we have $(IJ) \cdot Rb = I \cdot Rb = Ib \neq (0)$ and $J \cdot Rb = J \cdot b \subseteq Rb$ and $J \cdot Rb \neq (0)$ which give, (again by minimality of $K = Rb$ in F), that $J \cdot Rb = Rb$. Now Nakayama's lemma gives that $K = Rb = (0)$, contradiction to the assumption that $I \neq (0)$. Hence $I = J^n = (0)$.

3.4 Jacobson radical theorem:

Let R be a ring, $a \in R$. The followings are equivalent:

(a) a annihilates every simple left R-module;
(b) a annihilates every simple right R-module;
(c) a lies in every maximal left ideal of R;
(d) a lies in every maximal right ideal of R;
(e) $1 - xa$ has a left inverse for every $x \in R$;
(f) $1 - xa$ has a right inverse for every $x \in R$;
(g) $1 - xay$ is a unit for every $x, y \in R$.

3.5 Theorem Let R be a left (or right) Artinian ring. Then, $J(R)$ is a nilpotent ideal of R (i.e. $J(R)^n$ for some $n>0$) and is equal to the sum of all nilpotent ideals of R.

Proof:

Suppose $x \in R$ is nilpotent, say $x^n = 0$. Then, $(1-x)$ is a unit, indeed, $(1-x)(1+x+x^2+\cdots+x^{n-1})=1$. So if I is nilpotent ideal R, then every $x \in I$ satisfies condition (g) of the Jacobson radical theorem. This shows every nilpotent ideal of R is contained in $J(R)$. It therefore just remains to prove that $J(R)$ is itself a nilpotent ideal. Set $J = J(R)$. Consider the chain $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$ of two sided ideal of R. Since R is left Artinian, the chain stabilizes. So $J^n = J^{n+1} = \cdots$ for some n. Set $I = J^n$, so $I^2 = I$. We need to prove that $I = 0$. Suppose for a contradiction that $I \neq 0$. Choose a left ideal K of R minimal such that $IK \neq 0$ (use the fact that R is left Artinian). Take any $a \in K$ with $Ia \neq 0$. Then $I^2 a = Ia \neq 0$, so the left ideal Ia of R coincides with K by the minimality of K. Hence, $a \in K$ lies in Ia, so we can write $a = xa$ for some $x \in I$. So, $(1-x)a = 0$. But $x \in J$, so $(1-x)$ is a unit, hence $a = 0,$ which is a contradiction.

III. Rings With Nilpotent Ideals

In 1939 Levitzki stated that in a right Noetherian ring R every nil right ideal is nilpotent. Now a day this theorem is an easy consequence. It readily follows that R has a maximal nilpotent ideal N which is unique, the nilpotent radical and R/N is a semi- prime- ring. N is the intersection of the prime ideals of R and indeed is the intersection of the minimal prime ideals, which are finite in number.

It is impossible to generalize the procedures of [theorem3.1, 10] in order to obtain regular elements. Instead the concept has to be taken over factor rings.

Let A be an ideal of R and set

$$\ell'(A) = (c \in R;\ cx \in A, x \in R \text{ implies that } x \in A)$$
$$\ell'(A) = (c \in R;\ xc \in A, x \in R \text{ implies that } x \in A)$$
$$\ell(A) = \ell'(A) \cap \ell'(A).$$

Under conditions of [theorem3.1, 10] we have, $\ell(0) = \ell'(0) \cap \ell'(0)$.

4.1 Theorem: Let R be a right Noetherian ring with nilpotent radical N and let $p_1, \cdots, p_n$ be the minimal prime ideals of R. Then

1. $\ell'(0) \subset \ell'(N)$;
2. $\ell(N) = \ell(p_1) \cap \cdots \cap \ell(p_n)$;
3. $\ell(0) + N = \ell'(0)$;
4. Let $a \in R$, $c \in \ell'(0)$ then $a_1 \in R$, $c_1 \in \ell(N)$ exist with $ac_1 = ca_1$.

4.2 Theorem: A right Noetherian ring R has a right quotient ring Q, which is a right Artinian ring, if and only if $\ell(0) = \ell(N)$.

In this theorem $\ell(O) = \ell(N)$ is necessary. After this point the quotient problem for Noetherian rings become very difficult and no decisive results have been obtained. There are three aspects to the problem:

I. When does a Noetherian ring R have regular elements;
II. When are there enough regular elements to satisfy the right Ore condition;
III. What is the structure of a quotient ring?
For commutative Noetherian rings the matter is settled by the maximal primes of zero, these are the maximal annihilator ideals and they are finite in number. An element is regular if and only if it does not lie in any of these primes, such elements exit if and only if the ring is faithful ($Rx = 0$ implies that $x = 0$).

The second question is trivial and as for the third, a quotient ring is a semi-local ring in which the Jacobson radical has non-zero annihilator.

Rings with zero singular ideal are of some interest. Such a ring, if commutative, is semi-prime but in general the structure is very complicated. They are not known in the Artinian case except when indecomposable right ideals are uniserial. They are then determined as direct sums of blocked triangular matrix rings over division rings.

The assumption that R has zero singular ideal is useful technically, because $c \in \ell'(O)$ implies that cR is an essential right ideal and hence $l(c) = 0$. Thus $\ell(O) = \ell'(O)$. Djabali settles some cases of the quotient problem for rings with zero singular ideal.

In general case when the existence of the quotient ring is assumed, related properties can be studied.

4.3 Theorem:

Let R be a right Noetherian ring with a right quotient ring Q and N be the nilpotent radical of R. Then NQ is the nilpotent radical of Q. Let P be a prime ideal of R then either $PQ = Q$ or $\ell(O) \subset \ell(P)$ and PQ is a prime ideal of Q with $PQ \cap R = P$. Let P' be a prime ideal of Q, then $P' \cap R$ is a prime ideal of R and $(P' \cap Q)Q = P'$.

This theorem is well known for commutative rings but its generalization is not immediate. It depends on theorem (4.1) and so far is only known for Noetherian rings. Some necessary conditions for the existence of the quotient ring are obtained as follows. The transfer ideal of $\ell(O)$ is the largest ideal T such that $c + t \in \ell(O)$, $\forall c \in \ell(O)$, $t \in T$.

There is also a transfer right ideal T', defined in the same way. Clearly they are uniquely defined and $T \subset T'$. In order to fix the idea we remark that the Jacobson radical of a ring is the transfer ideal of the group of units.

Suppose that a Noetherian ring R has a right quotient ring Q and let T, T' be the transfer ideals of R. It is a consequence of theorem (4.3) that $J \cap R = T = T'$, where J is the Jacobson radical of Q. It follows that T is a semi-prime ideal of R. This may, of course, be true in a ring which does not have a quotient ring but at present the best result known is that $T \supset N$, by theorem (4.1).

We conclude our remarks on the quotient problem by Small which gives an example which is a right and left Noetherian ring but does not have a quotient ring on either side. This ring satisfies a polynomial identity. For relief we return to an easy case, the principal ideal rings. A ring R with unit element is a pri-ring when its right ideals are principal (single generator). Now we discuss about the semi-prime pri-ring.

1. A semi-prime pri-ring is a finite direct sum of prime pri-rings.
2. A prime pri-ring is isomorphic to a full ring of matrices over a right Ore domain.

A pri-ring which is left Noetherian is the direct sum of a semi-prime pri-ring and an Artinian pri-ring. It has a right quotient ring which is an Artinian pri-ring.

Difficulties arise with pri-rings which are not left Noetherian. A number of other problems have been studied from time to time. An old conjecture due to Jacobson enquires whether $\bigcap_{n=1}^{\infty} J^n = 0$, where J is the Jacobson radical, holds in a right Noetherian ring. The problem is still open for right and left Noetherian rings.

4.1 Primary ideal:

An ideal T of a ring R is a primary ideal if $AB \subset T$, where A, B are ideals of R, implies that either $A \subset T$ or $B^k \subset T$, together with the corresponding property when A and B are interchanged.

A strongly primary ideal T is a primary ideal such that R/T has an Artinian quotient ring. Is a primary ideal always strongly-primary? This seems unlikely and a counter-example should throw light on the nature of the quotient process.

Many difficulties stem from the lack of a representation of an ideal as an intersection of primary ideals. It is fashionable to avoid this property in commutative work, but we can not indulge this whim. Indeed primary decomposition does not hold in algebras, for if it held in group algebras then all finite groups would be soluble. The only decomposition theory which has anything to offer in the general case is the tertiary theory. There have been many attempts to deal with the questions of uniqueness and existence, most theories succeed in one place and fail in the other. The tertiary theory succeeds in both respects. Unfortunately, it is very difficult to relate to other aspects of the ring structure. Here we restrict our summary to the case of a right Noetherian ring R having unit element, which has some simplifying features.

Let M be a right $R-\text{mod}\,ule$. Index its essential sub modules as $M_\alpha(\alpha \in A)$ and define the radical of M to be $radM =$ ($x \in R;\ M_\alpha x = 0$, some $\alpha \in A$).

Clearly $radM$ is an ideal. When M is finitely generated, $radM$ is a finite intersection of prime ideals. Let I be a right ideal then the tertiary radical of I is $r(I) = rad(R-I)$. When I is an ideal of R we have, $r(I) \supset I$, but this need not hold for right ideals. Let R be a simple Artinian ring $I = eR$, $e^2 = e \neq 0$, 1, then $r(I) = 0$.

The definition can be rephrased for left modules and left ideals and indeed for bimodules. Thus for an ideal T we can define a radical $R(T)$ by the rule:

$R(T)$ is the set of elements $a \in R$ such that every ideal $B \not\subset T$ contains an ideal $B' \not\subset T$ with $B'a \subset T$.

This definition can be rephrased on the left as well. It follows that an ideal T has four tertiary radicals; as a right or left ideal and on the right or left as a (two sided) ideal. These can differ, certainly $R(T) \subset r(T)$, but equality occurs for Artinian rings. Whether $R(T) = r(T)$ for Noetherian rings is not yet settled. We observe that in a commutative ring these radicals all reduce to $\sqrt{T} =$ ($x \in R$; $x^n \in T$ for some $n > 0$).

A right ideal I is tertiary when $bRa \subset I$, $b \notin I$ implies that $a \in r(I)$.

Then $r(I)$ is a prime ideal. A right ideal V is a irreducible when $R-V$ is a uniform right $R-\text{mod}\,ule$. Then V is a tertiary right ideal. It follows that every proper right ideal is a finite intersection of tertiary right ideals.

4.4Theorem: Let I be proper right ideal of the right Noetherian ring R and $I = I_1 \cap \cdots \cap I_k$ where the I_j are tertiary right ideals with associated prime ideals P_j; $j = 1, \cdots, k$ .Then $r(I) = P_1 \cap \cdots \cap P_k$. The intersection of finite set tertiary right ideals, each having the same associated prime ideal P, is again $P-tertiary$. This enables a decomposition to be brought into reduced or normal form (the associated primes are distinct), as in the commutative theory. A uniqueness theorem now follows.

4.5 Theorem: Let $I = I_1 \cap \cdots \cap I_h = J_1 \cap \cdots \cap J_k$, where the decompositions are reduced, the I_α, I_β being tertiary right ideals. Then $h = k$ and the two sets of associated prime ideals coincide. These results can be carried out for left ideals and repeated for $2-sided$ ideals T, using $R(T)$ instead of $r(T)$.

In the formal sense the theory is entirely satisfactory and indeed is only possible theory for which these theorems hold. Nevertheless, it has proved difficult to apply to the study of the structure of Noetherian rings, because the nature of tertiary ideals is difficult to understand. Moreover, the tertiary radical has to destroy the partial order even for $2-sided$ ideals. For example, $r(p) = p$ for all prime ideals p of R. Since $(0) \subset p$, preservation of partial order would imply that $r(0) \subset p$ for all primes p, which would mean that $r(0)$ is the nilpotent radical. This is certainly not the case as it would lead to the existence of a primary decomposition for ideals.

IV. Conclusion

Theorem (2.1) states that for any ring R, the nil radical $N(R)$ exists and it is characterized by $N(R) = \{a \in R /$ the principal two-sided ideal (a) is a nil ideal$\}$. Theorem (2.2) shows that for any ring R, its left radical $J_l(R)$ is the intersection of the annihilators of all simple left modules over R. In particular, $J_l(R)$ is a 2-sided ideal of R. After that theorem (3.1) follows "let R be Artinian. Then $radR$ is the largest two-sided nilpotent ideal of R". Theorem (3.2) depicts that if R is Artinian, then $J(R)$ is a nilpotent ideal. Then it follows from theorem (3.3) that the Jacobson radical of an Artinian ring R is nilpotent. In fact, $J(R)$ is the largest nilpotent (left or right or 2-sided) ideal of R and consequently, $N(R) = J(R)$. With Jacobson radical theorem (3.5) has established that let R be a left (or right) Artinian ring. Then, $J(R)$ is a nilpotent ideal of R (i.e. $J(R)^n$ for some $n > 0$) and is equal to the sum of all nilpotent ideals of R. Finally, it has been discussed that Let $I = I_1 \cap \cdots \cap I_h = J_1 \cap \cdots \cap J_k$, where the decompositions are reduced, the I_α, I_β being tertiary right ideals. Then $h = k$ and the two sets of associated prime ideals coincide. These results can be carried out for left ideals and repeated for $2 - sided$ ideals T, using $R(T)$ instead of $r(T)$.

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