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Existence and Uniqueness Result for Boundary Value Problems Involving Capillarity Problems

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Abstract*: In this paper, we study a nonlinear boundary value problem (bvp) which generalizes capillarity problem. An existence and uniqueness result is obtained using the knowledge of range for nonlinear operator. Ours extends the result in [12].*

I. Introduction

A research on the existence and uniqueness result for certain nonlinear boundary value problems of capillarity problem has a close relationship with practical problems. Some significant work has been done on this, see Wei et al [1, 5, 2, 4, 3, 7, 10, 6]. In 1995, Wei and He [2] used a perturbation result of ranges for m-accretive mappings in Calvert and Gupta [1] to obtain a sufficient condition so that the zero boundary value problem, [1.1].

$$-\nabla_p u + g(x,u(x)) = f(x), a.e\ \ in\ \Omega$$

$$-\frac{\partial u}{\partial n} = 0,\ \ a.e\ \ in\ \Gamma,$$

has solutions in $L^P(\Omega)$, where $2 \le p < +\infty$. In 2008, as a summary of the work done in [5, 2, 4, 3, 7, 10, 6], Wei et al used some new technique to work for the following problem with so-called generalized $p - Laplacian$ operator:

(1.2)
$$-div[(c(x)+|\Delta u|^2)^{(p-2)/2}\Delta u] + \in /u/^{q-2}u + g(x,u(x)) = f(x), a.e\ in\ \Omega$$

$$-v(c(x)+|\Delta u|^2)^{\frac{(p-2)}{2}}\Delta u) \in \beta_x(u(x)),\ a.e\ in\ \Gamma$$

where $0 \le c(x) \in L^p(\Omega), \in$ is a non-negative constant and υ denotes the exterior normal derivatives of Γ. It was shown (7) that (1.2) has solutions in $L^p(\Omega)$ under some conditions where $2N/(N+1) < p \le s < +\infty, 1 \le q < +\infty\ if\ \ p \ge N,$ $and\ 1 \le q \le N_p/(N-p)\ if\ \ p < N,\ for\ N \ge 1.$ In Chen and Luo [8], the authors studied the eigenvalue problem for the following generalized capillarity equations.

(1.3)
$$-div\left[\left(1+\frac{|\Delta_u|^p}{\sqrt{1+|\Delta u|^{2p}}}\right)|\Delta u|^{(p-2)}\Delta u\right] = \lambda\left(|u|^{q-2}u+|u|^{r-2}u\right), in\ \Omega,$$

$$u = 0,\ a.e.\ on\ \partial\Omega$$

In their paper [10], Wei et al, borrowed the ideas dealing with the nonlinear elliptic boundary value problem with the generalized p-Laplacian operator to study the nonlinear generalized Capillarity equations with Neumann boundary conditions. They used the perturbation results of ranges for *m*-accretive mappings in [1] again to study.

[1.4]
$$-div\left[\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u\right] + \lambda\left(|u|^{q-2}u+|u|^{r-2}u\right) + g(x,u(x)) = f(x), a.e.\ in\ \Omega$$

$$-\left\langle \upsilon, \left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u \right\rangle \in \beta_x(u(x)), a.e\ on\ \Gamma$$

Motivated by [10, 12], we study the following boundary value problem:

(1.5)
$$-div\left[\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u\right]+\lambda\left(|u|^{q1-2}u+|u|^{q2-2}u+\ldots|u|^{qm-2}u\right)$$
$$+g(x,u(x),\nabla u(x))=f(x), a.e.\ in\ \Omega$$
$$-\left\langle \upsilon\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u\right\rangle\in\beta_x(u(x)),\ a.e\ in\Gamma$$

This equation generalized the Capillarity problem considered in [10]. We replaced the nonlinear term g(x, u(x)) by the term g(x, u(x), ∇u (x)) which is rather general. In this paper, we will use some perturbation results of the ranges for maximal monotone operators by Pascali and Shurlan [10] to prove that (1.5) has a unique solution in $W^{1,p}(\Omega)$ and later show that this unique solution is the zero of a suitably defined maximal monotone operator.

II. Preliminaries

We now list some basic knowledge we need. Let X be a real Banach space with a strictly convex dual space X^*. Using "↪" and "w-lim" to denote strong and weak convergence respectively. For any subset G of X, let intG denote its interior and G its closure. Let "X↪↪Y" denote that space X is embedded compactly in space Y and "X↪Y" denote that space X is embedded continuously in space Y. A mapping, T: D(T) = X → X^* is said to be hemi continuous on X if $w-\lim_{t\to 0}T(x+ty)=Tx,$ for any x,y $\in$ X Let J denote the duality mapping from X into 2^x, defined by

(2.1)
$$f(x)=f\in x^{\bullet}:(x,f)=\|x\|.\|f\|,\|f\|=\|x\|,x\in X$$

where (….) denotes the generalized duality paring between X and X* Let A: X → 2^x be a given multi-valued mapping. A is boundedly-inversely compact if for any pair of bounded subsets G and G' of X, the subset $G\cap A^{-1}(G')$ is relatively compact in X.

The mapping A: X → 2^x is said to be accretive if $((\upsilon_1-\upsilon_2),J(u_1-u_2))\geq 0,$ for any $ui\in D(A)\ and\ \upsilon i\in Au_i;\ i=1,2.$

The accretive mapping A is said be m-accretive if $R(1+\mu A)=X,$ for some $\mu>0$.

Let B: X → 2^{X^*} be a given multi-valued mapping, the graph of B, G(B) is defined by G(B) = $\{[u,w]\mid\mu\in D(B),\ w\in Bu\}.,\ B:X\to 2^{X^*}$ is said to be monotone [11] if G(B) is a monotone subset of X x X^* in the sense that

(2.2)
$$(u_1-u_2,w_1-w_2)\geq 0,\ for\ any\ [u_i,w_i]\in G(B);i=1,2.$$

The monotone operator B is said to be maximal monotone if G(B) is maximal among all monotone subsets of X x X^* in the sense of inclusion the mapping B is said to be strictly monotone if the equality in (2.2) implies that $u_1=u_2$. The mapping B is said to be coercive if

$$\lim_{n\to+\infty}((x_n,x_n^*)/\|x_n\|)=\infty\ for\ all\ [x_n,x_n^*]\in G(B)\ such that\ \lim_n\to+\infty\|x_n\|=+\infty.$$

Definition 2.1. The duality mapping $J:X\to 2^{X^*}$ is said to be satisfying condition (1) if there exists a function $\eta:X\to[0,+\infty]$ such that

(2.3) $\|Ju-J\upsilon\|\leq\eta(u-\upsilon),$ for all u, $\upsilon\in X$.

Definition 2.2. Let A: X $\to 2^X$ be an accretive mapping and $J:X\to X^*$ be a duality mapping. We say that A satisfies condition (*) if, for any $f\in R(A)\ and\ a\in D(A)$ and $a\in D(A),$ there exists a constant C(a, F) such that

(2.4)
$$(\upsilon-f,J(u-a))\geq C(a,f),\ for\ any\ u\in D(A),\ \upsilon\in Au.$$

Lemma 2.3. (Li and Guo) Let Ω be a bounded conical domain in R^N. Then we have the following results;

(1) If $mp>N\ then\ W^{m,p}(\Omega)$ ↪ $C_B(\Omega);if\ mp<N\ and\ q=Np/(N-mp),\ then\ W^{m,p}(\Omega)$ ↪ $L^q(\Omega);if\ mp=N,\ and\ p>1,\ then\ for\ 1\leq q<+\infty,\ W^{m,p}(\Omega)\to L^q(\Omega)$

(2) $If\ mp > N\ then\ W^{m,p}(\Omega) \hookrightarrow \hookrightarrow C_B(\Omega);\ if\ 0 < mp \le N\ and\ qo = Np/(N - mp),$ then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), 1 \le q < q0;$

Lemman 2.4. (Pascali and Sburlan [11]) if B: X $\to 2^{X^*}$ is an everywhere defined, monotone and hemi continuous operator, then B is maximal monotone.

Lemman 2.5. (Pascali and Sburlan [11]) if B: X $\to 2^{X^*}$ is maximal monotone and coercive, then R(B) = X^*

Lemman 2.6. (Pascali and Sburlan [11]) if Φ : X $\to (-\infty, +\infty)$ is a proper, convex and lower semi continuous function, then $\partial\Phi$ is maximal monotone from X to X^*.

Lemman 2.7. [11]. If B_1 and B_2 are two maximal monotone operators in X such that (int D $(B_{1)}) \cap D(B_2) \neq \phi$, then B_1 + B is maximal monotone.

Lemman 2.8. (Calvert and Gupta [1]). Let X = $L^P(\Omega)$ and Ω be a bounded in $\Re^N$. For $2 \le p < +\infty$, the duality mapping J_P: $L^P(\Omega) \to L^{P'}(\Omega)$ defined by $J_p u = |u|^{p-1} \operatorname{sgn}\ u \|u\|_p^{2-p},\ for\ u \in L^P(\Omega)$,satisfies condition (2.4); for 2N/(N + 1) < p $\le$ 2 and N ≥ 1, the duality mapping J_P: $L^P(\Omega) \to L^{P'}(\Omega)$ defined by $J_P u = |u|^{p-1} \operatorname{sgn}\ u,\ for\ u \in L^p(\Omega),\ satisfies\ condition\ (2.4),\ where\ (1/p) + (1/p') = 1$

III. Main Result

3.1 Notations and Assumptions of (1.5). We assume in this paper, that $2N/(N+1) < p < +\infty, 1 \le q1, q2; ..., qm < +\infty\ if\ p \ge N,\ and\ 1 \le q1, q2, ...qm \le Np/(N-p)\ if\ p < N,\ where\ N \ge 1.$ We use $\|.\|p', \|q'_1, \|.\|q'_2, ..., \|\ .\ \|\ q'_m\ and\ \|.\|_{1,p,\Omega}$ to denote the norms in $L^p(\Omega), L^{q1}(\Omega), L^{q2}(\Omega), ..., L^{qm}(\Omega)$ and $W^{1,P}(\Omega)$ respectively. Let (1/p) + (1/p$^/$) = 1, (1/q1) + (1/q$^/_1$) = 1, (1/q$_2$) + (1/q$^/_2$) = 1,…, (1/$_{qm}$) + (1/$^/_{q\,m}$) = 1

In (1.5), Ω is a bounded conical domain of a Euclidean space $\Re^N$ with its boundary $\Gamma \in C^1, (c.f. [4]).$

Let $|.|$ denote the Euclidean norm in $\Re^N$, $\langle .,. \rangle$ the Euclidean inner-product and υ the exterior normal derivative of Γ. λ is a nonnegative constant.

Lemman 3.1 Defining the mapping $B_{p,\,q1,\,q2,\,\dots qm}$: $W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$(\upsilon, B_{p,q1,q2,...qm} u) = \int_\Omega \left\langle \left(1 + \left(\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u, \nabla u\right)\right\rangle dx$$
$$+ \lambda \int_\Omega |u(x)|^{q1-2} u(x)\upsilon(x)dx + \lambda \int_\Omega |u(x)|^{q2-2} u(x)\upsilon(x)dx$$
$$+ ... + \lambda \int_\Omega |u(x)|^{qm-2} u(x)\upsilon(x)dx$$

for any $u, \upsilon \in W^{1,p}(\Omega)$. Then $B_{p,q1,q2,\dots qm}$ is everywhere defined, strictly monotone, hemi continuous and coercive.

The proof of the above lemma will be done in four steps.

Proof. Step 1: $B_{p,q1,q2\dots,qm}$ is everywhere defined.

From lemma 2.3, we know that $W^{1p}(\Omega) \hookrightarrow C_B(\Omega)$, when p > N. Also, $W^{1,p}(\Omega) \hookrightarrow L^{q1}(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^{q2}(\Omega)$,…, $W^{1,p}(\Omega) \hookrightarrow L^{qm}(\Omega)$, when p $\le N$.

Thus, for all $u, \upsilon \in W^{1,p}(\Omega), \|\upsilon\|_{q1} \le k_1 \|\upsilon\|_{1,p,\Omega}, \|\ \upsilon\ \|_{q2} \le k_2 \|\upsilon\|_{1,p,\Omega}, ..., \|\upsilon\|_{qm} \le k_m \|\upsilon\|_{1,p,\Omega}$

where $k_1, k_2, \dots, k_m$ are positive constants.

For $u, \upsilon \in W^{1,p}(\Omega)$, we have

$$|(\upsilon, B_{p,q1,q2,...,qm}\ u)| \le 2\int_\Omega |\nabla u|\ p-1 |\nabla \upsilon|\ dx + \lambda \int_\Omega |\mu|\ q1-1 |\upsilon|\ dx + \lambda \int_\Omega |\mu|\ q2-1 |\upsilon|\ dx + ... + \lambda \int_\Omega |\mu|\ qm-1 |\upsilon|\ dx$$

$$\leq\ 2\|\nabla u\|_p^{p/p'}\|\nabla \upsilon\|_p+\lambda\|\upsilon\|q1\|u\|_{q1}^{q1/q'1}+\lambda\|\upsilon\|_{q2}\|u\|_{q2}^{q2/q'2}+...+\lambda\|\upsilon\|_{qm}\|u\|_{qm}^{qm/q'm}$$

$$\leq\ 2\|u\|_{1,p,\Omega}^{p/p'}\|\upsilon\|_{1,p,\Omega}+k'_1\lambda\|\upsilon\|_{1,p,\Omega}\|u\|_{1,p,\Omega}^{q1/q'1}+k'_2\lambda\|\upsilon\|_{1,p,\Omega}\|u\|_{1,p,\Omega}^{q2/q'2}$$
$$+\ldots+k'_m\lambda\|\upsilon\|_{1,p,\Omega}\|u\|_{1,p,\Omega}^{qm/q'm}$$

Where $k'_1, k'_2, \ldots k'_m$ are positive constants. Thus $B_{p, q1, q2, \cdots qm}$ is everywhere defined.

Step 2: $B_{p,q1,q2,\ldots,qm}$ is strictly monotone

For $u,\ \upsilon\in W^{1,p}(\Omega)$, we have

$$\left|\left(u-\upsilon,\ B_{p,q1,q2,...,qm}u\right.\right.$$
$$\left.\left.-B_{p,q1,q2,...,qm}\upsilon\ \right]\right|$$

$$=\int_\Omega\left\langle\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u-\left(1+\frac{|\nabla\upsilon|^p}{\sqrt{1+|\nabla\upsilon|^{2p}}}\right)|\nabla\upsilon|^{p-2}\nabla\upsilon,\nabla u-\nabla\upsilon\right\rangle dx$$
$$+\ \lambda\int_\Omega\left(|\mu|q1-2_u-|\upsilon|^{q1-2}{}_\upsilon\right)(u-\upsilon)dx+\lambda\int_\Omega\left(|u|^{q2-2}u-|\upsilon|^{q2-2}\upsilon\right)(u-\upsilon)dx$$
$$+\ldots+\lambda\int_\Omega\left(|u|^{qm-2}u-|\upsilon|^{qm-2}\upsilon\right)(u-\upsilon)dx$$

$$=\int_\Omega\left\{\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^p-\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\nabla u\right\}$$
$$-\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla\upsilon|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\nabla\upsilon+\left(1+\frac{|\nabla\upsilon|^p}{\sqrt{1+|\nabla\upsilon|^{2p}}}\right)|\nabla\upsilon|^p\ \}dx$$
$$+\ \ldots+\lambda\int_\Omega\left(|\mu|qm-2_u-|\upsilon|^{qm-2}\upsilon\right)(u-\upsilon)dx$$

$$\geq\int_\Omega\left\{\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-1}-\left(1+\frac{|\nabla\upsilon|^p}{\sqrt{1+|\nabla\upsilon|^{2p}}}\right)|\nabla\upsilon|^{p-1}\right\}(|\nabla u|-|\nabla u|)dx$$
$$+\ \lambda\int_\Omega\left(|u|q1-1-|\upsilon|^{q1-1}\right)(|u|-|\upsilon|)dx+\lambda\int_\Omega\left(|u|^{q2-1}-|\upsilon|^{q2-1}\right)(|u|-|\upsilon|)dx$$
$$+\ \ldots+\lambda\int_\Omega\left(|u|qm-1_u-|\upsilon|^{qm-1}\right)(|u|)-|\upsilon|)dx$$

If we let $h(t)=\left(1+\frac{t}{\sqrt{(1+t^2)}}\right)t^{(p-1)/p},\ for\ t\geq 0.\ \ Then$

(3.1) $$h'(t)=\frac{t^{(p-1)/p}}{\left(1+t^2\right)^{3/2}}+t^{-(1/p)}\left(1+\frac{t}{\sqrt{1+t^2}}\right)\frac{p-1}{p}\geq 0,$$

Since $t\geq 0$. And, h'(t) = 0 if and only if t = 0. Then h(t) is strictly monotone. Thus we can say that $B_{p,q1,q2\ldots,qm}$ is strictly monotone

Step 3: $B_{p,q1,q2,\ldots qm}$ is hemi continuous

Need to show here that, for any

$u,\ v,\ w\in W^{1,p}(\Omega)\ and\ t\in[0,1],\ \left(w,\ B_{p,q1,q2}...,qm(u+t\upsilon)-B_{p,q1,q2,...qm}u\right)\to 0$ as $t\to 0$.

By Lebesgue's dominated convergence theorem, it follows that

$$0\leq \lim_{t\to 0}\left|(w, B_{p,q1,q2,\dots,qm}(u+t\upsilon)-B_{p,q1,q2,\dots,qm}u)\right|$$

$$\leq \int_{\Omega}\lim_{t\to 0}\left|(w, B_{p,q1,q2,\dots,qm}(u+t\upsilon)-B_{p,q1,q2,\dots,qm}u)\right|$$

$$\leq \int_{\Omega}\lim_{t\to 0}\left|\left(1+\frac{|\nabla u+t\nabla\upsilon|^{p}}{\sqrt{1+|\nabla u+t\nabla\upsilon|^{2}p}}\right)\right||\nabla u+t\nabla\upsilon|^{p-2}(\nabla u-t\nabla\upsilon$$

$$-\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2}p}}\right)|\nabla u|^{p-2}\nabla u \mid |\nabla w|dx+\lambda\int_{\Omega}\lim_{t-0}\|u+t\upsilon|^{q1}$$

$$+\lambda\int_{\Omega}\lim_{t-0}\|u+t\upsilon|^{q2-2}(u+t\upsilon)-|u|^{q2-2}\|w|dx+\dots+\lambda\int_{\Omega}\lim_{t-0}$$

$$=0$$

Therefore $B_{p,q1,q2,\dots qm}$ is hemi continuous

Step 4: $B_{p,q1,q2,\dots qm}$ is coercive

For $u\in W^{1,p}(\Omega)$, Lemma 2.4 implies that $\|u\|_{1,p,\Omega}\to\infty$ is equivalent to $\left\|u-(1/meas(\Omega))\int_{\Omega}udx\right\|_{1,p,\Omega}\to\infty$ and hence we have the following result:

$$\frac{(u,\ B_{p,q1,q2,..qm}u)}{\|u\|_{1,p,\Omega}}=\frac{\int_{\Omega}\left(1+(|\nabla|)^{p}/\sqrt{1+|\nabla u|^{2p}}\ \right)|\nabla u|^{p}dx}{\|u\|_{1,p,\Omega}}+\frac{\int_{\Omega}|u|^{q1}dx}{\|u\|_{1,p,\Omega}}$$

$$+\ \lambda\frac{\int_{\Omega}|u|^{q2}dx}{\|u\|_{1,p,\Omega}}+\dots+\lambda\frac{\int_{\Omega}|u|^{qm}dx}{\|u\|_{1,p,\Omega}}$$

$$=\frac{\int_{\Omega}\left(|\nabla u|^{p}+\sqrt{1+|\nabla u|^{2p}}\right)dx-\int_{\Omega}\left(\sqrt{1/|\nabla u|^{2p}}\right)dx}{\|u\|_{1,p,\Omega}}$$

$$+\ \lambda\frac{\int_{\Omega}|u|^{q1}dx}{\|u\|_{1,p,\Omega}}+\lambda\frac{\int_{\Omega}|u|^{q2}dx}{\|u\|_{1,p,\Omega}}+\dots+\lambda\frac{\int_{\Omega}|u|^{qm}dx}{\|u\|_{1,p,\Omega}}$$

$$\geq\frac{2\int_{\Omega}|\nabla u|^{p}dx-\int_{\Omega}\left(1/\sqrt{1+|\nabla u|^{2p}}\right)dx}{\|u\|_{1,p,\Omega}}+\lambda\frac{\int_{\Omega}|u|^{q1}dx}{\|u\|_{1,p,\Omega}}$$

$$+\ \lambda\frac{\int_{\Omega}|u|^{q2}dx}{\|u\|_{1,p,\Omega}}+\dots+\lambda\frac{\int_{\Omega}|u|^{qm}dx}{\|u\|_{1,p,\Omega}}\to+\infty,$$

as $\|u\|_{1,p,\Omega}\to+\infty,$ which implies that $B_{p,q1,q2,\dots,qm}$ is coercive

This completes the proof.

Definition 3. 2. Define a mapping A_p: $L^{p}(\Omega)\to 2^{Lp(\Omega)}$ as follows:

$D(A_p)=\{u\ \in L^{p}(\Omega)\mid there\ exist\ \ an\ \ f\ \in L^{p}(\Omega),\ such\ that\ \ f\ \in B_{p,q1,q2,\dots,qm}u+\partial\Phi_{p}(u)\}$

for $u\in D(A_{p}),\ let\ \ A_{p}u=\{f\ \ \in L^{p}(\Omega),\ such\ that\ \ f\ \in B_{p,q1,q2,\dots,qm}u+\partial\Phi_{p}(u)\}$

Definition 3.3.: The mapping

$A_{p}: L^{p}(\Omega)\to 2^{Lp(\Omega)}\ \ is\ \ m-\ accretive.$

Proof. (1) A_p is accretive

(a) Case 1:

If p $\geq$ 2, the duality mapping J_p: $L^{p'}(\Omega)$ is defined by $J_p u = |u|^{p-1}\operatorname{sgn} u\|u\|_p^{2-p}$ for $u \in L^p(\Omega)$.It then suffices to prove that for any $u_i \in D(A_p)$ and $\upsilon_i \in A_p u_i, i = 1,2,$

$$(\upsilon_1 - \upsilon_2,\ J_p(u_1 - \upsilon_2)) \geq 0$$

To do this, we are left to prove that both

$$\left(|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)\|u_1 - u_2\|_p^{2-p}, B_{p,q1,q2,\ldots,qm}u_1 - B_{p,q1,q2,\ldots,qm}u_2\right) \geq 0,$$

$$\left(|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)\right)\|u_1 - u_2\|_p^{2-p}, \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geq 0,$$

are available.

Now, take for a constant k > 0, $X_k: R \to R$ is defined by $X_k(t) = |t \Lambda k \vee (-k)^{p-1}\operatorname{sgn} t\ \|$ Then X_k is monotone, Lipschitz with $X_k(0) = 0$ and X_k is continuous except at finitely many points on R.

This gives that

$$\left(|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)\right)\|u_1 - u_2\|_p^{2-p}, \partial\Phi_p(u_1) - \partial\Phi_p(u_2))$$

$$= \lim_{k\to+\infty}\|u_1 - u_2\|_p^{2-p}\left(X_k(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right)$$

$$\geq 0,$$

Also

$$G = \|u_1 - u_2\|_p^{2-p}$$

$$\times \lim_{k\to+8}\int_\Omega\left\langle\left(1 + \frac{|\nabla u_1|^p}{\sqrt{1 + |\nabla u_1|^{2p}}}\right)|\nabla u_1|^{p-2}\nabla u_1 - \left(1 + \frac{|\nabla u_2|^p}{\sqrt{1 + |\nabla u_2|^{2p}}}\right)|\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\right\rangle$$

$$\times\ X'\kappa(u_1 - u_2)dx + \lambda\|u_1 - u_2\|_p^{2-p}\int_\Omega\left(|u_1|^{q1-2}u_1 - |u_2|^{q1-2}u_2\right)|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)dx$$

$$+ \lambda\|u_1 - u_2\|_p^{2-p}\int_\Omega\left(|u_1|^{q2-2}u_1 - |u_2|^{q2-2}u_2\right)|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)dx$$

$$+ \ldots + \lambda\|u_1 - u_2\|_p^{2-p}\int_\Omega\left(|u_1|^{qm-2}u_1 - |u_2|^{qm-2}u_2\right)|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)dx \geq 0$$

where

$$G = \left(|u_1 - u_2|^{p-1}\operatorname{sgn}(u_1 - u_2)\|u_1 - u_2\|_p^{2-p}, B_{p,q1,q1,\ldots,qm}u_1 - B_{p,q1,q2,\ldots,qm}u_2\right)$$

The last inequality is available since X_k is monotone and $X_k(0) = 0$

(b) Case 2

If 2N/(N+ 1) < p < 2, the duality mapping J_p: $L_p(\Omega) \to L_{p'}(\Omega)$ is defined by

$J_p(u) = |u|^{p-1}\operatorname{sgn} u$,

for $u \in L^p(\Omega)$. It then suffices to prove that for any $u_i \in D(A_p)$ *and* $\upsilon_i \in A_p u_i, i = 1,2$

$$(\upsilon_1 - \upsilon_2,\ J_p(u_1 - u_2)) \geq 0$$

To do this, we define the function X_n : R → R by

$$X_n(t) = \begin{cases} |t|^{p-1}\operatorname{sgn} t, \text{ if } |t| \geq \frac{1}{n} \\ \left(\frac{1}{n}\right)^{p-2} t, \text{ if } |t| \leq \frac{1}{n} \end{cases} \quad (3.2)$$

Then X_n is monotone, Lipschitz with $X_n(0) = 0$ and X'_n is continuous except at finitely many points on R. So $\left(X_n(u_1-u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right) \geq 0.$

Then, for $u_i \in D(A_p)$ and $\upsilon_i \in A_p u_{i,i} = 1,2.$ We have

$$\begin{aligned}(\upsilon_1-\upsilon_2,\ J_p(u_1-u_2)) &= \left(|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2), B_{p,q1,q2,...qm}u_1 - B_{p,q1,q2,...qm}u_2\right)\\ &+ \left(|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right)\\ &= \left(|u_1-u_2|^{p-1}\operatorname{sgn}(u_1-u_2), B_{p,q1,q2,...qm}u_1 - B_{p,q1,q2,...qm}u_2\right)\\ &+ \lim_{n\to\infty}\left(x_n(u_1-u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)\right) \geq 0\end{aligned}$$

Step 2 R $(l + \mu A_p) = L^p(\Omega)$, *for every* $\mu > 0.$

We first define mapping $l_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by $l_p u = u$ and $(\upsilon, I_p u)(W^{1,p}(\Omega))^* \times (W^{1,p}(\Omega) - (\upsilon,u)\upsilon(\Omega)$ for $u, \upsilon \in W^{1,p}(\Omega)$, where $\langle .,. \rangle_{L^2(\Omega)}$ denotes the inner product of $L^p(\Omega)$. The l_p is maximal monotone [7].

Secondly, for any $\mu > 0$, let the mapping $T_\mu : W^{1,p}(\Omega) \to 2^{(W1,p(\Omega))^*}$ be defined by

$$T\mu u = I_p \mu + \mu B_{p,q1,q2,...qm} u \mu \partial\Phi_p(u),$$

for $u \in W^{1,p}(\Omega)$. Then similar to that in [7], by lemmas 2.4, 2.6, 2.7 and 2.5 we see that T_μ is maximal monotone and coercive, so that $R(T_\mu) = (W^{1,p}(\Omega))^*$, *for any* $\mu > 0$

Therefore, for any $f \in L^p(\Omega)$, there exists $u \in W^{1,p}(\Omega)$, such that

$$f = T_\mu\ u = u + \mu B_{p,q1,q2,...qm}\ u\mu\partial\Phi_p(u) \tag{3.3}$$

From the definition of A_p. it follows that $R(I + \mu A_p) = Lp(\Omega)$, *for all* $\mu > 0.$ This completes the proof.

Lemma 3.4. The mapping A_p: $L^p(\Omega) \to 2L^p(\Omega)$, has a compact resolvent for $2N/(N + 1) < p < 2$ and $N \geq 1.$

Proof. Since A_p is m = accretive, we need to show that if $u + \mu A_p u = f(\mu > 0)$ and if $\{f\}$ is bounded in $L^p(\Omega)$, then $\{u\}$ is relatively compact in $L^p(\Omega)$. Now defined functions $X_n, \varsigma_n : R \to R$ by

$$X_n(t) = \begin{cases} |t|^{p-1}\operatorname{sgn} t, \text{ if } |t| \geq \frac{1}{n}, \\ \left(\frac{1}{n}\right)^{p-2} t, \text{ if } |t| \leq \frac{1}{n} \end{cases}$$

$$\zeta_n(t) = \begin{cases} |t|^{2-(2/p)}\operatorname{sgn} t, \text{ if } |t| \geq \frac{1}{n}, \\ \left(\frac{1}{n}\right)^{1-(2/p)} t, \text{ if } |t| \leq \frac{1}{n} \end{cases}$$

Noticing that $X'n(t) = (p-1) \times (p'/2)^p \times (\zeta' n(t))^p$, *for* $|t| \geq 1/n$, where $(1/p) + 1/p') = 1$ and $X_n(t) = (\zeta_n(t))^p$, for $|t| \leq 1/n.$ We know that $(X_n(u), \partial\Phi_p(u)) \geq 0$ for $u \in W^{1,p}(\Omega)$ since X_n is monotone, Lipschitz with $X_n(0)$ 0 and $X'n$ is continuous except at finitely many points on R.

$$\begin{aligned}\left(|U|^{P-1}\operatorname{sgn} u,\ A_p u\right) &= \lim_{n\to\infty}(X_n(u), A_p u) \geq \lim_{n\to\infty}(X_n(u), B_{p,q1,q2,...qm}u)\\ &= \lim_{n\to}\int_\Omega \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^p X'_n(u)dx + \lambda \lim_{n\to\infty}\int_\Omega |u|^{q1-2} uXn(u)dx\\ &+ \lambda \lim_{n\to\infty}\int_\Omega |u|^{q2-2} uXn(u)dx + ... + \lambda \lim_{n\to\infty}\int_\Omega |u|^{qm-2} uXn(u)dx\end{aligned}$$

$$\geq \lim_{n\to} \int_{\omega} |\nabla u|^{p} X'n(u)dx$$
$$\geq const. \lim_{n\to\infty} \int_{\Omega} |grad|(\varsigma(u))|^{p} dx$$
$$\geq const. \int_{\Omega} |grad(|u|^{2-(2/p)} \operatorname{sgn} u)|^{p} dx$$

From $f = u + \mu A_p u,$ we have;

$$\|f\|_p \left\| |u|^{2-(2/p)} Sgn\ u \right\|_{p^2/2(p-1)}^{p^2/2(p-1)p'} \geq \left(|u|^{p-1} \operatorname{sgn}\ u,\ f\right) = \left(|u|^{p-1} \operatorname{sgn}\ u, u\right) + \mu\left(|u|^{p-1} \operatorname{sgn}\ u,\ A_p u\right)$$
$$\geq \left\| |u|^{2-(2/p)} Sgn\ u \right\|_{p^2/2(p-1)}^{p^2/2(p-1)p'} + \mu.const. \left\| grad|u|^{2-(2/p)} Sgn\ u \right\|_{p'}^{p}$$

Which gives that

$$\left\| |u|^{2-(2/p)} Sgn\ u \right\|_{p}^{p/2(p-1)} \leq \left\| |u|^{2-(2/p)} Sgn\ u \right\|_{p^2/2(p-1)}^{p/2(p-1)} \|f\|_p$$
$$\leq const.$$

in view of the fact that $p < \dfrac{p^2}{2(p-1)}$ when 2N (N+1) < p < 2 for $N \geq 1.$ Again we have that,

$$\left\| grad\left(|u|^{2-(2/p)} \operatorname{sgn}\ u\right) \right\|_p \leq const.$$

Hence, $\{f\}$ bounded in $L^p(\Omega)$ implies that $\left\{|u|^{2-(2/p)} \operatorname{sgn}\ u\right\}$ is bounded in $W^{1,p}(\Omega)$

We notice that $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{p^2/2(p-1)}(\Omega)$ when N$\geq$ 2 and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when N = 1 by lemma (3.1), therefore $\left\{|u|^{2-(2/p)} \operatorname{sgn}\ u\right\}$ is relatively compact in $L^{p^2/2(p-1)}(\Omega)$. This gives that $\{u\}$ is relatively compact in $L^p(\Omega)$ since the Nemytskii mapping $u \in L^{p/2(p-1)}(\Omega) \to |u|^{p^{2/2(p-1)}} \operatorname{sgn}\ u \in L^p(\Omega)$ is continuous.

This completes the proof.

Remark 3.5. Since $\Phi_p(u+a) = \Phi_p(u)$, for any $u \in W^{1,p}(\Omega)$ and $\alpha \in C_0^\infty(\Omega)$, we have $f \in A_p u$ implies that $f = B_{p,q1,q2,\ldots,qm}$ in the sense of distributions.

Proposition 3.6. For $f \in L^p(\Omega),$ if there exists $u \in L^p(\Omega)$ such that $f \in A_p u$, then u is the unique solution of (1.7).

Proof. First we show that

$$-div\left[\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u\right] + \lambda\left(|u|^{q1-2}u + |u|^{q2-2}u + \ldots + |u|^{qm-2}u\right) = f(x), a.e.x\ in\ \Omega$$

is available.

$f \in A_p u$ implies that $f = B_{p,q1,q2,\ldots,qm}\ u + \partial\Phi_p(u).$ For all $\varphi \in C_0^\infty(\Omega)$, by remark (3.12), we have;

$$(\varphi, f) = \left(\varphi, B_{p,q1,q2,\ldots,qm}u + \partial\Phi_p(u)\right)$$
$$= \left(\varphi, B_{p,q1,q2,\ldots,qm}u\right) = \int_{\Omega} \left\langle \left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u, \nabla\varphi \right\rangle dx$$
$$+ \lambda \int_{\Omega} |u|^{q1-2} u\varphi dx + \lambda \int_{\Omega} |u|^{q2-2} u\varphi dx + \ldots + \lambda \int_{\Omega} |u|^{qm-2} u\varphi dx$$

$$= \int_{\Omega} -div\left[\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u\right]\varphi dx$$

$$+\lambda\int_{\Omega}{}^{|u|q1-2u\varphi dx} + \lambda\int_{\Omega}{}^{|u|q2-2\ u\varphi dx+...+} \lambda\int_{\Omega}{}^{|u|qm-2\ u\varphi dx}$$

which implies that (3.25) is true.
Secondly, we show that

(3.4) $$-\left\langle \upsilon, \left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|}^{2p}}\right)|\nabla u|^{(p-2)}\nabla u, \right\rangle \in \beta_x(u(x)),\ a.e\ x\in\Gamma$$

This will be proved under the condition that $|\beta_x(u)| \le a|u|^{p/p'} + b(x),$ where $b(x) \in L^{p'}(\Gamma)$ and $a \in R.$ From (3.25), $f \in A_p u$ implies that

$$f(x) = -div\left[\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u\right] + \lambda|u|^{q1-2}u + \lambda|u|^{q2-2}u + ... + \lambda|u|^{qm-2}u \in L^{p}(\Omega).$$

Using Green's Formula, we have that for any $\upsilon \in W^{1,p}(\Omega),$

$$\int_{\Gamma}\left\langle \upsilon, \left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u, \right\rangle \upsilon|_{\Gamma}\, d\Gamma(x)$$

$$= \int_{\Omega} div\left[\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u\ |^{(p-2)}\nabla u\right]\upsilon d(x)$$

$$+\int_{\Omega}\left\langle \left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u\ |^{(p-2)}\nabla u, \nabla\upsilon\right\rangle d(x)$$

Then

$$-\left\langle \upsilon, \left(1+|\nabla u|^{p}/\sqrt{1+|\nabla u|^{2p}}\right)|\nabla u|^{(p-2)}\nabla u, \right\rangle \in W^{(1/p)p'}(\Gamma) = \left(W^{1/p.p'}(\Gamma)\right)^{*},$$

where $W^{1/p.p}(\Gamma)$ is the space of races of $W^{1,p}(\Omega)$. Let the mapping $B: L^{p}(\Gamma) \to L^{p'}(\Gamma)$ be defined by Bu = g(x), for any $u \in L^{p}(\Gamma)$, where g(x) = $\beta_x(u(x))$ a.e. on Γ. Clearly, $B = \partial\psi$ where

$$\psi(u) = \int_{\Gamma}{}^{\varphi x(u(x))d\Gamma(x)}$$

is a proper, convex and lower-semi continuous function on $L^{p}(\Gamma)$

Now define the mapping K: $W^{l,p}(\Omega) \to L^{p}(\Gamma)$ by

$$K(\upsilon) = \upsilon/\Gamma \text{ for any } \upsilon \in W^{l,p}(\Omega)$$

Then

$$K^{*}BK : W^{l,p}(\Omega) \to \left(W^{l,p}(\Omega)\right)^{*}$$

is maximal monotone since both K, B are continuous. Finally, for any $u,\ \upsilon \in W^{l,p}(\Omega)$, we have

$$\psi(K\upsilon) - \psi(Ku) \quad = \quad \int_{\Gamma}{}^{[\varphi_x(\upsilon/\Gamma(x)) - \varphi_x(u|\Gamma(x))]d\Gamma(x)}$$

$$\geq \int_{\Gamma} \beta_x (u|\Gamma(x)(\upsilon|\Gamma(x)) - u|\Gamma(x\ \ ))d\Gamma(x)$$
$$= (BKu, K\upsilon - Ku) = (K^{*}BKu, \upsilon - u).$$

Hence we get that K $_*$ BK $\subset \partial\Phi_p$ and so K * BK $= \partial\Phi_p$. Therefore, we have that

$$-\left\langle \upsilon\left(1+\frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{(p-2)}\nabla u, \right\rangle \in \beta_x(u(x)),\ a.e \in \Gamma$$

Next we show that u is unique.

If $f \in A_p u\ and\ f \in A_p\upsilon$, where $u, \upsilon \in D(A_p)$ then

(3.5) $$0 \leq (u-\upsilon, B_{p,q1,q2,\ldots,qm}\ u - B_{p,q1,q2,\ldots,qm}\upsilon)$$

(3.6) $$= (u-\upsilon, \partial\Phi_p(\upsilon) - \partial\Phi_p(u)) \leq 0$$

$B_{p,q1,q2,\ldots,qm}$ being strictly monotone and $\partial\Phi_p$ maximal monotone, implies that $u(x) = \upsilon(x)$. This completes the proof.

Remark 3.7. If $B_x = 0\ for\ any\ x \in \Gamma\ then\ \partial\Phi_p(u) \equiv 0,\ for\ all\ u \in W^{1,p}(\Omega)..$

Proposition 3.8. *If* $B_x \equiv 0\ \ for\ any\ x \in \Gamma\ then\ \{f \in L^p(\Omega)\ |\}\int_{\Omega} fdx = 0\ \} \subset R(A_p)$.

Proof. In view of lemmas 2.4, 2.5 and 3.1 we note that $R(B_{p,q1,q2,\ldots,qm}) = (W^{1,p}(\Omega))^*$. Note also that for any $f \in L^p(\omega)\ with \int_{\Omega} fdx = 0,$ the linear function $u \in W^{1,p}(\omega) \to \int_{\Omega} fudx$ is an element of $(W^{1,p}(\Omega))^*.$ So there exists a $u \in W^{1,p}(\Omega)$ such that

$$+\ \ \lambda\int_{\Omega} |u|q1^{-2}u\upsilon dx + \lambda\int_{\Omega} |u|q2^{-2}u\upsilon dx + \ldots + \lambda\int_{\Omega} |u|qm^{-2}u\upsilon dx$$

for any $\upsilon \in W^{1,p}(\Omega)$. Therefore, $f = A_p u$ in view of Remark 3.12. This completes the proof.

Definition 3.9. (see [1, 7]). For $t \in R_t,\ x \in \Gamma, let\ B_x^0(t) \in B_x(t)$ be the element with least absolute value if $\beta_x(t) \neq 0\ and\ \beta_x^0(t) = \pm\infty,\ where\ t > 0\ or\ t < 0$ respectively, in case $B_x(t) = \emptyset.\ finally, let\ \beta_x(t) = \lim_{t\to\infty}\beta_x^0(t)$ (in the extended sense) for $x \in \Gamma.\beta_x(t)$ define measurable functions on Γ, in view of our assumptions on β_x.

Proposition 3.10. Let $f \in L^p(\Omega)$ such that

(3.7) $$\int_{\Gamma} \beta-(x)d\Gamma(x) < \int_{\Omega} fdx < \int_{\Omega} \beta+(x)d\Gamma(x)$$

Then $f \in IntR(A_p)$.

Proof. Let $f \in L^p(\Omega)$ and satisfy (3.31), by proposition 3.5, there exists $u_n \in L^p(\Omega)$ such that, for each $n \geq 1,\ f = (1/\upsilon)u_n + A_p u_n$. In the same reason as that in [1], we only need to prove that $\|u_n\|_p <$ const for all $n \geq 1.$

Indeed suppose to the contrary that $1 \leq \|u_n\|_p \to \infty$. Let $\upsilon_n = \frac{u_n}{\|u\|_p}$. *Let* ψ: R $\to$R be defined by ψ (t) = $|t| = |t|^p, \partial\psi : R \to R$ be its sub differential and for $\mu > 0, \partial\psi_u : R \to R$ denote the Yosida-approximation of $\partial\psi$. Let $\theta_u: R \to R$ denote the indefinite integral of $[(\partial\psi\mu)']^{1/p}$ with $\theta_u(0) =$ 0 so that $(\theta'_p)^p = (\partial\psi_\mu)'$. In view of [1] we have

(3.8) $$\left(\partial\psi_\mu(\upsilon n),\partial\Phi(u_n)\right)\geq\int_\Gamma\beta_x\left((1+\mu\partial\psi)^{-1}(u_n/\Gamma(x))\right)x\,\partial\psi_\mu(\upsilon_n/\Gamma(x))d\Gamma(x)\geq 0.$$

Now multiplying the equation $f=(1/n)u_n+A_p u_n \; by \; \partial\psi_u(u_n),$ we get that

(3.9)

$$\left(\partial\psi_u(\upsilon_n),f\right) = \left(\partial\psi_u(\upsilon_n),\frac{1}{n}u_n\right)+\left(\partial\psi_\mu(\upsilon_n),B_{p,q1,q2\ldots,qm}u_n\right)+\left(\partial\psi_\mu(\upsilon_n),\partial\phi_p(\mu_n)\right).$$

Since $\partial\psi_\mu(0)=0,$ it follows that $\left(\partial\psi_\mu(\upsilon_n),u_n\right)\geq 0.$ Also we have that

$$\left(\partial\psi_\mu(\upsilon_n),B_{p,q1,q2,\ldots,qm}u_n\right) =$$

$$\int_\Omega\left\langle\left(1+\frac{|\nabla u_n|^p}{\sqrt{1+|\nabla u_n|^{2p}}}\right)|\nabla u_n|^{(p-2)}\nabla u_n,\nabla\upsilon_n,\right\rangle(\partial\psi_\mu)'\;(\upsilon_n)dx$$

$$+\lambda\int_\Omega |u_n|q1-2_{u_n}\,\partial\psi\mu(\upsilon_n)dx+\lambda\int_\Omega |u_n|q2-2u_n\partial\psi\mu(\upsilon_n)dx$$

$$+\ldots+\lambda\int_\Omega (u_n)qm-2u_n\partial\psi\mu(\upsilon_n)dx\geq\int_\Omega\frac{|\nabla u|^p}{\|u_n\|_p}(\partial\psi u)'(\upsilon_n)dx$$

$$=\|u_n\|_p^{p-1}\int_\Omega\left|grad(\theta_\mu(\upsilon_n))\right|^p dx$$

Then we get from (3.33) that

$$\|u_n\|_p^{p-1}\int_\Omega\left|grad(\theta_\mu(\upsilon_n))\right|^p dx+\int_\Gamma\beta_x\left((1+\mu\partial\psi)^{-1}(u_n|\Gamma(x)|)\right)\times\partial\psi_\upsilon(\upsilon_n/\Gamma(x))d\Gamma(x)$$
$$\leq\left(\partial\psi_\mu(\upsilon_n),f\right)$$

since $|\partial\psi_\mu(t)\leq|\partial\psi(t)|$ for any $t\in\Re \; and \; \mu>0,$ we see from $\|\upsilon_n\|_p=1, \; that \; \|\partial\psi_\mu(\upsilon_n)\|_p\leq c$ for $\mu>0$ where c is a constant which does not depend on $n \; or \; \mu$ and $\left(\frac{1}{p}\right)+\left(\frac{1}{p^t}\right)=1$ From (3.36), we have that

(3.10) $$\int_\Omega\left|grad(\theta_\mu(\upsilon_n))\right|^p dx\leq\frac{c}{\|u_n\|_p^{p-1}}$$

for $\mu>0,\; n\geq 1$ Now, we know that $(\theta'_\mu)^p=(\partial\psi_\mu)\to(\partial\psi)',$ as $\mu\to o \; a.e. \; on \; \Re.$ Letting $\mu\to 0$ we see from Fatou's lemma and (3.37) that

(3.11) $$\int_\Omega\left|grad\left(|\upsilon_n|^{2-\left(\frac{2}{p}\right)}\operatorname{sgn}\upsilon_n\right)\right|^p dx\leq\frac{c}{\|u_n\|_p^{p-1}}$$

From (3.38), we know that $|\upsilon_n|^{2-\left(\frac{2}{p}\right)}\operatorname{sgn}\,\upsilon_n\to k\;(a\,cons\tan t)\;in\;L^p(\Omega)\;as\;n\to+\infty.$

Next, we show that $k\neq 0 \; is \; in \; L^p(\Omega)$ from two aspects:

(1) If $p\geq 2,\;\sin ce$

$$\left\||\upsilon_n|^{2-\left(\frac{2}{p}\right)}\operatorname{sgn}\upsilon_n\right\|^p=\|\upsilon_n\|_{2p-2}^{2-\left(\frac{2}{p}\right)}\geq\|\upsilon_n\|^{2-\left(\frac{2}{p}\right)}=1$$

it follows that $k\neq 0 \; in \; L^p(\Omega)$

(2) If

$$2N/(N+1) < p < 2, \left\| |\upsilon_n|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n \right\|_p = \|\upsilon_n\|_{2p-2}^{2-\left(\frac{2}{p}\right)} \geq \|\upsilon_n\|_p^{2-\left(\frac{2}{p}\right)} = 1$$

Then $\left(|\upsilon_n|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n\right)$ is bounded in $W^{1,p}(\Omega)$. By lemma (1.3) $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when N = 1 and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{p2/2(p-1)}(\Omega)$ when $N \geq 2$. So $\left(|\upsilon_n|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n\right)$ is relatively compact in $L^{p^{2/2(p-1)}}(\Omega)$.

Then there exists a subsequence of $\left(|\upsilon_n|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n\right)$, satisfying $\left(|\upsilon_n|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n\right) \to g \ in \ L^{p2/2(p-1)}(\Omega)$. Noticing that $p \leq p^2/2(p-1)$ when 2N/(N + 1) < p < 2, it follows that k = g almost everywhere on Ω.

Now,

$$1 = \|\upsilon_n\|_p^p = \int_\Omega \|\upsilon_n\| \left|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n\right|^{p^2/2(p-1)dx}$$

$$\leq const \int_\Omega \|\upsilon_n\| \left|^{2-\left(\frac{2}{p}\right)} \operatorname{sgn}\upsilon_n - g\right|^{p2/2(p-1)dx}$$

$$+ \ const \ \|g\|_{p^{2/2(p-1)}}^{p^{2/2(p-1)}}$$

It follows that $g \neq 0$ in $L^p(\Omega)$ and then $k \neq 0$ in $L^p(\Omega)$. Assume, now, k > 0, we see from (3.36) that

$$\int_\Gamma \beta_x \left((1+\mu\partial\psi)^{-1}(u_n/\Gamma(x))\times\partial\psi_\mu(\upsilon_n|\Gamma(x)\right)d\Gamma(x) \leq (\partial\psi_\mu)(\upsilon_n), f)$$

Choosing a subsequence so that $u_n/\Gamma(x) \to +\infty \ a.e. \ on \ \Gamma,$ we see letting n $\to +\infty$ so that $\int_\Gamma B + (x) d\Gamma(x) \leq \int_\Omega f(x)dx$ which is a contradiction.

Thus $f \in \operatorname{int} R(A_p)$.

This completes the proof.

Proposition 3.11. $A_p + B_1$: $L^p(\Omega) \to L^p(Omega)$ *is m – accretive and has a compact resolvent.*

Proof. Using a theorem in Corduneanu, we know that $A_p + B_1$: $L^P(\Omega) \to L^{P'}(\Omega)$ is m-accretive. To show that $A_p + B_1$: $L^P(\Omega) \to L^{P'}(\Omega)$ has a compact resolvent, we only need to prove that if $w \in A_p u + B_1 u$ with (w) being bounded in $L^p(\Omega)$, then (u) is relatively compact in $L^p(\Omega)$.

Now, we discuss it from it from two aspects.

(1) If $p \geq 2,$ since

$$\int_\Omega^{|\nabla u|^p dx} \leq (u, \ B_{p,q1,q2,\ldots,qm}u)$$

$$= (u, \ A_p u) - (u, \ \partial\Phi p(u))$$

$$\leq (u, \ A_p u) + (u, \ B, \ u) = (u, w) \leq \|u\|_p \|u\|_{p'} \leq const.$$

It follows that (u) is bounded in $W^{1,p}(\Omega) \ where \left(\frac{1}{p}\right) + \left(\frac{1}{p'}\right) = 1$ Then (u) is relatively compact in $L^p(\Omega)$ since $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$

(2) If $2N/(N+1) < p < 2$, since $\omega \in A_p u + B_1 u$ with (ω) and (u) being bounded in $L^p(\Omega)$, we have w-B_1u $\in$ A_pu with w-B_1u and u being bounded in $L^p(\Omega)$ which gives that u is relatively compact in $L^p(\Omega)$ since A_p is m-accretive by proposition (3.8) and has a compact resolvent by lemma (2.9)
This completes the proof.

Theorem: Let $f \in L^p(\Omega)$ be such that

$$\int_\Gamma \beta-(x)d\Gamma(x) + \int_\Omega g-(x)dx < \int_\Omega f(x)dx < \int_\Gamma \beta+(x)d\Gamma(x) + \int_\Omega g+(x)dx$$

Then, (1.4) has a unique solution in $L^p(\Omega)$, where $2N/(N+1) < p < +\infty$ and $N \geq 1$

Proof. We want to use theorem (1.9) to finish our proof. From the propositions we use see that all of the conditions in theorem (1.9) are satisfied. It suffices to show that $f \in \operatorname{int}\lfloor R(A_p) + R(B_1) \rfloor$ *which ensure that* $f \in R(A_p + B_1 + B_2)$. Thus proposition (2.11) tells us that (1.4) has a unique solution $L^p(\Omega)$.

Using the similar methods as those in [2,4,7], by dividing it into two cases and using propositions (2.13) and (2.15) respectively, we know that $f \in \operatorname{int}[R(A_p) + R(B_1)]$. This completes the proof.

Remark: Compared to the work done in [1.7], not only the existence of the solution of (1.4) is obtained but also the uniqueness of the solution is obtained. Furthermore, our work extended the work of [12].

References

[1]. B.D. Calvert and C.P. Gupta; "Nonlinear elliptic boundary value problems in L^P spaces and sums of ranges of accretive operators". Nonlinear Analysis, vol. 2, no. 1, pp 1-26.1978.

[2]. L. Wei and Z. He; "The Applications of sums of ranges of accretive operators to nonlinear equations involving the p-Laplacian operator". Nonlinear Analysis: Theory, Methods and Applications, vol.24, no.2, pp 185-193, 1995.

[3]. L. Wei; "Existence of solutions of nonlinear elliptic boundary value problems". Mathematics in Practice and Theory, vol.31, no.3 pp 360-364, 2001 (Chinese).

[4]. L. Wei and Z. Hen; "The Applications of Theories of accretive operators to nonlinear elliptic boundary value problems in L^P spaces". Nonlinear Analysis: Theory, Methods and Applications. Vol. 46, no. 2, pp 199-211, 2001.

[5]. L. Wei and H.Y. Zhou; "Existence of solutions of a family of nonlinear boundary value problems in L^2 spaces", Applied Mathematics, a journal of Chinese universities, vol.20, no.2, pp 175-182, 2005.

[6]. L. Wei and H.Y. Zhou; "The existence of solutions of nonlinear boundary value problems involving the p-Laplacian operators in L^s spaces", Journal of Systems science and complexity, vol. 18, no. 4, pp 511-521, 2005.

[7]. L. Wei and R.P. Agrawal, "Existence of solutions to nonlinear Neumann boundary value problems with generalized p-Laplacian operator". Computers and Mathematics with Applications, vol. 56, no.2, pp 530-541, 2008.

[8]. Z.C. Chen and T. Luo ; "The eigenvalue problem of p-Laplacian-like equations. Acta Mathematica Sinica, vol. 46, no. 4, pp 631-638, 2003(Chinese).

[9]. L.K. Li and Y.T. Guo; " The Theory of Sobolov Spaces". Shanghai Science and Technology Press, Shanghai, China 1981.

[10]. L. Wei, L. Duan and H. Zhou; "Study of the existence and uniqueness of solution of generalized capillarity problem". Abstract and Applied Analysis, vol. 2012, Article ID 154307, 18 pages.

[11]. D. Pascali and S. Sburlan; "Nonlinear Mappings of Monotone Type", Martinus Nijhoff, The Hagne, The Netherlands, 1978.

[12]. L. Wei, L. Duan and H. Zhou; "Solution of nonlinear elliptic boundary value problems and its iterative construction", Abstract and Applied Analysis, vol.2012, Article ID 210325, 14 pages.