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Lacunary Cesáro $C_{,1.1}$ −Statistical Convergence of Difference Double Sequences

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Absract: *Recently, Esi [2] defined the statistical analogue for double difference sequences $x = (x_{k,l})$ as follows: A double sequence $x = (x_{k,l})$ is said to be P −statistically Δ −convergent to L provided for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} \{the\ number\ of\ (k,l): k < m, l < n, |\Delta x_{k,l} - L| \geq \varepsilon\} = 0.$$

Using this definition, Esi [2] introduced and studied lacunary statistical convergence for difference double sequences and also gave some inclusion theorems. In this paper we in analogy to Esi, defined and proved some inclusion theorems for lacunary Cesáro $C_{,1.1}$ −statistical convergence of difference double sequences.

***Key Words**: Statistical convergence, Cesáro $C_{,1.1}$ −Statistical convergence, Double sequences Mathematics subject classification: Primary 40F05, 40J05, 40G05.*

I. Introduction

A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by $P - lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such a $x = (x_{k,l})$ is "P −convergent" Pringsheim [7]. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l,

$$\|x\| = sup|x_{k,l}| < \infty$$

It should be noted that in contrast to the case for single sequences, a convergent double sequence need not be bounded. The concept of statistical convergence was introduced by Fast [3]. A complex number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |k \leq n: |x_k - L| \geq \varepsilon| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. Mursaleen and Edely [6] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be P −statistical convergence to L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |(k,l): k < m, l < n, |x_{k,l} - L| \geq \varepsilon| = 0.$$

In this case, we write $St_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all P −statistical convergent double sequences by St_2.

By a lacunary $\theta = (k_r), r = 0,1,2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as$r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent space N_θ was defined by Freedman et. al. [4] as follows:

$$N_\theta = \left\{ x = (x_k): \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, for\ some\ L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_r = l_s - l_{s-1} \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s,\ h_r \bar{h}_r$ *and* $\theta_{r,s}$ is determined by

$$I_r = \{(k,l): k_{r-1} < k \leq k_r\ and\ l_{s-1} < l \leq l_s\},$$

$q_r := \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}}$ *and* $q_r \bar{q}_s$. Savas and Patterson [9]

The set of all double lacunary sequences denoted by $N_{\theta_{r,s}}$ and defined by Savas and Patterson [10] as follows:

$$N_{\theta_{r,s}} = \left\{x = (x_{k,l}): P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k,l} - L| = 0, for\ some\ L\right\}.$$

Lacunary Statistical convergence of Difference Double Sequences

The following definitions and results are by Esi [2].

Definition 2.1(Esi [2]): The double sequence $x = (x_{k,l})$ is $\Delta-$bounded if there exists a positive number M such that $\|\Delta x_{k,l}\| < M$ for all $k, and\ l$,

$$\|x\|_\Delta = \sup_{k,l} |\Delta x_{k,l}| < \infty.$$

Where $\Delta x_{k,l} = x_{k,l} - x_{k,l+1} - x_{k+1,l} - x_{k+1,l+1}$. We will denote the set of all bounded double difference sequences by $l_\infty^n(\Delta)$.

Definition 2.2: (Esi [2]) A real double sequence $x = (x_{k,l})$ is said to be $P-$statistical $\Delta-$convergence to L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{(k,l): k < m, l < n, |\Delta x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $St_2 - \lim_{k,l} x_{k,l} = L\ and$ we

Definition 2.3:(Esi [2]) The double sequence $x = (x_{k,l})$ is strong double diference Cesáro summable to L if

$$w_\Delta^{ll} = \left\{x = (k,l): P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} |\Delta x_{k,l} - L| = 0, for\ some\ L \in \mathbb{C}\right\}.$$

The class of all strongly double difference Cesáro summable sequences is denoted by w_Δ^{ll}.

Definition 2.4. (Esi [2]): Let $\theta_{r,s}$ be a double lacunary sequence. The double number sequence $x = (x_{k,l})$ is $N_{\theta_{r,s}}\Delta - P-$convergent to L provided that for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |\Delta x_{k,l} - L| = 0.$$

We will denote the set of all $N_{\theta_{r,s},}\Delta - P-$converngent sequences by $N_{\theta_{r,s,\Delta}}$. We now consider the double difference lacunary statistical convergence.

Definition 2.5(Esi [2]): Let $\theta_{r,s}$ be a double lacunary sequence. The double number sequence $x = (x_{k,l})\ is\ S_{\theta_{r,s}}\Delta - P-$convergent to L provided that for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s}: |\Delta x_{k,l} - L| \geq \varepsilon\}| = 0.$$

We will denote the set of all $S_{\theta_{r,s},}\Delta - P-$converngent sequences by $S_{\theta_{r,s,\Delta}}$.

Theorem 2.1: Let $\theta_{r,s}$ be a double lacunary sequence. Then

(i) $N_{\theta_{r,s,\Delta}} \subset S_{\theta_{r,s,\Delta}}$ and the inclusion is strict,

(ii) If $x = (x_{k,l}) \in l_\infty^n(\Delta) \cap S_{\theta_{r,s,\Delta}}$ then $x = (x_{k,l}) \in N_{\theta_{r,s,\Delta}}$

(iii) $l_\infty^n(\Delta) \cap S_{\theta_{r,s,\Delta}} = l_\infty^n(\Delta) \cap N_{\theta_{r,s,\Delta}}$.

Theorem 2.2: Let $\theta_{r,s}$ be a double lacunary sequence. Then

(i) $St_{2,\Delta} \subset S_{\theta_{r,s,\Delta}}$ if $\lim inf q_r > 1$ and $\lim inf\ \bar{q}_s > 1$.

(ii) $S_{\theta_{r,s,\Delta}} \subset St_{2,\Delta}$ if $\lim inf q_r < \infty$ and $\lim inf\ \bar{q}_s < \infty$,

(iii) $St_{2,\Delta} = S_{\theta_{r,s,\Delta}}$ if $1 < \lim inf q_r < \infty$ and $1 < \lim inf\ \bar{q}_s < \infty$.

3. Lacunary cesáro $C_{1.1}$ −Statistical convergence of difference double sequences.

Let $A = [a_{jk}^{mn}]_{j,k=0}^\infty$ be a doubly infinite matrix of real number for $m, n = 1,2, \ldots$ forming the sum

$$y_{mn} = \sum_{j=0}^\infty \sum_{k=0}^\infty a_{jk}^{mn} x_{jk} \qquad (1)$$

Called the $A-$means of the double sequence x yielded a method of summability. We say that a sequence x is $A-$summable to the limit S of the A mean exist for all $m, n = 0,1, \ldots$ and converges in pringsheim sense

$$\lim_{pq\to\infty} \sum_j^p \sum_k^q a_{jk}^{mn} x_{jk} = y_{mn}$$

And

$$\lim_{m,n\to\infty} y_{mn} = s$$

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence with the same limit. Robison [8] presented a form dimensional analogue of regularly for double sequences in which he added an additional assumption of boundedness. A four dimensional matrix A is said to be bounded

regularly on RH −sequence if it maps every bounded p −convergent sequence (or convergent in Pringsheim sense) into a P −convergent sequence with same limit (see Hamilton [5] and Robison [8]).
Define the first means

$$\sigma_{mn}^{x} = \frac{1}{mn}\sum_{j=0}^{m}\sum_{k=0}^{n} x_{jk} \qquad and$$

$$A\sigma_{mn}^{x} = \frac{1}{mn}\sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}^{mn}$$

We say that $x = \left(x_{jk}\right)$ is statistical summable $(C, 1.1)$ to ℓ, if the sequence $\sigma = \left(\sigma_{jk}^{x}\right)$ is statistically convergent to ℓ in Pringsheim's sense, that is, $st_2 - \lim_{mn} \sigma_{(ij)}^{x} = \ell$. We denote by $C_{1.1}(st_2)$, the set of all double sequence which one statistically summable$(C, 1.1)$.

Definition 3.1: A double sequence $x = x_{kl}$ is $C_{,1.1}\ \Delta$ −bounded if there exists a positive number M such that

$$|\Delta\sigma_{kl}| < M \quad for\ all\ k\ and\ \ell$$

$$\|\sigma\|_{\Delta} = \sup_{kl} |\Delta\sigma_{kl}| < \infty.$$

Where

$$\Delta\sigma_{kl} = \sigma_{kl} - \sigma_{k,l+1} - \sigma_{k+1,l} + \sigma_{k+1,l+1}.$$

We denote by
$l_{\infty}^{\sigma}(\Delta)$, the set of all bounded double $C_{,1.1}$−statistically convergent difference sequences.

Definition 3.2: A real sequence $x = (x_{k,l})$ is said to be $C_{,1.1}$−Statistical Δ −convergent to l provided that for each $\varepsilon > 0$.

$$p - \lim_{mn} |\{(k,l): k < m, l < n; |\Delta\sigma_{kl} - l| \geq \varepsilon\}| = 0$$

and we write

$$C_{,1.1}\left(st_{2,\Delta}\right) - \lim_{k,l} \sigma_{kl} = L.$$

we shall denote the set of all $C_{,1.1}$−statistical Δ −convergent double sequences by $C_{,1.1}\ (st_{2,\Delta})$.

Definition 3.3: A double sequences $x = (x_{kl})$ is strongly double difference Cesáro summable to L if

$$C_{,1.1}^{\omega^{ll}}\left(st_{2,\Delta}\right) = \left\{x = (x_{kl}): P - \lim_{mn}\left|\Delta\sigma_{k,l} - L\right| = 0\,, for\ some\ L \in \mathbb{C}\right\}$$

We denote by $C_{,1.1}^{\omega^{ll}}\left(st_{2,\Delta}\right)$, the class of all strongly double difference Cesáro summable sequences.

Definition 3.4: Let $\theta_{r,s}$ be a double lacunary sequence. The double sequence $x = (x_{kl})$ is $N_{\theta_{r,s,\Delta}}$ − $C_{,1.1}$−convergent to L provided that for every $\varepsilon > 0$,
$P - \lim_{r,s} \frac{1}{h_{r,s}}\left|\Delta\sigma_{k,l} - L\right| = 0.$
denote by $C_{,1.1}N_{\theta_{r,s,\Delta}}$, the set of all $N_{\theta_{r,s,\Delta}} - C_{,1.1}$−convergent sequences.

Definition 3.5.: Let $\theta_{r,s}$ be a double lacunary sequence. The double sequence $x = (x_{kl})$ is $S_{\theta_{r,s,\Delta}}$ − $C_{,1.1}$−convergent to L if for every $\varepsilon > 0$,
$P - \lim_{r,s} \frac{1}{h_{r,s}}\left|\left\{(k,l) \in I_{r,s}: \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right| = 0.$
we shall denote by $C_{,1.1}S_{\theta_{r,s,\Delta}}$, the set of all $S_{\theta_{r,s,\Delta}} - C_{,1.1}$−convergent sequences.

Theorem 3.1: Let $\theta_{r,s}$ be a double lacunary sequence. Then:

i. $C_{,1.1}(N_{\theta_{r,s,\Delta}}) \subset C_{,1.1}(S_{\theta_{r,s,\Delta}})$
ii. If $\sigma = \sigma_{kl} \in l_{\infty}^{\sigma}(\Delta) \cap C_{,1.1}(S_{\theta_{r,s,\Delta}})$ then, $\sigma = \sigma_{kl} \in C_{,1.1}(N_{\theta_{r,s,\Delta}})$
iii. $l_{\infty}^{\sigma}(\Delta) \cap C_{,1.1}(S_{\theta_{r,s,\Delta}}) \cap l_{\infty}^{\sigma}(\Delta) \cap C_{,1.1}(N_{\theta_{r,s,\Delta}})$

Proof: (i) Since

$$\left|\left\{(k,l) \in I_{r,s}: \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right| \leq \sum_{(k,l)\in I_{rs}\ \&\ |\Delta\sigma_{k,l}-L|\geq\varepsilon} \left|\Delta\sigma_{k,l} - L\right| \leq \sum_{(k,l)\in I_{rs}} \left|\Delta\sigma_{k,l} - L\right|$$

and so if $\sigma = \left(\sigma_{k,l}\right) \in C_{,1.1}(N_{\theta_{r,s,\Delta}})$ then we have $\sigma = \left(\sigma_{k,l}\right) \in C_{,1.1}(S_{\theta_{r,s,\Delta}})$. To show the inclusion is strict, we define $\sigma = \left(\sigma_{k,l}\right)$ as follows:

$$\Delta\sigma_{k,l} = \begin{pmatrix} 1 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \cdots \\ 2 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \cdots \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 2 & [\sqrt[3]{h_{r,s}}] & [\sqrt[3]{h_{r,s}}] & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

It is clear that $\sigma = (\sigma_{k,l})$ is not Δ −bounded double sequence and for $\varepsilon > 0$

$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s}: \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right| = P - \lim_{r,s} \frac{1}{h_{r,s}} \frac{[\sqrt[3]{h_{r,s}}]}{h_{r,s}} = 0$

So $\sigma = (\sigma_{k,l}) \in C_{1.1}(S_{\theta_{r,s,\Delta}})$. But

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left|\Delta\sigma_{k,l} - L\right| = P - \lim_{r,s} \frac{1}{h_{r,s}} \frac{[\sqrt[3]{h_{r,s}}]\left([\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]+1)\right)}{2h_{r,s}} = \frac{1}{2}.$$

Therefore, $\sigma = (\sigma_{k,l}) \notin C_{1.1}N_{\theta_{r,s,\Delta}}$. Hence the proof for (i).

(ii) Suppose that $\sigma = (\sigma_{k,l}) \in l_\infty^\sigma(\Delta) \cap C_{,1.1}(S_{\theta_{r,s,\Delta}})$. Then $\left|\Delta\sigma_{k,l}\right| < M$ for all k and l, also for given $\varepsilon > 0$ and sufficiently large r and s, we obtain the following

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left|\Delta\sigma_{k,l} - L\right| = \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \,\&\, \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon} \left|\Delta\sigma_{k,l} - L\right| + \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \,\&\, \left|\Delta\sigma_{k,l} - L\right| \leq \varepsilon} \left|\Delta\sigma_{k,l} - L\right| \leq \frac{M}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s}: \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right| + \varepsilon.$$

Therefore, $\sigma = (\sigma_{k,l}) \in l_\infty^\sigma(\Delta) \cap C_{,1.1}(S_{\theta_{r,s,\Delta}})$ implies $\sigma = (\sigma_{k,l}) \in C_{,1.1}(N_{\theta_{r,s,\Delta}})$.

(iii) It follows from (i) and (ii).

Theorem 3.2: Let $\theta_{r,s}$ be a double lacunary sequence. Then,

(i) $C_{1.1}(St_{2,\Delta}) \subset C_{1.1}(S_{\theta_{r,s,\Delta}})$ if lim inf $q_r > 1$ and lim inf $\overline{q_s} = 1$

(ii) $C_{1.1}(S_{\theta_{r,s,\Delta}}) \subset C_{1.1}(St_{2,\Delta})$ if lim inf $q_r < \infty$ and lim inf $\overline{q_s} < \infty$

(iii) $C_{1.1}(St_{2,\Delta}) \subset C_{1.1}(S_{\theta_{r,s,\Delta}})$ if 1< lim inf $q_r < \infty$ and 1< lim inf $\overline{q_s} < \infty$

Proof: (i) Suppose that lim inf $q_r > 1$ and lim inf $\overline{q_s} = 1$. then exists a $\delta > 0$ such that both $q_r > 1 + \delta$ and $\overline{q_s} > 1 + \delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h_s}}{l_s} \geq \frac{\delta}{1+\delta}$. If

$\sigma = (\sigma_{k,l}) \in C_{1.1}(St_{2,\Delta})$, then for each $\varepsilon > 0$ and for sufficiently large r and s we obtain the following

$\frac{1}{k_{r,s}} \left|\left\{(k,l) \in I_{r,s}: k \leq k_r \text{ and } l \leq l_s, \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right| \geq \frac{1}{k_{r,s}} \left|\left\{(k,l) \in I_{r,s}: \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right| = \frac{h_{r,s}}{k_{r,s}} \frac{1}{h_{r,s}} \left|\{(k,l) \in\right.$
Ir,s:∆σk,l−L≥ε≥δ1+δ21hr,sk,l∈Ir,s:∆σk,l−L≥ε.

Therefore, $\sigma = (\sigma_{k,l}) \in C_{1.1}(S_{\theta_{r,s,\Delta}})$.

(ii) Suppose that lim inf $q_r < \infty$ and lim inf $\overline{q_s} < \infty$, then there exists $K > 0$ such that $q_r \leq K$ and $\overline{q_s} \leq K$ for all r and s. Let $\sigma = (\sigma_{k,l}) \in S_{\theta_{r,s,\Delta}}$ and $N_{r,s}$=$\left|\left\{(k,l) \in I_{r,s}: \left|\Delta\sigma_{k,l} - L\right| \geq \varepsilon\right\}\right|$. So given $\varepsilon > 0$ there exists a positive integer r_0 such that $\frac{N_{r,s}}{h_{r,s}} < \varepsilon$ for all $r, s > r_0$. Let $M = \max\{N_{r,s}: 1 \leq r, s \leq r_0\}$. Let m *and* n be such that $k_{r-1} < m \leq k_r$ and ; $l_{s-1} < n \leq l_s$. therefore, we obtain

$$\left|\left\{k \le m \text{ and } l \le n: \left|\Delta\sigma_{k,l} - L\right| \ge \varepsilon\right\}\right| \le \frac{1}{k_{r-1}l_{s-1}}\left|\left\{(k,l) \in I_{r,s}: k \le k_r \text{ and } l \le l_s, \left|\Delta\sigma_{k,l} - L\right| \ge \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1}l_{s-1}}\sum_{i,j=1,1}^{r,s} N_{i,j} \le \frac{Mr_0^2}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\sum_{i,j=1,1}^{r,s} N_{i,j} = \frac{Mr_0^2}{k_{r-1}l_{s-1}}$$

$$+ \frac{1}{k_{r-1}l_{s-1}}\sum_{i,j=1,1}^{r,s} N_{i,j}\frac{h_{i,j}}{h_{i,j}} \le \frac{Mr_0^2}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\left(\begin{matrix} sup \\ i,j \ge r_o r_o \end{matrix} \frac{N_{i,j}}{h_{i,j}}\right)\left(\sum_{i,j=r_o+1,r_o+1}^{r,s} h_{i,j}\right)$$

$$\le \frac{Mr_0^2}{k_{r-1}l_{s-1}} + \varepsilon \sum_{i,j=r_o+1,r_o+1}^{r,s} h_{i,j} \le \frac{Mr_0^2}{k_{r-1}l_{s-1}} + \varepsilon K^2.$$

and the results follows immediately

(iv) The proof of (iii) follows from combining (i) and (ii).

Reference

[1]. Esi, A. Lacunary statistical convergence of difference double sequences. *Acta Universitatis Apulensis,* 28(2011), 143-148.

[2]. Esi, A. On some new difference double sequence spaces via Orlicz function. *Journal of Advanced Studies in Topology,* 2(2) (2011), 16-25.

[3]. Fast, H. Sur la convergence statistique, Colleg. Math, 2(1951), 241-244.

[4]. Freedman, A. R., Sember, I. J. & Raphael, M. Some Cesάro type summability spaces, Proc. London Math. Soc. 37(1978), 508-520.

[5]. Hamilton, H. J. Transformations of multiple sequences. *Duke Math. J., 2* (1936), 29-60.

[6]. Mursaleen, M. & Edely, O. H. Statistical convergence of double sequences. J. Math. Anal. Appl., 288(1) (2003), 223-231.

[7]. Pringsheim, A. On the theory of doubly infinite sequences of numbers, *Math. Ann.,* 53(1900), 289-321.

[8]. Robison, G. M. Divergent double sequences and series. Amer. Math. Soc. Trans, 28(1926), 50-73.

[9]. Savas, E & Patterson, R. F. On some double almost lacunary sequence spaces defined by Orlicz functions, FILOMAT, 19(2005), 35-44.

[10]. Savas, E & Patterson, R. F. Lacunary statistical convergence of multiple sequences, Applied Mathematics Letters, 19(2006), 527-534.