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# Lacunary Triple Sequence $\Gamma^3$ of Fibonacci Numbers over Probabilisitic p-Metric Spaces

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***Abstract:*** *In this paper we study the concept of almost asymptotically lacunary statistical $\Gamma^3$ over probabilistic p-metric spaces defined by sequence of Orlicz functions and discuss some general topological properties of above sequence spaces.*

***Key words and phrases :****analytic sequence, triple sequences, $\Gamma^3$ -space, Fibonacci number, Orlicz function, probabilistic p-metric space, asymptotically equivalence, p-convergent, double lacunary sequence, statistical convergent.*
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## 1. Introduction

Throughout ω, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $\omega^3$ for the set of all complex triple sequences $(x_{mnk})$, where $m, n, k \in \aleph$, the set of positive integers. Then, $\omega^3$ is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in *Apostol [1]* and double sequence spaces is found in *Hardy [7]*, *Subramanian et al. [12-14]*, and many others. Later on investigated by some initial work on triple sequence spaces is found in *Sahiner et al. [11]* , *Esi et al. [2-6]*, *Subramanian et al. [15-23]* and many others.

Let $(x_{mnk})$ be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$ is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} \,(m,n,k = 1,2,3,...) \,.$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} \left|x_{mnk}\right|^{\frac{1}{m+n+k}} < \infty$$

The vector space of all triple analytic sequences are usually denoted by $\Lambda^3$. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$\left|x_{mnk}\right|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty$$

The vector space of all triple entire sequences are usually denoted by $\Gamma^3$. Let the set of sequences with this property be denoted by $\wedge^3$ and $\Gamma^3$ is a metric space with the metric

$$d(x,y) = sup_{m,n,k} \left\{ \left|x_{mnk} - y_{mnk}\right|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \tag{1.1}$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in $\Gamma^3$. Let $\phi$ = {finite sequences}.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq}$ for all m,n,k$\in \aleph$, where $\delta_{mnk}$ is a three dimensional matrix with 1 in the $(m,n,k)^{th}$ position and zero otherwise.

## 2. Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $\tilde{p}$ - *limx*=0)

(i.e) $\left(\left(m+n+k\right)!\left|x_{mnk}\right|\right)^{1/m+n+k} \to 0$ as m,n,k $\to \infty$. We shall write more briefly as *P* $\tilde{-}$ *convergent to* 0.

**2.1 Definition**

A modulus function was introduced by Nakano [24]. We recall that a modulus *f* is a function from $[0,\infty) \to [0,\infty)$, such that

(1) *f* (*x*)=0 if and only if *x*=0

(2) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0, y \ge 0$,
(3) $f$ is increasing,
(4) $f$ is continuous from the right at 0. Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from here that $f$ is continuous on $[0, \infty)$.

**2.2 Definition**
Let $A = (a^{\mu\nu}_{\kappa,\ell})$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $(k,\ell)^{th}$ term of $Ax$ is as follows:

$$(Ax)_{\kappa\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$$

such transformation is said to be non-negative if $(a_{kl}^{mn})$ is non-negative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which both added an adiditional assumption of boundedness. This assumption was made since a double sequence which is $p-$ convergent is not necessarily bounded.

Let λ and μ be two sequence spaces and $(a_{kl}^{mn})$ be a four dimensional infinite matrix of real numbers $(a_{kl}^{mn})$, where $m, n, k, \ell \in \aleph$. Then, we say $A$ defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mnk}) \in \lambda$ the sequence $Ax = \{(Ax)_{\kappa\ell}\}$, the $A$ transform of $x$, is in μ. By $(\lambda : \mu)$, we denote the class of all matrices $A$ such that $A : \lambda \to \mu$ Thus $A \in (\lambda : \mu)$ if and only if the series converges for each $k, \ell \in \aleph$. A sequence $x$ is said to be $A$ – summable to α if $Ax$ converges to α which is called as the $A$ – limit of $x$.

Let $(q_{mnk})$ be a sequence of positive numbers and

$$Q_{abc} = \sum_{m=0}^{a} \sum_{n=0}^{b} \sum_{k=0}^{c} q_{mnk} \left(a, b, c \in \aleph\right) \tag{2.1}$$

Then, the matrix $R^q = \left(r_{abc}^{mnk}\right)^q$ of the Riesz mean is given by

$$\left(r_{abc}^{mnk}\right)^q = \begin{cases} \dfrac{q_{mnk}}{Q_{abc}} & if\, 0 \le m;n;k \le a;b;c \\ 0 & if\, (m;n;k) > abc \end{cases} \tag{2.2}$$

**2.3 Definition**

A function $f : \Re \times \Re \times \Re \to \Re_0^{+} \times \Re_0^{+} \times \Re_0^{+}$ is called a distribution if it is non-decreasing and left continuous with $inf_{t \in \Box\times\Box\times\Box} f(t) = 0$ and $sup_{t \in \Box_0^+ \times \Box_0^+ \times \Box_0^+} f(t) = 1.$ We will denote the set of all distribution functions by $D$.

**2.4 Definition**

A triangular metric, briefly t-over probabilisitic $p-$ metric spaces, is a binary operation on [0,1] which continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continous mapping * : [0, 1] × [0, 1] × [0, 1] → [0, 1] × [0, 1] × [0, 1] such that $a, b, c \in [0,1]$:

(1) $a * 1 = 1$,
(2) $a * b = b * a$
(3) $c * d \ge a * b$ if $c \ge a$ and $d \ge b$,
(4) $(a * b) * c = a * (b * c)$.

**2.5 Definition**

A triple $(X, P, *)$ is called a probabilistic $p$ - metric space or shortly $PP$ - space if $X$ is a real vector space, $P$ is a mapping from $X \times X \times X \to D \times D \times D$ (for $x \in X$, the distribution function P(x) is denoted by $P_x$ and $P_x(t)$ is the value of $P_x$ at ($t \in \aleph \times \aleph \times \aleph$) and. is a $t$- $p$- metric satisfying the following conditions:

(i) $P_x\left(|(d_1(x_1,0),\ldots,d_n(x_n,0))|_p\right) = 0$ if and and only if $P_x\left(d_1(x_1,0),\ldots,d_n(x_n,0)\right)$ are linearly dependent,

(ii) $P_x\left(t\,|(d_1(x_1,0),\ldots,d_n(x_n,0))|_p\right) = 1$ is invariant under permutation,

(iii) $P_x\left(\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p\right) = |\alpha|\, P_x\left(\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p\right), \alpha \in \Box \setminus \{0\}$.

(iv) $P_x\left(d_p((x_1,y_1),\ (x_2,y_2)\ldots(x_n,y_n))\right) = P_x\left((d_X(x_1,x_2,\ldots x_n))^p + P_x(d_Y(y_1,y_2,\ldots y_n)^p)^{1/p}\right)$ for $1 \le p < \infty$; (or)

(v) $P_x\left(d\left((x_1,y_1),(x_2,y_2),\ldots(x_n,y_n)\right)\right) := \sup\left\{P_x\left(\left\{d_X(x_1,x_2,\ldots x_n)^p\right) + P_x(d_Y,(y_1,y_2,\ldots y_n)\ \right\}\right)\right\}$, for $x_1, x_2, \ldots x_n \in X$, $y_1, y_2, \ldots y_n \in Y$.

**2.6 Definition**

A triple $(X, P, *)$ be a *PP*- space. Then a sequence $x = (x_{mn})$ is said to convergent $\overline{0} \in X$ with respect to the probabilistic *p*-metric *P* if, for every $\in > 0$ and $\theta \in (0, 1)$, there exists a positive integer $m_0n_0$ such that $P_{x_{mnk}-\overline{0}}(\in) > 1-\theta$ whenever $m, n \geq m_0 n_0 k_0$. It is denoted by *P-limx*= $\overline{0}$ or $x_{mnk} \xrightarrow{P} \overline{0}$ as $m, n, k \to \infty$.

**2.7 Definition**

A triple $(X, P, *)$ be a *PP*- space. Then a sequence $x = (x_{mnk})$ is called a Cauchy sequence with respect to the probabilistic *p* - metric *P*-if, for every $\in>0$ and $\theta\in(0,1)$ there exists a positive integer $m_0n_0k_0$ such that $P_{x_{mnk}-x_{rst}}(\in) > 1-\theta$ for all $m, n \geq m_0 n_0 k_0$ and $r, s, t > r_0 s_0 t_0$.

**2.8 Definition**

A triple $(X, P, *)$ be a *PP*- space. Then a sequence $x = (x_{mnk})$ is said to analytic in *X*, if there is a $u \in \aleph$ such that $P_{x_{mnk}}(u) > 1-\theta,$ where $\theta \in (0,1).$ We denote by $\Lambda^{3P}$ the space of all analytic sequences in *PP*- space.

**2.9 Definition**

Two non-negative sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ are asymptotically equivalent $\overline{0}$ if

$$lim_{mnk} \frac{x_{mnk}}{y_{mnk}} = \overline{0}$$

and it is denoted by $x \equiv \overline{0}$.

**2.10 Definition**

Let *K* be the subset of $\aleph \times \aleph \times \aleph$ , the set of natural numbers. Then the asymptotically density of *K*, denoted by δ(K), is defined as

$$\delta(K) = \lim_{a,b,c} \frac{1}{abc} \left| \{m,n,k \leq a,b,c : m,n,k \in K\} \right|,$$

where the vertical bars denote the cardinality of the enclosed set.

**2.11 Definition**

A number of triple sequence $x = (x_{mnk})$ is said to be statistically convergent to the number $\overline{0}$ if for each $\in>0$, the set

$K(\in) = \left\{ m \leq a, n \leq b, k \leq c : \left| x_{mnk} - \overline{0} \right|^{1/m+n+k} \geq \varepsilon \right\}$ has asympototic density zero

$$\lim_{abc} \frac{1}{abc} \left| \left\{ m \leq a, n \leq b, k \leq c : \left( \left| x_{mnk} - \overline{0} \right| \right)^{1/m+n+k} \geq \varepsilon \right\} \right| = 0$$

In this case we write *St-lim x* = $\overline{0}$.

**2.12 Definition**

The two non-negative triple sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ are said to be asymptotically double equivalent of multiple *L* provided that for every $\in > 0$,

$$\lim_{abc} \frac{1}{abc} \left| \left\{ (m,n,k) : m \leq a, n \leq b, k \leq c \left| \frac{x_{mnk}}{y_{mnk}} - L \right| \geq \in \right\} \right| = 0.$$

and simply asymptotically double statistical equivalent if *L*=1. Furthermore, let $S_{\theta_{rst}}^L$ denote the set of all sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ such that *x is equivalenttoy*.

**2.13 Definition**

Let $\theta_{rst} = \{(m_r, n_s, k_t)\}$ be a triple lacunary sequence; the two triple sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ are said to be asymptotically triple lacunary statistical equivalent of multiple *L* provided that for every $\in > 0$,

$$lim_{r,s,t} \frac{1}{h_{r,s,t}} \left| \left\{ (m,n,k) \in I_{r,s,t} : \left| \frac{x_{mnk}}{y_{mnk}} - L \right| \geq \varepsilon \right\} \right| = 0$$

and simply asymptotically triple lacunary statistical equivalent if *L*=1. Furthermore, let $S_{\theta_{rst}}^L$ denote the set of all sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ such that *xisequivalenttoy*.

**2.14 Definition**

Let $\theta_{rst} = \{(m_r, n_s, k_t)\}$ be a triple lacunary sequence; the two triple sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ are said to be strong asymptotically double lacunary equivalent of multiple *L* provided that

$$lim_{r,s,t} \frac{1}{h_{r,s,t}} \sum_{(m,n,k)?I_{r,s,t}} \left| \frac{x_{mnk}}{y_{mnk}} - L \right| = 0,$$

that is *x* is equivalent to *y* and it is denoted by $N_{\theta_{rst}}^{L}$ and simply strong asymptotically triple lacunary equivalent if *L*=1. In addition, let $N_{\theta_{rst}}^{L}$ denote the set of all sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ such that *x* is equivalent to *y*.

**2.15 Definition**

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$m_0 = 0, h_i = m_i - m_{r-1} \to \infty$ as $i \to \infty$ and

$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty$ as $l \to \infty$

$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty$ as $j \to \infty$

Let $m_{i,l,j} = m_i n_l k_j, h_{i,l,j} = h_i \overline{h_l h_j},$ and $\theta_{i,l,j}$ is determined by

$I_{i,l,j} = \{(m,n,k) : m_{i-1} < m < m_i \text{ and } n_{l-1} < n < n_l \text{ and } k_{j-1} < k \le k_j\}$,

$q_k = \frac{m_k}{m_{k-1}}, \overline{q_l} = \frac{n_l}{n_{l-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.

**2.16 Definition**

Let M be an sequence of Orlicz functions. The two non-negative triple sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ are said to be strong M-asymptotically double equivalent of multiple $\overline{0}$ provided that

$$\left[\Gamma_M^{3F}, \| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0) \|_p \right] = F_\mu(x)$$

$$= \lim_{a,b,c\to\infty} \frac{1}{abc}$$

$$\left\{ \sum_{m=1}^{a} \sum_{n=1}^{n} \sum_{k=1}^{c} \left[ M \left( f_{abc}^{mnk} \left( \left| \frac{x_{mnk}}{y_{mnk}} - \overline{0} \right| \right)^{1/m+n+k}, \left\| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0) \right\|_p \right) \right] = 0 \right\}$$

$$= \lim_{a,b,c\to\infty} \frac{1}{abc}$$

$$\left\{ \frac{1}{f(a+3)(b+3)(c+3)} \sum_{m=1}^{a} \sum_{n=1}^{b} \sum_{k=1}^{c} \left[ M \left( f_{abc}{}^{mnk} \left( \left| \frac{x_{mnk}}{y_{mnk}} - \overline{0} \right| \right)^{1/m+nK}, \left\| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0)) \right\| \right) \right] \right\}$$

and

$$\left[\Gamma_M^{3F}, \| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0) \|_p \right] = F_\eta(x)$$

$$= \sup_{a,b,c} \frac{1}{abc}$$

$$\left\{ \sum_{m=1}^{a} \sum_{n=1}^{n} \sum_{k=1}^{c} \left[ M \left( f_{abc}^{mnk} \left( \left| \frac{x_{mnk}}{y_{mnk}} - \overline{0} \right| \right)^{1/m+n+k}, \left\| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0) \right\|_p \right) \right] < \infty \right\}$$

$$= \sup_{a,b,c} \frac{1}{abc}$$

$$\left\{\frac{1}{f(a+3)(b+3)(c+3)-1}\sum_{m=1}^{a}\sum_{n=1}^{b}\sum_{k=1}^{c}\left[M\left(f_{abc}{}^{mnk}\left(\left|\frac{x_{mnk}}{y_{mnk}}-\bar{0}\right|\right)^{1/m+nK},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0))\|\right)\right]\right\}$$

it is denoted by $[M]^{\bar{0}}$ and simply M-asymptotically triple.

**2.17 Definition**

Let M be an sequence of Orlicz functions and $\theta_{rst}=\{(m_r,n_s,k_t)\}$ be a triple lacunary sequence; the two triple sequences $x=(x_{mnk})$ and $y=(y_{mnk})$ are said to be strong M-asymptotically double equivalent of multiple $\bar{0}$

$$\left[\Gamma_M^{3F},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0)\|_p\right]=F_\mu(x)$$

$$=\lim_{r,s,t\to\infty}\frac{1}{h_{r,s,t}}$$

$$\left\{\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left(f_{abc}^{mnk}\left(\left|\frac{x_{mnk}}{y_{mnk}}-\bar{0}\right|\right)^{1/m+n+k},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0)\|_p\right)\right]\right\}$$

$$=\lim_{r,s,t\to\infty}\frac{1}{h_{r,s,t}}$$

$$\left\{\frac{1}{f(a+3)(b+3)(c+3)-1}\sum_{m=1}^{a}\sum_{n=1}^{b}\sum_{k=1}^{c}\left[M\left(f_{abc}{}^{mnk}\left(\left|\frac{x_{mnk}}{y_{mnk}}-\bar{0}\right|\right)^{1/m+nK},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0))\|\right)\right]\right\}$$

and

$$\left[\wedge^{3F}{}_{M,\|(d(x_1,0),d(x_2,0),.....,d(x_{n-1},0))\|_p}\right]=F_\eta(x)=\sup_{r,s,t\to\infty}\frac{1}{h_{r,s,t}}$$

$$\left\{\sum_{m\in I_{r,s,t}}\sum_{n\in I_{r,s,t}}\sum_{k\in I_{r,s,t}}\left[M\left(f_{abc}^{mnk}\left(\left|\frac{x_{mnk}}{y_{mnk}}-\bar{0}\right|\right)^{1/m+n+k},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0)\|_p\right)\right]\right\}$$

$$=\sup_{r,s,t\to\infty}\frac{1}{h_{r,s,t}}$$

$$\left\{\frac{1}{f(a+3)(b+3)(c+3)-1}\sum_{m\in I_{rst}}\sum_{n\in I_{rst}}\sum_{k\in I_{rst}}\left[M\left(f_{abc}{}^{mnk}\left(\left|\frac{x_{mnk}}{y_{mnk}}-\bar{0}\right|\right)^{1/m+nK},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0))\|\right)\right]\right\}$$

$$\left\{\frac{1}{f(a+3)(b+3)(c+3)-1}\sum_{m\in I_{rst}}\sum_{n\in I_{rst}}\sum_{k\in I_{rst}}\left[M\left(f_{abc}{}^{mnk}\left(\left|\frac{x_{mnk}}{y_{mnk}}-\bar{0}\right|\right)^{1/m+nK},\left\|\left(d(x_1,0),d(x_2,0),...,d(x_{n-1},0)\right)\right\|\right)\right]\right\}$$

provided that is denoted by $N_{\theta_{r,s,t}}^{[M]^{\bar{0}}}$ and simply strong M-asymptotically triple lacunary.
Consider the metric space

$\left[\Lambda_M^{3F},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0)\|_p\right]$ with the metric

$d(x, y) = \sup_{abc}\{M(F_\eta(x) - F_\eta(y)) : m, n, k = 1, 2, 3,...\}$
Consider the metric space

$\left[\Gamma_M^{3F},\|(d(x_1,0),d(x_2,0),...,d(x_{n-1},0)\|_p\right]$ with the metric

$d(x, y) = \sup_{abc}\{M(F_\eta(x) - F_\eta(y)) : m, n, k = 1, 2, 3,...\}$

## 3. Almost asymptotically lacunary convergence of PP-Triple Sequence Spaces

The idea of statistical convergence was first introduced by Steinhaus in 1951 and then studied by various authors. In this paper has studied the concept of statistical convergence in prabablistic p-metric spaces.

**3.1 Definition.**

A triple $(X,P,*)$ be a *PP*- space. Then a sequence $x=(x_{mn})$ is said to statistically convergent to $\bar{0}$ with respect to the probabilistic p-metric P-provided that for every ε > 0 and γ ∈ (0,1)

$$\delta\left(\left\{m,n,k\in\aleph:P_{(|x_{mnk}|)^{1/m+n+k}}(\in)\leq 1-\gamma\right\}\right)=0$$

or equivalently

$$\lim_{abc}\frac{1}{abc}\left|\left\{m\leq a,n\leq b,k\leq c:P_{(|x_{mnk}|)^{1/m+n+k}}(\varepsilon)\leq 1-\gamma\right\}\right|=0$$

In this case we write $St_P - \lim_x = \bar{0}$.

**3.2 Definition**

A triple $(X,P,*)$ be a *PP*- space. Then a sequence $x=(x_{mnk})$ and $y=(y_{mnk})$ are said to be almost asymptotically statistical equivalent of multiple $\bar{0}$ in PP-space X if for every ε > 0 and γ ∈ (0,1).

$$\delta\left(\left\{m,n,k\in\aleph:P_{(|x_{mnk}|)^{1/m+n+k}-\bar{0}}(\in)\leq 1-\gamma\right\}\right)=0$$

or equivalently

$$\lim_{abc}\frac{1}{abc}\left|\left\{m\leq a,n\leq b,k\leq c:P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/M+n+k}-\bar{0}}(\varepsilon)\leq 1-\gamma\right\}\right|=0.$$

In this case we write $x \overset{S_{\hat{\theta}}(PP)}{\equiv} y.$

3.3 Definition A triple $(X,P,*)$ be a *PP*- space and $\theta=(m_r n_s k_t)$ be a lacunary sequnce. The two non-negative triple sequences $x=(x_{mnk})$ and $y=(y_{mnk})$ are said to be a almost asymptotically lacunary statistical equivalent of multiple $\bar{0}$ in *PP*- space $X$ if for every $\in>0$ and $\gamma\in(0,1)$

$$\delta_\theta\left(\left\{m,n,k \in I_{rst} : P_{\left(|x_{mnk}|\right)^{1/m+n+k}-\bar{0}}(\in) \le 1-\gamma\right\}\right) = 0 \quad \ldots \quad \ldots \quad \ldots \qquad (3.1)$$

or equivalently

$$\lim_{rst} \frac{1}{h_{rst}}\left|\left\{m,n,k \in I_{rst} : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/M+n+k}-\bar{0}}(\varepsilon) \le 1-\gamma\right\}\right| = 0 .$$

In this case we write $x \overset{S_{\hat{\theta}}(PP)}{\equiv} y.$

**3.4 Lemma.** A triple $(X, P, *)$ be a *PP*- space. Then for every ε > 0 and γ ∈ (0, 1), the following statements are equivalent.

(1) $\lim_{rst} \frac{1}{h_{rst}}\left|\left\{m,n,k \in I_{rst} : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/M+n+k}-\bar{0}}(\varepsilon) \le 1-\gamma\right\}\right| = 0$

(2) $\delta_\theta\left(\left\{m,n,k \in I_{rst} : P_{\left(|x_{mnk}|\right)^{1/m+n+k}-\bar{0}}(\in) \le 1-\gamma\right\}\right) = 0$

(3) $\delta_\theta\left(\left\{m,n,k \in I_{rst} : P_{\left(|x_{mnk}|\right)^{1/m+n+k}-\bar{0}}(\in) \le 1-\gamma\right\}\right) = 1$

(4) $\lim_{rst} \frac{1}{h_{rst}}\left|\left\{m \le a, n \le b, k \le c : P_{(|x_{mnk}|)^{1/m+n+k}}(\varepsilon) \le 1-\gamma\right\}\right| = 1$

## 4. Main results

**4.1 Theorem**

A triple $(X, P, *)$ be a *PP*- space. If two sequences $x = (x_{mnk})$ and $y = (y_{mnk})$ are almost asymptotically lacunary statistical equivalent of multiple $\bar{0}$ with respect to the probabilistic p-metric P, then $\bar{0}$ is null sequence.

**Proof:**

Assume that $x \overset{S_{\hat{\theta}}(PP)}{\equiv} y.$ For a given λ > 0 choose γ ∈ (0, 1) such that (1 - γ) > 1 - λ. Then, for any ε > 0, define the following set:

$$K = \left\{m,n,k \in I_{r,s,t} : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) \le 1-\gamma\right\}$$

Then, clearly

$$\lim_{rst} \frac{K \cap \bar{0}}{h_{rst}} - = 1 ,$$

so K is non-empty set, since $x \overset{S_{\hat{\theta}}(PP)}{\equiv} y.$ $\delta_\theta(K) = 0$ for all ε > 0, which implies $\delta_\theta(\varepsilon - K) = 1$ . If $m, n \in \aleph$ - K, then we have

$$P_{\bar{0}}(\varepsilon) = P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > (1-\gamma) \ge 1-\lambda$$

since λ is arbitrary, we get $P_{\bar{0}}(\varepsilon) = 1$.

This completes the proof.

**4.2 Theorem**

A triple $(X,P,*)$ be a *PP*- space. For any lacunary sequence θ = ($m_r n_s k_t$), $\hat{S}_\theta(PP) \subset \hat{S}(PP)$ if $\limsup_{rst} q_{rst} < \infty$.

**Proof:**

If $\limsup_{rst} q_{rst} < \infty$ then there exists a B > 0 such that $q_{rst} < B$ for all r, s, t ≥ 1. Let $x \overset{S_\theta^{\wedge}(PP)}{\equiv} y.$ and ε > 0. Now we have to prove $\hat{S}(PP)$. Set

$$K_{rs} = \left|\left\{ m,n,k \in I_{r,s,t} : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) \le 1-\gamma \right\}\right|$$

Then by definition, for given ε > 0, there exists $r_0 s_0 t_0 \in K$ such that

$$\frac{K_{rst}}{h_{rst}} < \frac{\varepsilon}{2B} \text{ for all } r > r_0,\ s > s_0 \text{ and } t > t_0$$

Let M = max{$K_{rst}$: 1 ≤ r ≤ $r_0$, 1 ≤ s ≤ $s_0$, 1 ≤ t ≤ $t_0$} and let uvw be any positive integer with $m_{r-1} < u \le m_r$, $n_{s-1} < v < n_s$ and $k_{t-1} < w \le k_t$ Then

$$\frac{1}{uvw}\left|\left\{ m \le u, n \le v, k \le w : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > 1-\gamma \right\}\right| \le$$

$$\frac{1}{m_{r-1}n_{s-1}k_{t-1}}\left|\left\{ m \le m_r, n \le n_s, k \le k_t : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > 1-\gamma \right\}\right| =$$

$$\frac{1}{m_{r-1}n_{s-1}k_{t-1}}\{K_{111} + \ldots K_{rst}\} \le \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{3B} q_{rst} \le \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{3}$$

This completes the proof.

**4.3 Theorem.** A triple $(X,P,*)$ be a *PP*- space. For any lacunary sequence θ = ($m_r n_s k_t$), $\hat{S}_\theta(PP) \subset \hat{S}(PP)$ if limin f $f_{rst} q_{rst} > 1$.

**Proof:** If limin $f_{rst} q_{rst} > 1$, then there exists a β > 0 such that $q_{rst} > 1 + \beta$ for all sufficiently larger r, s, t which implies

$$\frac{h_{rst}}{K_{rst}} \ge \frac{\beta}{1+\beta}$$

Let $x \overset{S_\theta^{\wedge}(PP)}{\equiv} y$, then for every ε > 0 and for sufficiently large r, s, t we have

$$\frac{1}{m_r n_s k_t}\left|\left\{ m \le m_r, n \le n_s, k \le k_t : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > 1-\gamma \right\}\right| \ge$$

$$\frac{1}{m_r n_s k_t}\left|\left\{ m,n,k \in I_{rst} : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > 1-\gamma \right\}\right| \ge$$

$$\frac{\beta}{1+\beta}\frac{1}{h_{rst}}\left|\left\{ m,n,k \in I_{rst} : P_{\left(\left|\frac{x_{mnk}}{y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > 1-\gamma \right\}\right|. \text{ Therefore } x \overset{S_\theta^{\wedge}(PP)}{\equiv} y.$$

This completes the proof.

**4.4. Corollary.** A triple $(X,P,*)$ be a *PP*- space. For any lacunary sequence θ = ($m_r n_s k_t$), with 1 < limin f $f_{rst} q_{rst} \le \limsup_{rst} q_{rst} < \infty$, then

$$S_\theta^{\wedge}(PP) = S(PP).$$

**Proof:** The result clearly follows from Theorem 4.2 and Theorem 4.3.

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