IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. I (Mar. - Apr. 2016), PP 09-17
www.iosrjournals.org

Numerical Solution of Two-Point Boundary Value Problems By Using Reliable Iterative Method

M. A. Al-Jawary[1], A. H. Abass[2]
[1](*Head of Department of Mathematics, College of Education for Pure Science (Ibn AL-Haytham) / Baghdad University, Iraq, Baghdad*)
[2](*Department of Mathematics, College of Education for Pure Science (Ibn AL-Haytham) / Baghdad University, Iraq, Baghdad*)

Abstract: *In this paper, we present a reliable iterative method proposed by Daftardar-Gejji and Hossein Jafari namely (DJM) to solve linear and nonlinear two-point boundary value problems. In DJM, the solution is obtained in the series form with easily computed components. The results of the maximal error remainder values show that the present method is very effective and reliable. Numerical examples are given to demonstrate the efficiency of the proposed method. The software used for the calculations in this study was MATHEMATICA® 10.0.*
Keywords: *Iterative method, Two-point boundary value problem, Numerical solution.*

I. Introduction

The boundary value problems are considered one of the important equations because it has been used in different wide fields [1]. Boundary value problems appear in applied mathematics, several branches of physics, engineering and have attracted much attention [2-8]. There are many methods used to solve this problem for instance: Homotopy perturbation method [1], Variational iteration method [9], differential transformation method [10] and extended Adomain decomposition method [11]. There is a new technique for solving linear and nonlinear ordinary and partial differential equations and the other problems proposed by Daftardar-Gejji and Hossein Jafari in 2006 namely (DJM)[12-15]. This method has been used in solving many equations such as the Fornbery-Whitham equation [16], linear and nonlinear Klein-Gordon equations [15], Gas dynamic equation [14] and many others nonlinear equations [13]. In the present paper, we use this method to solve two-point boundary value problems of the form:

$$u'' = f(x, u, u'), \qquad\qquad a < x < b,$$

with boundary conditions

$$u\,(a) = \alpha, \qquad u\,(b) = \beta.$$

Where f is continuous on the set $D = \{(x, u, u') \mid a \le x \le b\ ,\ u, u' \in R\}, a, b, \alpha$ and β are constants. For this propose, we assume an unknown parameter for the value of the derivative with use the boundary point.
The new iterative method is very active, accurate and simple in principles and computer. Also this method has distinguishing features to solve the difficulties in Adomian decomposition method [11] without using Adomian polynomial in solving nonlinear equations. It is well-known the nonlinear two-point boundary value problems has many solutions and at least one positive solution is exist [17-19].

This paper is organized as followers: In section 2, the DJM is presented. In section 3, we present the application of DJM for two-point boundary value problems. In section 4, some examples are and finally in section 5 the conclusion is presented.

II. A new iterative method (DJM)

According to the DJM [12-15], consider the following general function equation:

$$u = N(u) + f \tag{1}$$

Where N is a nonlinear operator from a blanch space $B \to B$ and f is a known function. We are looking for solution u of Eq. (1) having the series form:

$$u = \sum_{i=0}^{\infty} u_{i} \tag{2}$$

The nonlinear operator can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_{i}\right) = N(u_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} u_{j}) - N(\sum_{j=0}^{i-1} u_{j})\} \tag{3}$$

From Eqs. (2) and (3), Eq. (1) is equivalent to

$$\sum_{i=0}^{\infty} u_{i} = f + N(u_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} u_{j}) - N(\sum_{j=0}^{i-1} u_{j})\} \tag{4}$$

We define the recurrence relation:

$$\begin{cases} u_0 = f \\ u_1 = N(u_0) \\ u_{m+1} = N(u_0 + u_1 + \ldots + u_m) - N(u_0 + u_1 + \ldots + u_{m-1}), \quad m = 1,2,\ldots \end{cases} \tag{5}$$

Then

$$(u_1 + \ldots + u_{m+1}) = N(u_0 + \ldots + u_m), \qquad m = 1, 2, \ldots \tag{6}$$

and

$$u = f + \sum_{i=1}^{\infty} u_i \tag{7}$$

III. Solving two-point boundary value problem by using DJM

In this section, we used the DJM to solve two-point boundary value problems to obtain an approximate solution.
Let us consider the following boundary value problem:

$$u'' = N(u) + f \tag{8}$$

With boundary conditions

$$u\,(a) = \alpha, \qquad u\,(b) = \beta.$$

Where N linear or nonlinear parameter, f is given function and $a, b,$ α and β are constants.
Now integrate both sides of Eq.(8) from a to x we get

$$u'(x) = u'(a) + \int_a^x N\big(u(t)\big)dt + \int_a^x f(t)dt \tag{9}$$

Integrate both sides of Eq.(9) from a to x and by using the boundary condition at $x=a$ we get

$$u(x) = \alpha + (x-a)A + \int_a^x \int_a^x N\big(u(t)\big)dtdt + g(x),$$

Where $A = u'(a)$, and $g(x) = \int_a^x \int_a^x f(t)dtdt$, since $u'(a)$ is unknown.

Following (5) we get

$u_0 = \alpha$+(x-a)A+g(x)

$$u_1 = N(u_0) = \int_a^x \int_a^x u_0(t)dtdt\,,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_a^x \int_a^x (u_0+u_1)(t)dtdt - \int_a^x \int_a^x u_0(t)dtdt,$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \int_a^x \int_a^x (u_0+u_1+u_2)(t)dtdt - \int_a^x \int_a^x (u_0+u_1)(t)dtdt\,,$$

$$u_i = \int_a^x \int_a^x (u_0+u_1+\cdots+u_{i-1})(t)dtdt - \int_a^x \int_a^x (u_0+\cdots+u_{i-2})(t)dtdt,$$

Then,

$$u(x) = \sum_{i=0}^{\infty} u_i\,. \tag{10}$$

Now, we impose the second boundary condition at $x = b$ in Eq.(10) to find A when the equation is linear there is no problem in finding the series but when it is nonlinear equation there will be several values, then we choose at least one positive real number of them, for more details of existence and uniqueness, we refer the reader to [17-19]. To assess the accuracy of our approximate solutions, the appropriate function of the error remainder will be [20,21].

$$ER_n(x) = u'' - N(u) - f\,,$$

and the maximal error reminder is

$$MER_n = \max_{0\le x\le 1} |ER_n(x)|.$$

IV. Illustrative examples

In this section, four examples will be solved by DJM, two examples are linear and two others are nonlinear to illustrate the performance of the DJM. The calculations below are done by using MATHEMATICA® 10.0.

Example 1. Consider the linear two-point boundary value problem of the form [22]

$$u'' = x + 2u, \qquad 0 < x < 1 \tag{11}$$

With boundary conditions

$$u(0) = 1, \qquad u(1) = 0, \tag{12}$$

Integrate both sides of Eq.(11) from 0 to x we get

$$u'(x) = u'(0) + \frac{1}{2}x^2 - \int_0^x 2u(t)dt \tag{13}$$

It is important here to note $u'(0)$ is not given. However, $u'(0)$ will be determined later by using the boundary condition at $x = 1$.

Integrate both sides of Eq.(13) from 0 to x and using the boundary condition at x=0 we get

$$u(x) = 1 + \mathrm{A}x + \frac{1}{6}x^3 - \int_0^x \int_0^x 2u(t)dtdt \tag{14}$$

where $\mathrm{A} = u'(0)$.

According to the DJM, we achieve the following components:

$$u_0(x) = 1 + \mathrm{A}x + \frac{1}{6}x^3,$$

$$u_1(x) = N(u_0) = -\int_0^x \int_0^x 2u_0(t)dtdt = -x^2 - \frac{\mathrm{A}x^3}{3} - \frac{x^5}{60},$$

$$u_2(x) = N(u_0 + u_1) - N(u_0) = -\int_0^x \int_0^x 2(u_0 + u_1)(t)dtdt - u_1 = \frac{x^4}{6} - \frac{\mathrm{A}x^5}{30} - \frac{x^7}{1260},$$

$$u_3(x) = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = -\frac{x^6}{90} - \frac{\mathrm{A}x^7}{630} - \frac{x^9}{45360},$$

$$u_4(x) = \frac{x^8}{2520} - \frac{\mathrm{A}x^9}{22680} - \frac{x^{11}}{2494800},$$

Hence $u(x) = \sum_{i=0}^{4} u_i(x)$,

$$u(x) = 1 + \mathrm{A}x - x^2 + \frac{x^3}{6} - \frac{\mathrm{A}x^3}{3} + \frac{x^4}{6} - \frac{x^5}{60} + \frac{\mathrm{A}x^5}{30} - \frac{x^6}{90} + \frac{x^7}{1260} - \frac{\mathrm{A}x^7}{630} - \frac{x^8}{2520} - \frac{x^9}{45360} + \frac{\mathrm{A}x^9}{22680} + \frac{x^{11}}{2494800}. \quad (15)$$

Now, we impose the second boundary condition at x=1 on an approximant Eq.(15) to find A we get

$$u = \frac{5314}{17325} - \frac{22063\mathrm{A}}{3240} = 0 \quad (16)$$

We solve Eq.(16), we got

$$\mathrm{A} = -\frac{382608}{871255}. \quad (17)$$

Now, substituting Eq.(17) in (15), we obtain

$$u(x) = 1 - \frac{382608x}{871255} - x^2 + \frac{1636471x^3}{5227530} + \frac{x^4}{6} - \frac{1636471x^5}{52275300} - \frac{x^6}{90} + \frac{1636471x^7}{1097781300} + \frac{x^8}{2520} - \frac{1636471x^9}{39520126800} + \frac{x^{11}}{2494800}. \quad (18)$$

Table (1) and Figure (1) show the convergence of maximal error remainder, where the points in Figure (1) are lay on a straight lines which mean we achieved exponential rate of convergence.

n	MER_n
1	0.0197076
2	0.0000330407
3	2.20829×10^{-8}
4	7.89768×10^{-12}

Table.1: the maximal error remainder: MER_n by the DJM where $n = 1, \dots, 4$

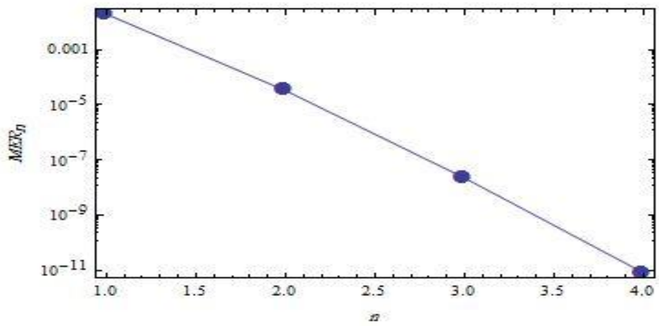

Fig.1: Logarithmic plots of MER_n versus n is 1 through 4

Example 2. Consider the linear two-point boundary value problem of the form [11]

$$u'' = u + \cos(x), \qquad 0 < x < 1, \quad (19)$$

Subject to the boundary conditions

$$u(0) = 1, \qquad u(1) = 1, \tag{20}$$

Integrate both side of Eq.(19) from 0 to x we get

$$u'(x) = u'(0) + \sin(x) + \int_0^x u(t)dt. \tag{21}$$

Integrate both side of Eq.(21) from 0 to x, and using the boundary condition at x=0 we get

$$u(x) = 2 + \mathrm{A}x - \cos(x) + \int_0^x \int_0^x u(t)dtdt \tag{22}$$

where $\mathrm{A} = u'(0)$.

Following (5), we get the recurrence relation:

$$u_0 = 2 + \mathrm{A}x - \cos x,$$

$$u_1 = N(u_0) = \int_0^x \int_0^x u_0(t)dtdt = -1 + x^2 + \frac{\mathrm{A}x^3}{6} + \cos(x),$$

$$u_2 = N(u_0 + u_1) - N(u_0) = \int_0^x \int_0^x (u_0 + u_1)(t)dtdt - u_1 = 1 - \frac{x^2}{2} + \frac{x^4}{12} + \frac{\mathrm{A}x^5}{120} - \cos(x),$$

$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$
$$= \int_0^x \int_0^x (u_0 + u_1 + u_2)(t)dtdt - \int_0^x \int_0^x (u_0 + u_1)(t)dtdt = -1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{360} + \frac{\mathrm{A}x^7}{5040} + \cos(x),$$

$$u_4 = \int_0^x \int_0^x (u_0 + u_1 + u_2 + u_3) - \int_0^x \int_0^x (u_0 + u_1 + u_2) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{20160} + \frac{\mathrm{A}x^9}{362880} - \cos(x)$$

Hence $\quad u(x) = \sum_{i=0}^{4} u_i(x)$,

$$u(x) = 2 + \mathrm{A}x + \frac{x^2}{2} + \frac{\mathrm{A}x^3}{6} + \frac{x^4}{12} + \frac{\mathrm{A}x^5}{120} + \frac{x^6}{720} + \frac{\mathrm{A}x^7}{5040} + \frac{x^8}{20160} + \frac{\mathrm{A}x^9}{362880} - \cos(x) \tag{23}$$

Now, we impose the boundary condition at x=1 in Eq.(23) we have

$$u = \frac{52109}{20160} + \frac{426457\mathrm{A}}{362880} - \cos(1). \tag{24}$$

Here we solve Eq.(24) to find a parameter A, we get

$$\mathrm{A} = -\frac{595174533}{638115961}, \tag{25}$$

By substituting Eq.(25) in (23) we get

$$u = 2 - \frac{198391511x}{638115961} + \frac{x^2}{2} - \frac{198391511x^3}{1276231922} + \frac{x^4}{12} - \frac{198391511x^5}{25524638440} + \frac{x^6}{720} - \frac{198391511x^7}{1072034814480} + \frac{x^8}{20160} - \frac{198391511x^9}{77186506642560} - \cos(x) \tag{26}$$

It can be seen clearly from Table 2, Figure 2 that by increasing the iterations (n from 1 to 4) we achieved good accuracy with exponential rate of convergence.

n	MER_n
1	0.00484871
2	4.09033×10^{-6}
3	1.37063×10^{-9}
4	2.4547×10^{-13}

Table.2: the maximal error remainder: MER_n by the DJM where $n = 1, \dots, 4$

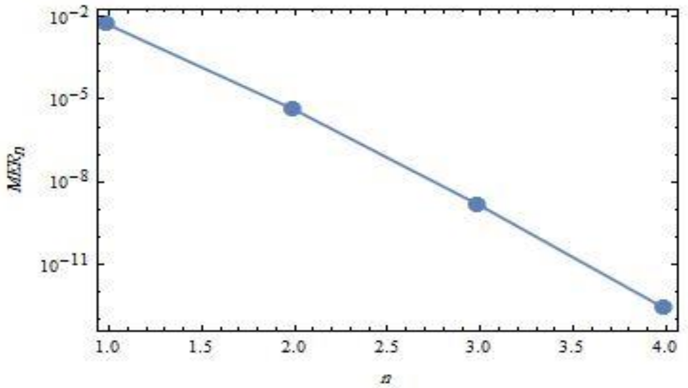

Fig.2: Logarithmic plots of MER_n versus n is 1 through 4

Example 3. Consider the nonlinear boundary value problem of the form [23]

$$u'' = -(u')^3, \tag{27}$$

With boundary condition

$$u(0) = \sqrt{2}, \qquad u(1) = 2. \tag{28}$$

Integrate both sides of Eq.(27) from 0 to x we get

$$u'(x) = u'(0) - \int_0^x (u'(t))^3 dt \tag{29}$$

Integrate both sides of Eq.(29) from 0 to x and using the boundary condition at x=0 we get

$$u(x) = \sqrt{2} + Ax - \int_0^x \int_0^x (u'(t))^3 dtdt, \tag{30}$$

where $\mathrm{A} = u'(0)$.

According to the DJM, we obtain the following components:

$$u_0 = \sqrt{2} + \mathrm{A}x,$$

$$u_1 = N(u_0) = -\int_0^x \int_0^x ({u'}_0(t))^3 dtdt = -\frac{1}{2}A^3x^2,$$

$$u_2 = N(u_0 + u_1) - N(u_0) = -\int_0^x \int_0^x ((u_0 + u_1)'(t))^3 dtdt - u_1 = \frac{A^3x^2}{2} - \frac{1}{4}A\left(x - \frac{1 + (-1 + A^2x)^5}{5A^2}\right),$$

$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = -\int_0^x \int_0^x ((u_0 + u_1 + u_2)'(t))^3 dtdt + \int_0^x \int_0^x ((u_0 + u_1)'(t))^3 dtdt,$$
$$= -\frac{A^3x^2}{2} + \frac{A^5x^3}{2} - \frac{5A^7x^4}{8} + \frac{13A^9x^5}{20} - \frac{3A^{11}x^6}{5} + \frac{27A^{13}x^7}{56} - \cdots$$

$$u_4 = N(u_0 + u_1 + u_2 + u_3) - N(u_0 + u_1 + u_2)$$
$$= -\int_0^x \int_0^x ((u_0 + u_1 + u_2 + u_3)'(t))^3 dtdt + \int_0^x \int_0^x ((u_0 + u_1 + u_2)'(t))^3 dtdt,$$

$$= \frac{9A^9x^5}{40} - \frac{3A^{11}x^6}{5} + \frac{39A^{13}x^7}{35} - \frac{3879A^{15}x^8}{2240} + \frac{381A^{17}x^9}{160} - \frac{2403A^{19}x^{10}}{800} + \cdots$$

The approximation solution for $u = \sum_{i=0}^{4} u_i$

$$u = \sqrt{2} + Ax + \frac{A^3x^2}{2} - \frac{A^5x^3}{2} + \frac{A^7x^4}{8} + \frac{A^9x^5}{8} - \frac{A^{11}x^6}{5} - \cdots \tag{31}$$

Here we impose the boundary condition at x=1 in Eq. (31) we get

$$u = \sqrt{2} + A - \frac{A^3}{2} + \frac{A^5}{2} - \frac{A^7}{8} + \frac{A^9}{8} - \cdots \quad (32)$$

By solving the equation (32) for the parameter A, we obtain several values, and then we may choose at least one positive real number of them.

$$A = 0.706248. \quad (33)$$

By substituting Eq. (33) in (31) we get

$$u = 1 + 0.70624x - 0.1763x^2 - 0.08785x^3 + 0.05477x^4 + 0.03824x^5 - 0.026164x^6 + \cdots \quad (34)$$

Table. 3 and Figure 3 illustrate the convergence of the solution through the use of the maximal error reminder.

n	MER_n
1	0.0501262
2	0.00345636
3	0.00016207
4	5.72912×10^{-6}

Table.3: the maximal error remainder: MER_n by the DJM where $n = 1, \ldots, 4$

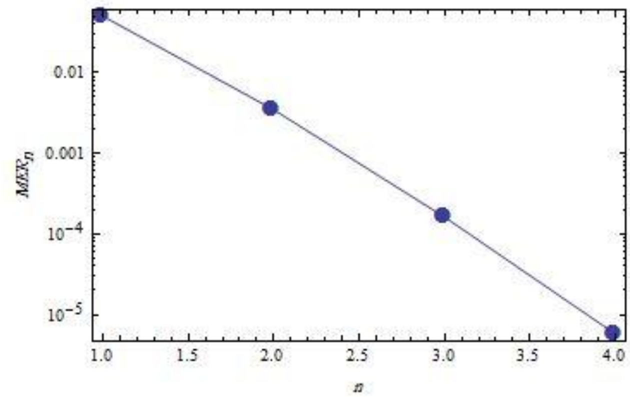

Fig.3: Logarithmic plots of MER_n versus n is 1 though 4

Example 4. Let use consider the following nonlinear two-point boundary value problem given in [23]

$$u'' = -2uu', \qquad 0 < x < 1 \quad (35)$$

With the boundary condition:

$$u(0) = 1, \quad u(1) = \frac{1}{2}. \quad (36)$$

By taking the integration twice to Eq. (35) from 0 to x and substituting the boundary condition at $x = 0$ we get

$$u(x) = 1 + Ax - \int_0^x \int_0^x 2u(t)u'(t)dtdt \quad (37)$$

where $A = u'(0)$, following (5) we get

$u_0 = 1 + Ax,$

$$u_1 = N(u_0) = -\int_0^x \int_0^x 2u_0(t)u'_0(t)dtdt = -Ax^2 - \frac{A^2x^3}{3},$$

$u_2 = N(u_0 + u_1) - N(u_0)$

$$= -\int_0^x \int_0^x 2(u_0+u_1)(t)(u_0+u_1)'(t)dtdt - u_1 = \frac{2Ax^3}{3} + \frac{2A^2x^4}{3} - \frac{A^2x^5}{5} + \frac{2A^3x^5}{15} - \frac{A^3x^6}{9} - \frac{A^4x^7}{63},$$

$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1)$

$$= -\int_0^x\int_0^x 2(u_0+u_1+u_2)(t)(u_0+u_1+u_2)'(t)dtdt + \int_0^x\int_0^x 2(u_0+u_1)(t)(u_0+u_1)'(t)dtdt$$

$$= -\frac{Ax^4}{3} - \frac{8A^2x^5}{15} + \frac{13A^2x^6}{45} - \frac{4A^3x^6}{15} - \frac{4A^2x^7}{63} + \frac{12A^3x^7}{35} - \cdots,$$

$u_4 = N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2)$

$$= -\int_0^x\int_0^x 2(u_0+u_1+u_2+u_3)(t)(u_0+u_1+u_2+u_3)'(t)dtdt$$

$$+ \int_0^x\int_0^x 2(u_0+u_1+u_2)(t)(u_0+u_1+u_2)'(t)dtdt$$

$$u_4 = \frac{2Ax^5}{15} - \frac{13A^2x^6}{45} + \frac{8A^2x^7}{45} + \frac{8A^3x^7}{35} - \frac{A^2x^8}{14} + \frac{67A^3x^8}{210} + \cdots,$$

$$u_5 = -\frac{2Ax^6}{45} - \frac{38A^2x^7}{315} + \frac{7A^2x^8}{90} - \frac{163A^3x^8}{1260} - \frac{101A^2x^9}{2835} + \cdots.$$

and so on. Therefore, we get

$$u = u_0+u_1+u_2+\cdots = 1 + Ax - Ax^2 + \frac{2Ax^3}{3} - \frac{A^2x^3}{3} - \frac{Ax^4}{3} + \frac{2A^2x^4}{3} + \frac{2Ax^5}{15} - \frac{11A^2x^5}{15} + \cdots \quad (38)$$

Now, we substituting the boundary condition at x=1 in Eq. (38) we get

$$u = 1 + \frac{19A}{45} - \frac{3802A^2}{51975} + \frac{457859A^3}{255405150} - \frac{3871235447A^4}{742462771050} - \cdots \quad (39)$$

Now, when we solve Eq.(39), the value of A is obtain among many roots, we choose a real positive number at least.

$$A = 4.13283. \quad (40)$$

By substituting Eq. (40) in (39) we obtain

$$u = 1 + 4.13283x - 4.13283x^2 - 2.93821x^3 + 10.0093x^4 - 2.56251x^5 - \cdots$$

Table 4 and Figure 4 show the analysis of the maximal error remainders for MER_n, where the points are lay on a straight lines which mean we achieved exponential rate of convergence.

n	MER_n
1	3.114
2	0.382912
3	0.0311592
4	0.00188825
5	0.0000911657

Table.4: the maximal error remainder: MER_n by the DJM where $n = 1, \ldots, 5$

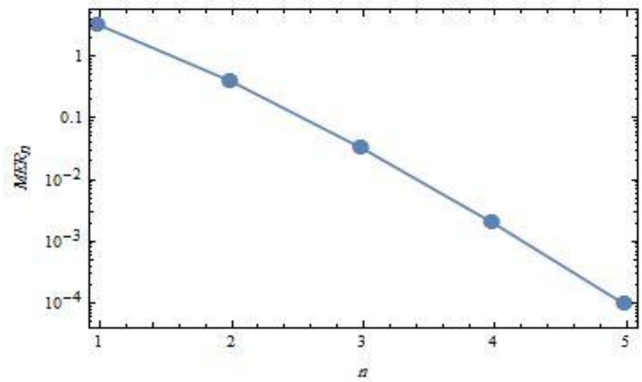

Fig.4: Logarithmic plots of MER_n versus n is 1 through 4

V. Conclusion

The main objective of this paper has been achieved by solving the linear and nonlinear two-point boundary value problems. For linear problems the solution is exist and unique, however, for nonlinear case one positive solution is obtained. We have also achieved the rapid convergence of our approximate solutions by the sequence of the curves of the error remainder functions with logarithmic plots of the maximal error parameters. Moreover, by solving some examples, it is seems that the DJM appears to be very accurate to employ with reliable results. The software used for the calculations in this study was MATHEMATICA® 10.0.

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