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Fourth And Fifth Orders Approximate Solutions Of Stationary Exterior Fields Of Einstein's Equations

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Abstract: *Approximate solutions of stationary exterior fields of Einstein's equations are obtained by expanding the metric in powers of a certain parameter and solving explicitly the first few orders in terms of two harmonic functions. Earlier approximate solutions up to third order were found. In the present paper we obtain the new fourth and fifth order equations and find their approximate solutions for the particular choice of the harmonic functions. There is a physical interpretation of the approximate solutions at the end of the paper.*
Keywords: *Einstein's Field Equations, Approximate Solutions, Asymptotically Flat Solutions*

I. Introduction

Till now a general axisymmetric stationary (rotating) solution of Einstein's exterior (vacuum) field equations has not been found. The two known classes of solutions- Lewis [1] and Papapetrou [2] are of particular interest, apart from a few particular solutions such as those of Kerr [3] and Tomimatsu and Sato [4]. The Lewis and Papapetrou solutions depend on a harmonic function (solution of the flat-space axisymmetric Laplace equation), while the Kerr and Tomimatsu-Sato solutions are asymptotically flat. Along the axis of symmetry extending to infinity, the Lewis solutions have a line singularity and in the case of Papapetrou solutions, though they have a subclass containing asymptotically flat solutions, they correspond to zero mass of the source. For this reason these solutions are considered to be not physically interesting, although some constructive use of the solutions might be possible, as indicated by the work of Herlt [5]. A class of exact solutions of the static (non-rotating) axisymmentric Einstein exterior (vacuum) equations was found by Weyl [6] which is well known to depend on a single harmonic function. Islam [7,8] obtained a class of approximate stationary solutions in terms of two harmonic functions σ and ζ by expanding the metric in terms of a parameter. The property of this class is that it reduced to the Weyl class when ζ =0 and to the Papapetrou class when σ =0. Earlier Salam [9,10] obtained these classes of solutions upto third order. Making use of some different procedures in this paper (hereinafter referred to as "Paper"III) we obtained the solutions more higher orders than earlier [11].

In section II field equations and their approximate solutions up to third order have been given for the required purpose of the present paper. The derivation of more higher (fourth and fifth) orders equations and their solutions are given in section III. The physical interpretation of the approximate solutions is also given in section IV.

II. Field Equations and Their Approximate Solutions up to Third Order

In an earlier Paper-I [10], we considered a general formulation of the axisymmetric stationary Einstein's exterior field equations beginning with Weyl-Lewis – Papapetrou form of the metric [7,9]:

$$ds^2 = fdt^2 - 2kdtd\phi - ld\phi^2 - e^{\mu}\left(d\rho^2 + dz^2\right), \tag{1}$$

where (ρ , ϕ , z) are cylindrical-polar-like co-ordinates ($x = \rho\cos\phi$, $y = \rho\sin\phi$, z), t is the time, and c is the velocity of light where, c = 1, and f, k, l, μ are all functions of ρ and z. Here 'cylinder polar' means that for asymptotically flat solutions for a large distance from the source the metric tends to the following flat space metric

$$ds^2 = dt^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2\,, \tag{2}$$

where $k\to 0$, $l\to \rho^2$, $\mu \to 0$ at infinity [7,9,10].

It was shown by Weyl for the static field (k = 0), and the procedure extended for stationary fields ($k \neq 0$), we can impose for the field equations the following algebraic condition on the functions f, k and l:

$$D^2 \equiv fl + k^2 = \rho^2\,. \tag{3}$$

We label the coordinates $\left(x^0 = ct, x^1, x^2, x^3\right) = \left(t, \rho, z, \phi\right)$ $\left(with \quad c = 1\right)$. Three of Einstein's exterior field equations are as follows [7,9,10,11]:

$$2e^{\mu}D^{-1}R_{00} = \left(D^{-1}f_{\rho}\right)_{\rho} + \left(D^{-1}f_z\right)_z + D^{-3}f\left(f_{\rho}l_{\rho} + f_z l_z + k_{\rho}^2 + k_z^2\right) = 0\,, \tag{4a}$$

$$-2e^{\mu}D^{-1}R_{03}=\left(D^{-1}k_{\rho}\right)_{\rho}+\left(D^{-1}k_{z}\right)_{z}+D^{-3}k\left(f_{\rho}l_{\rho}+f_{z}l_{z}+k_{\rho}^{2}+k_{z}^{2}\right)=0\,, \quad (4b)$$

$$-2e^{\mu}D^{-1}R_{33}=\left(D^{-1}l_{\rho}\right)_{\rho}+\left(D^{-1}l_{z}\right)_{z}+D^{-3}l\left(f_{\rho}l_{\rho}+f_{z}l_{z}+k_{\rho}^{2}+k_{z}^{2}\right)=0\,, \quad (4c)$$

where $f_{\rho}\equiv\frac{\partial f}{\partial\rho}, f_{z}\equiv\frac{\partial f}{\partial z}$, etc. Only two of (4 a,b,c) are independent because of (3). For convenient Papapetrou used the function w instead of k, defined by [6]

$$w=\frac{k}{f}\,. \quad (5)$$

Eliminating k and l from (4a,b), with the help of (5), the field equations become

$$f\left(f_{\rho\rho}+f_{zz}+\rho^{-1}f_{\rho}\right)-f_{\rho}^{2}-f_{z}^{2}+\rho^{-2}f^{4}\left(w_{\rho}^{2}+w_{z}^{2}\right)=0, \quad (6a)$$

$$f\left(w_{\rho\rho}+w_{zz}-\rho^{-1}w_{\rho}\right)+2f_{\rho}w_{\rho}+2f_{z}w_{z}=0 \quad [2,3,4,5]. \quad (6b)$$

Making use of (3), Einstein's other non-trivial equations are:

$$2R_{11}=-\mu_{\rho\rho}-\mu_{zz}+\rho^{-1}\mu_{\rho}+\rho^{-2}\left(f_{\rho}l_{\rho}+k_{\rho}^{2}\right)=0, \quad (7a)$$

$$2R_{12}=\rho^{-1}\mu_{z}+\frac{1}{2}\rho^{-2}\left(f_{\rho}l_{z}+f_{z}l_{\rho}+2k_{\rho}k_{z}\right)=0, \quad (7b)$$

$$2R_{22}=-\mu_{\rho\rho}-\mu_{zz}-\rho^{-1}\mu_{\rho}+\rho^{-2}\left(f_{z}l_{z}+k_{z}^{2}\right)=0. \quad (7c)$$

From (7a,b,c) we can express μ_{ρ}, μ_{z} in terms of f and w as follows:

$$\mu_{\rho}=-f^{-1}f_{\rho}+\frac{1}{2}\rho f^{-2}\left(f_{\rho}^{2}-f_{z}^{2}\right)-\frac{1}{2}\rho^{-1}f^{2}\left(w_{\rho}^{2}-w_{z}^{2}\right), \quad (8a)$$

$$\mu_{z}=-f^{-1}f_{z}+\rho f^{-2}f_{\rho}f_{z}-\rho^{-1}f^{2}w_{\rho}w_{z}\,. \quad (8b)$$

The consistency of (8a,b) is guaranteed by (6a,b). Therefore the basic equations are (6a,b), since μ can be obtained trivially from (8a,b) once f and w are known [9]. In the absence of rotation $w = 0$ ($k = 0$), (6b) becomes redundant and (6a) can be solved by putting $f=e^{\sigma}$, where σ is harmonic and satisfies the Laplace equation

$$\nabla^{2}\sigma=\sigma_{\rho\rho}+\sigma_{zz}+\rho^{-1}\sigma_{\rho}=0, \quad (9)$$

and the Weyl (resulting) metric can be written as:

$$ds^{2}=e^{\sigma}dt^{2}-e^{-\sigma}\left[e^{\chi}\left(d\rho^{2}+dz^{2}\right)+\rho^{2}d\phi^{2}\right]\,, \quad (10)$$

$$\text{with } \chi_{z}=\rho\sigma_{\rho}\sigma_{z}, \chi_{\rho}=\frac{1}{2}\rho\left(\sigma_{\rho}{}^{2}-\sigma_{z}{}^{2}\right). \quad (11)$$

The consistency of (11) is guaranteed by (9). Equations (9–11) give the Weyl (1917) class of solutions which represents axially symmetric static (non-rotating) exterior fields [6,7,8,9,10].

A class of exact solutions of (6a,b) was found by Papapetrou (1953) in terms of a harmonic function is given by

$$w=A\rho\zeta_{\rho}, \nabla^{2}\zeta=\zeta_{\rho\rho}+\zeta_{zz}+\rho^{-1}\zeta_{\rho}=0, \quad (12a)$$

where A is an arbitrary constant, and

$$h=f^{-1}=\alpha\cosh\zeta_{z}+\beta\sinh\zeta_{z}, A^{2}=\alpha^{2}-\beta^{2}\,, \quad (12b)$$

where α, β are also arbitrary constants and related to A as above [2,7,8,9,10]. For asymptotic flatness h must tend to unity and

$$w=\text{-}A\,\rho^{2}r^{-3} \quad (13)$$

must tends to zero at infinity, where $r^{2}=\rho^{2}+z^{2}$ [7]. The unphysical nature of the solution (12a,b) can be demonstrated by taking $\zeta=r^{-1}$, in which the metric is asymptotically flat, w having the behaviour as in (13), but $g_{00}=h^{-1}=f$ has no term proportional to r^{-1} for large r, i.e., there is no mass. The Weyl class of solutions contains asymptotically flat solutions with mass, that can be shown by considering the harmonic function

$$\sigma=-2mr^{-1},\; r^{2}=\rho^{2}+z^{2}, \quad (14)$$

where m is the mass. This is the Curzon solution (1924) [12]. Thus if we could somehow combine the Weyl and Papapetrou solutions, we might get asymptotically flat solutions with non-zero mass. There is a class of approximate solutions depending on two harmonic functions σ and ζ was found by Islam (1976a, 1985) [7,8] such that if $\zeta = 0$ one gets the Weyl class of solutions and if $\sigma = 0$ the approximate solutions reduce to the (exact) Papapetrou class of solutions. Earlier, Salam (1988, 1997) [9,10] extended this approximate class of solutions to a higher (3rd) order for a specific choice of σ, ζ. Using some different procedures in this paper, we extend this approximate solution to two higher orders.

Making use the symbol $h = f^{-1}$ instead of f, the equations (6a,b) transform to the following:

$$h\,(h_{\rho\rho} + h_{zz} + \rho^{-1}h_\rho) - h_\rho^2 - h_z^2 - \rho^{-2}\,(w_\rho^2 + w_z^2) = 0, \tag{15a}$$

$$h\,(w_{\rho\rho} + w_{zz} - \rho^{-1}w_\rho) - 2h_\rho w_\rho - 2h_z w_z = 0. \tag{15b}$$

We obtain the approximate class of solutions in the following way: Assume the solutions we are seeking depend on harmonic functions σ, ζ represented symbolically by $F(\sigma, \zeta)$ so that $F(\sigma, 0)$ is the Weyl solution and $F(0, \zeta)$ is the Papapetrou solution. We can replace σ, ζ by $\lambda\sigma$, $\lambda\zeta$ where λ is a constant parameter. This can be done since $\lambda\sigma$, $\lambda\zeta$ are also harmonic. Then we expand $F(\lambda\sigma, \lambda\zeta)$ in a power series in λ which amounts to expanding the functions h and w in a power series in λ and solving (15a,b) successively in terms of two harmonic functions. Hence we expand h,w as follows :

$$h = 1 + \lambda h^{(1)} + \lambda^2 h^{(2)} + \lambda^3 h^{(3)} + \lambda^4 h^{(4)} + \ldots\ldots\ldots\ldots + \lambda^n h^{(n)} + \ldots\ldots, \tag{16a}$$

$$w = \lambda w^{(1)} + \lambda^{2w(2)} + \lambda^3 w^{(3)} + \lambda^4 w^{(4)} + \ldots\ldots\ldots\ldots + \lambda^n w^{(n)} + \ldots\ldots, \tag{16b}$$

where $h^{(i)}$, $w^{(i)}$ are all functions of ρ and z and $i = 1, 2, 3, 4, \ldots\ldots\ldots\ldots$ When we assume $\lambda = 0$, we get the Minkowski space. This is consistent with the fact that when $\lambda = 0$ (16 a, b) give respectively $h = 1$ and $w = 0$, and the space becomes the (flat space-time) Minkowski space-time for these values of h and w.

In the Weyl and Papapetrou solutions replacing σ, ζ by $\lambda\sigma$, $\lambda\zeta$ respectively we have

$$h = \exp(\lambda\sigma),\ w = 0, \tag{17a}$$

$$h = \alpha \cosh \lambda\,\zeta_z + \beta \sinh \lambda\,\zeta_z\ ,\ w = \left(\alpha^2 - \beta^2\right)^{\frac{1}{2}} \lambda\rho\zeta_\rho . \tag{17b}$$

The earlier Weyl and Papapetrou solutions with the trivial change of sign of σ are also given respectively by (17a) and (17b). Our aim is to get a power series solution in terms of σ and ζ such that in each order we get the Weyl solution when $\zeta = 0$ and the Papapetrou solution when $\sigma = 0$. If $\lambda = 0$, we get $h = 1$, $\alpha = 1$, from (17 a, b) [9,10]. For the Weyl solution (16a), (17a) imply [7,10,11]:

$$h^{(n)} = \frac{1}{n!}\sigma^n,\ w^{(n)} = 0, \text{ for all } n, \tag{18}$$

and for the Papapetrou solution, it is readily verified from (16a,b) and (17a,b) that (with $\alpha = 1$, $\beta < 1$) :

$$h^{(1)} = \beta\ \zeta_z\ ,\ h^{(2)} = \frac{1}{2}\ \zeta_z^2\ ,\ h^{(3)} = \frac{1}{6}\beta\ \zeta_z^3 \tag{19a}$$

$$w^{(1)} = (1-\beta^2)^{\frac{1}{2}}\rho\zeta_\rho\ ,\ w^{(n)} = 0,\ n \geq 2. \tag{19b}$$

In Paper-I [10], we have explained and found the order of equation up to third order and their solutions. For the required purpose of the present paper we only mentioned them below:

Making use of (15a,b) and (16a,b), we have found first, second and third order (coefficients of $\lambda, \lambda^2, \lambda^3$) equations and their solutions which are respectively as follows [9,10 (Paper-I)]:

$$\nabla^2 h^{(1)} = 0, \Delta w^{(1)} = 0\ , \tag{20}$$

the solutions of which can be taken as

$$h^{(1)} = \sigma + \beta\zeta_z\ ,\ w^{(1)} = (1-\beta^2)^{\frac{1}{2}}\rho\zeta_\rho\ , \tag{21}$$

where σ, ζ, ζ_z are harmonic, and $|\beta| < 1$ and three-dimensional (axisymmetric) Laplacian operator ∇^2 in the cylindrical polar coordinates defined by

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} + \rho^{-1}\frac{\partial}{\partial\rho}\right) \text{ and } \Delta \equiv \left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} - \rho^{-1}\frac{\partial}{\partial\rho}\right). \tag{22}$$

$$\nabla^2 h^{(2)} + h^{(1)}\nabla^2 h^{(1)} - h_\rho^{(1)2} - h_z^{(1)2} - \rho^{-2}\left(w_\rho^{(1)2} + w_z^{(1)2}\right) = 0\ . \tag{23}$$

Using (20) and (21), equation (23) reduces to the following [9]:

$$\nabla^2 h^{(2)} = \sigma_\rho^2 + \sigma_z^2 + \zeta_{\rho z}^2 + \zeta_{zz}^2 + 2\beta(\sigma_\rho \zeta_{\rho z} + \sigma_z \zeta_{zz}), \tag{24}$$

which has the solution

$$h^{(2)} = \frac{1}{2}\sigma^2 + \frac{1}{2}\zeta_z^2 + \beta\sigma\zeta_z. \tag{25}$$

To obtain this solution we can use the identity

$$\nabla^2(GH) = G\nabla^2 H + H\nabla^2 G + 2(G_\rho H_\rho + G_z H_z), \tag{26}$$

where G, H are any two functions of ρ, z.

$$\Delta w^{(2)} + h^{(1)}\Delta w^{(1)} - 2h_\rho^{(1)} w_\rho^{(1)} - 2w_z^{(1)} h_z^{(1)} = 0. \tag{27}$$

Using (21) into (27), we have

$$\Delta w^{(2)} = 2(1-\beta^2)^{\frac{1}{2}} \rho(\sigma_z \zeta_{\rho z} - \sigma_\rho \zeta_{zz}). \tag{28}$$

By simple integration, the solution of (28) becomes:

$$w_\rho^{(2)} = (1-\beta^2)^{\frac{1}{2}} \rho(\sigma_z \zeta_z - \sigma\zeta_{zz}), \tag{29a}$$

$$w_z^{(2)} = (1-\beta^2)^{\frac{1}{2}} \rho(\sigma\zeta_{\rho z} - \sigma_p \zeta_z). \tag{29b}$$

$$\nabla^2 h^{(3)} + h^{(1)}\nabla^2 h^{(2)} + h^{(2)}\nabla^2 h^{(1)} - 2[h_\rho^{(1)} h_\rho^{(2)} + h_z^{(1)} h_z^{(2)} + \rho^{-2}(w_\rho^{(1)} w_\rho^{(2)} + w_z^{(1)} w_z^{(2)})] = 0 \tag{30a}$$

$$\Delta w^{(3)} + h^{(1)}\Delta w^{(2)} + h^{(2)}\Delta w^{(1)} - 2[h_\rho^{(1)} w_\rho^{(2)} + h_\rho^{(2)} w_\rho^{(1)} + h_z^{(1)} w_z^{(2)} + h_z^{(2)} w_z^{(1)})] = 0 \tag{30b}$$

Making use of aforesaid equations into (30a,b) and by some manipulations, we have

$$\nabla^2 h^{(3)} = \sigma(\zeta_{\rho z}^2 + \zeta_{zz}^2) + 2\beta^2 \zeta_z(\sigma_\rho \zeta_{\rho z} + \sigma_z \zeta_{zz}) + (\sigma + \beta\zeta_z)(\sigma_\rho^2 + \sigma_z^2)$$
$$+ 2\beta\sigma(\sigma_\rho \zeta_{\rho z} + \sigma_z \zeta_{zz}) + \beta\zeta_z(\zeta_{\rho z}^2 + \zeta_{zz}^2) \tag{31a}$$

$$\Delta w^{(3)} = 2(1-\beta^2)^{\frac{1}{2}} \rho(\sigma + \beta\zeta_z)(\sigma_z \zeta_{\rho z} - \sigma_\rho \zeta_{zz}). \tag{31b}$$

To obtain the approximate solutions of the foregoing equations for the particular choice of harmonic functions σ, ζ, let

$$\sigma = ar^{-1}, \zeta = br^{-1}, r = (\rho^2 + z^2)^{\frac{1}{2}}, \tag{32}$$

where a, b as arbitrary constants.
Now by using (32) in the above equations and by some simplifications, we have

$$h^{(1)} = ar^{-1} - \beta bzr^{-3},\ w^{(1)} = -(1-\beta^2)^{\frac{1}{2}} b\rho^2 r^{-3},\ \nabla^2 h^{(2)} = a^2 r^{-4} - 4\beta abzr^{-6} + b^2(r^{-6} + 3z^2 r^{-8}).$$

$$h^{(2)} = \frac{1}{2}a^2 r^{-2} - \beta abzr^{-4} + \frac{1}{2}b^2 z^2 r^{-6}, \Delta w^{(2)} = -2(1-\beta^2)^{\frac{1}{2}} ab\rho^2 r^{-6}, w^{(2)} = -\frac{1}{2}(1-\beta^2)^{\frac{1}{2}} ab\rho^2 r^{-4},$$

$$\nabla^2 h^{(3)} = a^3 r^{-5} - 5\beta a^2 bzr^{-7} + ab^2[r^{-7} + (3+4\beta^2)z^2 r^{-9}] - \beta b^3(zr^{-9} + 3z^3 r^{-11})$$

$$h^{(3)} = \frac{1}{6}a^3 r^{-3} - \frac{1}{2}\beta a^2 bzr^{-5} + ab^2[\frac{1}{35}(1-\beta^2)r^{-5} + \frac{1}{14}(3+4\beta^2)z^2 r^{-7}] - \frac{1}{6}\beta b^3 z^3 r^{-9}$$

$$\Delta w^{(3)} = 2(1-\beta^2)^{\frac{1}{2}}(-a^2 b\rho^2 r^{-7} + ab^2\beta\rho^2 zr^{-9}),\ w^{(3)} = (1-\beta^2)^{\frac{1}{2}}[-\frac{1}{5}a^2 b\rho^2 r^{-5} + \frac{1}{7}\beta a\ b^2\rho^2 zr^{-7}] \tag{33}$$

III. Derivation of Fourth and Fifth Order Equations and Their Solutions for Special Choice of Harmonic Functions

To obtain the fourth and fifth order equations (coefficients of λ^4 and λ^5) using the power series (16a,b) into the basic equations (15a,b) and by some simplifications, we have respectively:

$$\nabla^2 h^{(4)} = -h^{(1)}\nabla^2 h^{(3)} - h^{(2)}\nabla^2 h^{(2)} + 2h_\rho^{(1)} h_\rho^{(3)} + h_\rho^{(2)2} + 2h_z^{(1)} h_z^{(3)} + h_z^{(2)2}$$
$$+ \rho^{-2}[2w_\rho^{(1)} w_\rho^{(3)} + w_\rho^{(2)2} + 2w_z^{(1)} w_z^{(3)} + w_z^{(2)2}], \tag{34a}$$

$$\Delta w^{(4)} = -h^{(1)}\Delta w^{(3)} - h^{(2)}\Delta w^{(2)} + 2(h_\rho^{(1)} w_\rho^{(3)} + h_\rho^{(2)} w_\rho^{(2)} + h_\rho^{(3)} w_\rho^{(1)}) + 2(h_z^{(1)} w_z^{(3)} + h_z^{(2)} w_z^{(2)} + h_z^{(3)} w_z^{(1)}) \tag{34b}$$

$$\nabla^2 h^{(5)} = -h^{(1)}\nabla^2 h^{(4)} - h^{(2)}\nabla^2 h^{(3)} - h^{(3)}\nabla^2 h^{(2)} + 2h_\rho^{(1)}h_\rho^{(4)} + 2h_\rho^{(2)}h_\rho^{(3)} + 2h_z^{(1)}h_z^{(4)} + 2h_z^{(2)}h_z^{(3)} + \rho^{-2}[2(w_\rho^{(1)}w_\rho^{(4)} + w_\rho^{(2)}w_\rho^{(3)} + w_z^{(1)}w_z^{(4)} + w_z^{(2)}w_z^{(3)})], \tag{35a}$$

$$\Delta w^{(5)} = -h^{(1)}\Delta w^{(4)} - h^{(2)}\Delta w^{(3)} - h^{(3)}\Delta w^{(2)} + 2(h_\rho^{(1)}w_\rho^{(4)} + h_\rho^{(2)}w_\rho^{(3)} + h_\rho^{(3)}w_\rho^{(2)} + +h_\rho^{(4)}w_\rho^{(1)}) + 2(h_z^{(1)}w_z^{(4)} + h_z^{(2)}w_z^{(3)} + h_z^{(3)}w_z^{(2)} + +h_z^{(4)}w_z^{(1)}). \tag{35b}$$

The above fourth and fifth order equations clearly have the following form [11]:

$$\nabla^2 h^{(4)} = H_4(\rho,z);\ \Delta\ w^{(4)} = G_4(\rho,z), \tag{36a}$$

$$\nabla^2 h^{(5)} = H_5(\rho,z);\ \Delta\ w^{(5)} = G_5(\rho,z), \tag{36b}$$

where *Hi* , *Gi* are linear combinations of terms of the form :

$$\rho^n z^m r^{-p},\ r = (\rho^2 + z^2)^{\frac{1}{2}}, \tag{37}$$

where *n*, *m*, *p* are integers (in general, positive). The solutions $h^{(i)}, w^{(i)}$ are also linear combinations of terms of the same form (37). In this connection, the following results are useful:

$$(\rho^n z^m r^{-p})_\rho = z^m(n\rho^{n-1}r^{-p} - p\rho^{n+1}r^{-p-2}),\ (\rho^n z^m r^{-p})_z = \rho^n(m z^{m-1}r^{-p} - pz^{m+1}r^{-p-2}),$$

$$(\rho^n z^m r^{-p})_{\rho\rho} = z^m[n(n-1)\rho^{n-2}r^{-p} - p(2n+1)\rho^n r^{-p-2} + p(p+2)\rho^{n+2}r^{-p-4}],$$

$$(\rho^n z^m r^{-p})_{zz} = \rho^n[m(m-1)z^{m-2}r^{-p} - p(2m+1)z^m r^{-p-2} + p(p+2)z^{m+2}r^{-p-4}],$$

$$\nabla^2(\rho^n z^m r^{-p}) = (\rho^n z^m r^{-p})_{\rho\rho} + (\rho^n z^m r^{-p})_{zz} + \rho^{-1}(\rho^n z^m r^{-p})_\rho = [n^2\rho^{n-2}z^m r^{-p} + m(m-1)\rho^n z^{m-2}r^{-p} + p(p-2n-2m-1)\rho^n z^m r^{-p-2}],$$

$$\Delta\ (\rho^n z^m r^{-p}) = (\rho^n z^m r^{-p})_{\rho\rho} + (\rho^n z^m r^{-p})_{zz} - \rho^{-1}(\rho^n z^m r^{-p})_\rho = [n(n-2)\rho^{n-2}z^m r^{-p} + m(m-1)\rho^n z^{m-2}r^{-p} + p(p-2n-2m+1)\rho^n z^m r^{-p-2}]. \tag{38}$$

Substituting the cited various expressions and their derivatives into (34a,b) and by some manipulations, we get respectively:

$$\begin{aligned}\nabla^2 h^{(4)} = &-(ar^{-1} - \beta bzr^{-3})\{a^3r^{-5} - 5\beta a^2bzr^{-7} + ab^2[r^{-7} + (3+4\beta^2)z^2r^{-9}] - \beta b^3(zr^{-9} + 3z^3r^{-11})\} \\ &-(\frac{1}{2}a^2r^{-2} - \beta abzr^{-4} + \frac{1}{2}b^2z^2r^{-6})\{a^2r^{-4} - 4\beta abzr^{-6} + b^2(r^{-6} + 3z^2r^{-8})\} \\ &+ 2(-a\rho r^{-3} + 3\beta b\rho zr^{-5})\{-\frac{1}{2}a^3\rho r^{-5} + \frac{5}{2}\beta a^2b\rho zr^{-7} + ab^2[-\frac{1}{7}(1-\beta^2)\rho r^{-7} - \frac{1}{2}(3+4\beta^2)\rho z^2r^{-9}] \\ &+ \frac{3}{2}\beta b^3\rho z^3r^{-11}\} + \{-a^2\rho r^{-4} + 4ab\beta\rho zr^{-6} - 3b^2\rho z^2r^{-8}\}^2 + 2(-azr^{-3} - \beta br^{-3} + 3\beta bz^2r^{-5}) \\ &\{-\frac{1}{2}a^3zr^{-5} - \frac{1}{2}\beta a^2br^{-5} + \frac{5}{2}\beta a^2bz^2r^{-7} + ab^2[\frac{1}{7}(2+5\beta^2)zr^{-7} - \frac{1}{2}(3+4\beta^2)z^3r^{-9}] \\ &+ \beta b^3(-\frac{1}{2}z^2r^{-9} + \frac{3}{2}z^4r^{-11})\} + \{-a^2zr^{-4} + \beta ab(-r^{-4} + 4z^2r^{-6}) + b^2(zr^{-6} - 3z^3r^{-8})\}^2 \\ &- \frac{2(1-\beta^2)}{\rho^2}(2b\rho r^{-3} - 3b\rho^3r^{-5})\{-\frac{2}{5}a^2b\rho r^{-5} + a^2b\rho^3r^{-7} + \beta ab^2(\frac{2}{7}\rho zr^{-7} - \rho^3zr^{-9})\} \\ &+ \frac{(1-\beta^2)}{\rho^2}a^2b^2\{-\rho\ r^{-4} + 2\rho^3r^{-6}\}^2 + \frac{6(1-\beta^2)}{\rho^2}b\rho^2zr^{-6}\{a^2b\rho^2zr^{-7} \\ &+ \beta\ a\ b^2(\frac{1}{7}\rho^2r^{-7} - \rho^2z^2r^{-9})\} + \frac{4(1-\beta^2)}{\rho^2}a^2b^2\rho^4z^2r^{-12}.\end{aligned} \tag{39a}$$

$$\begin{aligned}\Delta w^{(4)} = 2(1-\beta^2)^{\frac{1}{2}}\Big[&-(a\ r^{-1} - \beta bzr^{-3})(-a^2b\ \rho^2r^{-7} + \beta ab^2\rho^2z\ r^{-9}) + a\,b\ (\frac{1}{2}a^2r^{-2} - \beta abz\ r^{-4} + \frac{1}{2}b^2z^2r^{-6})(\rho^2r^{-6}) \\ &+ (-a\rho r^{-3} + 3\beta b\rho z\ r^{-5})\{a^2b(-\frac{2}{5}\rho r^{-5} + \rho^3r^{-7}) + \beta ab^2(\frac{2}{7}\rho zr^{-7} - \rho^3zr^{-9})\} + ab(-a^2\rho r^{-4} \\ &+ 4\beta\ ab\rho zr^{-6} - 3b^2\rho z^2r^{-8})(-\rho r^{-4} + 2\rho^3r^{-6}) + b\{-\frac{1}{2}a^3\rho r^{-5} + \frac{5}{2}\beta a^2\ b\ \rho z\ r^{-7}\end{aligned}$$

$$+ab^2[-\frac{1}{7}(1-\beta^2)\rho r^{-7}-\frac{1}{2}(3+4\beta^2)\rho z^2r^{-9}]+\frac{3}{2}\beta b^3\rho z^3r^{-11}\}(-2\rho r^{-3}+3\rho^3r^{-5})$$
$$+ab(-azr^{-3}-\beta br^{-3}+3\beta bz^2r^{-5})\{a\rho^2zr^{-7}+\beta b(\frac{1}{7}\rho^2r^{-7}-\rho^2z^2r^{-9})\}+2ab\ \{-a^2z\ r^{-4}$$
$$+\beta ab(-r^{-4}+4z^2r^{-6})+b^2(zr^{-6}-3z^3r^{-8})\}(\rho^2z\ r^{-6})+3b\ \{-\frac{1}{2}a^3zr^{-5}+\beta a^2b(-\frac{1}{2}r^{-5}+\frac{5}{2}z^2r^{-7})$$
$$+ab^2[\frac{1}{7}(2+5\beta^2)z r^{-7}-\frac{1}{2}(3+4\beta^2)z^3r^{-9}]+\frac{1}{2}\beta\ b^3(-z^2r^{-9}+3z^4r^{-11})\}(\rho^2zr^{-5})\ \Big] \quad (39b)$$

(39a,b), are polynomials in *a*, *b* which conveniently help to analyze the expressions of $\nabla^2h^{(4)},\Delta w^{(4)}$. Here $\nabla^2h^{(4)}$ has terms proportional to a^4,a^3b,a^2b^2,ab^3,b^4 and $\Delta w^{(4)}$ is a linear combination of terms in a^3b,a^2b^2,ab^3,b^4. Accordingly, gathering and 'reading off' terms with these coefficients and simplifying we get expressions for $\nabla^2h^{(4)},\Delta w^{(4)}$ which implicitly can be written as follows:

$$\nabla^2h^{(4)}=\{a^4h_1(\rho,z)+a^3bh_2(\rho,z)+a^2b^2h_3(\rho,z)+a\ b^3h_4(\rho,z)+b^4h_5(\rho,z)\}, \quad (40a)$$
$$\Delta w^{(4)}=\{a^3bg_1(\rho,z)+a^2b^2g_2(\rho,z)+a\ b^3g_3(\rho,z)+b^4g_4(\rho,z), \quad (40b)$$

where the $h_i(\rho,z)$ and $g_i(\rho,z)$ are linear combinations of expressions of the form (37). The solutions can be expressed as

$$h^{(4)}=\{a^4h_1'(\rho,z)+a^3bh_2'(\rho,z)+a^2b^2h_3'(\rho,z)+a\ b^3h_4'(\rho,z)+b^4h_5'(\rho,z)\}, \quad (41a)$$
$$w^{(4)}=\{a^3bg_1'(\rho,z)+a^2b^2g_2'(\rho,z)+a\ b^3g_3'(\rho,z)+b^4g_4'(\rho,z)\}. \quad (41b)$$

From (40a,b), (41a,b) it is clear that

$$\nabla^2h_i'(\rho,z)=h_i(\rho,z),i=1,......,5, \quad (42a)$$
$$\Delta g_i'(\rho,z)=g_i(\rho,z),i=1,......,4\ . \quad (42b)$$

Here $h_i'(\rho,z)$, $g_i'(\rho,z)$ are also linear combinations of terms of the form (37). These can be determined with the use of the results given in (38). We shall not display these explicitly, but making use of them implicitly for the fifth order, which we now proceed to consider in some detail.

The functions h_i, g_i, h_i', g_i' also depend on the parameter β which are either linear or quadratic in β. Analyzing $\nabla^2h^{(4)},\Delta w^{(4)}$, etc., as polynomials in *a*, *b* is one approach; another is to consider these, as indicated, sums of terms of the forms (37), and gather together sets of terms with the same *m*, *n*, *p* and then use the results (38) to solve the equations. The explicit expressions for $h^{(4)},w^{(4)}$ needed to evaluate (35a,b), are as follows [11]:

$$h^{(4)}=\frac{1}{24}a^4r^{-4}-\frac{a^2b^2(7+15\beta^2)}{450}r^{-6}-\frac{1}{6}\beta a^3bzr^{-6}+\frac{a^2b^2(3+4\beta^2)}{21}z^2r^{-8}$$
$$-\frac{a^2b^2(1-\beta^2)}{210}\rho^2r^{-8}-\frac{1}{6}\beta a\ b^3z^3r^{-10}+\frac{1}{24}b^4z^4r^{-12}, \quad (43a)$$
$$w^{(4)}=(1-\beta^2)^{\frac{1}{2}}\Bigg[\frac{1}{20}a^3b\rho^2r^{-6}+\frac{17}{840}\beta a^2b^2\rho^2zr^{-8}-\frac{1}{10}\beta a^2b^2zr^{-8}-\frac{1}{210}ab^3\rho^2r^{-8}$$
$$-\frac{1}{5}\beta a^2b^2\rho^2zr^{-10}-\frac{1}{21}a\ b^3\rho^2z^2r^{-10}+\frac{4}{5}\beta a^2b^2\rho^2z^3r^{-10}\Bigg]. \quad (43b)$$

Inserting the various expressions and derivatives on the right hand sides of (35a,b), we get the following expressions:

$$\nabla^2h^{(5)}=\Bigg[\frac{1}{6}\frac{a^5}{r^7}+\left(\frac{32+122\beta^2}{105}\right)\frac{a^3b^2}{r^9}-\frac{7}{6}\frac{\beta a^4bz}{r^9}+\frac{(1-\beta^2)}{105}\frac{ab^4}{r^{11}}+\frac{(522-1950\beta^2)}{525}\frac{\beta\ a^2b^3z}{r^{11}}$$
$$-\frac{(4658+16300\beta^2)}{525}\frac{a^3b^2z^2}{r^{11}}+\frac{(112-420\beta^2)}{525}\frac{a^3b^2\rho^2}{r^{11}}-\frac{(73-178\beta^2)}{210}\frac{ab^4z^2}{r^{13}}$$
$$-\frac{(5354-3730\beta^2)}{525}\frac{\beta\ a^2b^3z^3}{r^{13}}-\frac{(2499-6195\beta^2)}{2100}\frac{\beta a^2b^3\rho^2z}{r^{13}}+\frac{(862+3478\beta^2)}{105}\frac{a^3b^2\rho^2z^2}{r^{13}}$$

$$-\frac{8}{105}\frac{(1-\beta^2)a^3b^2\rho^4}{r^{13}}+\frac{(174+694\beta^2)}{21}\frac{a^3b^2z^4}{r^{13}}+\frac{26(1-\beta^2)}{10}\frac{\beta\ a^2b^3z}{r^{13}}-\frac{18(1-\beta^2)}{105}\frac{ab^4\rho^2}{r^{13}}$$
$$-\frac{1}{6}\frac{\beta\ b^5z^3}{r^{15}}+\frac{(91+560\beta^2)}{42}\frac{a\ b^4z^4}{r^{15}}-\frac{(3506+3354\beta^2)}{105}\frac{\beta\ a^2b^3\rho^2z^3}{r^{15}}-\frac{26(1-\beta^2)}{35}\frac{\beta\ a^2b^3\rho^4z}{r^{15}}$$
$$\left.+\frac{6}{7}\frac{(1-\beta^2)ab^4\rho^2z^2}{r^{15}}-\frac{(844+528\beta^2)}{21}\frac{\beta\, a^2b^3z^5}{r^{15}}+\frac{2(1-\beta^2)}{5}\frac{\beta\, a^2b^3\rho^2z}{r^{15}}-\frac{1}{2}\frac{\beta\, b^5z^5}{r^{17}}\right] \tag{44a}$$

$$\Delta w^{(5)}=2(1-\beta^2)^{\frac{1}{2}}\left[-\frac{61}{60}\frac{a^4b\rho^2}{r^9}-\frac{471}{56}\frac{\beta\, a^3b^2\rho^2z}{r^{11}}+\frac{21}{10}\frac{\beta\ a^3b^2z}{r^{11}}-\frac{(624+1885\beta^2)}{4200}\frac{a^2b^3\rho^2}{r^{11}}+\frac{1}{10}\frac{\beta^2a^2b^3}{r^{11}}\right.$$
$$+\frac{14}{5}\frac{\beta\ a^3b^2\rho^2z}{r^{13}}+\frac{(27556+38845\beta^2)}{4200}\frac{a^2b^3\rho^2z^2}{r^{13}}+\frac{225}{42}\frac{\beta\ a^3b^2\rho^2z^3}{r^{13}}+\frac{1}{35}\frac{\beta\, a\ b^4\rho^2z}{r^{13}}-\frac{31}{10}\frac{\beta^2a^2b^3z^2}{r^{13}}$$
$$+\frac{(87+355\beta^2)}{525}\frac{a^2b^3\rho^4}{r^{13}}+\frac{305}{42}\frac{\beta\ a^3b^2\rho^4z}{r^{13}}-\frac{4}{5}\frac{\beta\, a^3b^2z^3}{r^{13}}+\frac{1}{5}\frac{\beta^2a^2b^3\rho^2}{r^{13}}-\frac{2}{5}\frac{\beta^2a^2b^3\rho^2z^2}{r^{15}}+\frac{52}{105}\frac{\beta\ ab^4\rho^2z^3}{r^{15}}$$
$$\left.-\frac{(725+951\beta^2)}{105}\frac{a^2b^3\rho^2z^4}{r^{15}}-\frac{(713+963\beta^2)}{105}\frac{a^2b^3\rho^4z^2}{r^{15}}+\frac{4}{35}\frac{\beta\ a\ b^4\rho^4z}{r^{15}}+\frac{4}{35}\frac{(1-\beta^2)a^2b^3\rho^6}{r^{15}}+\frac{12}{5}\frac{\beta^2a^2b^3z^4}{r^{15}}\right] \tag{44b}$$

We assume $h^{(5)}, w^{(5)}$ are of the following forms:

$$h^{(5)}=\left[\frac{k_1}{r^5}+\frac{k_2}{r^7}+\frac{k_3z}{r^7}++\frac{k_4}{r^9}+\frac{k_5z}{r^9}+\frac{k_6z^2}{r^9}+\frac{k_7\rho^2}{r^9}+\frac{k_8z^2}{r^{11}}+\frac{k_9z^3}{r^{11}}+\frac{k_{10}\rho^2z}{r^{11}}+\frac{k_{11}\rho^2z^2}{r^{11}}+\frac{k_{12}\rho^4}{r^{11}}+\frac{k_{13}z^4}{r^{11}}\right.$$
$$\left.+\frac{k_{14}z}{r^{11}}+\frac{k_{15}\rho^2}{r^{11}}+\frac{k_{16}z^3}{r^{13}}+\frac{k_{17}z^4}{r^{13}}+\frac{k_{18}\rho^2z^3}{r^{13}}+\frac{k_{19}\rho^4z}{r^{13}}+\frac{k_{20}\rho^2z^2}{r^{13}}+\frac{k_{21}z^5}{r^{13}}+\frac{k_{22}\rho^2z}{r^{13}}+\frac{k_{23}z^5}{r^{15}}\right] \tag{45a}$$

$$w^{(5)}=\left[\frac{k'_1\rho^2}{r^7}+\frac{k'_2\rho^2z}{r^9}+\frac{k'_3z}{r^9}++\frac{k'_4\rho^2}{r^9}+\frac{k'_5}{r^9}+\frac{k'_6\rho^2z}{r^{11}}+\frac{k'_7\rho^2z^2}{r^{11}}+\frac{k'_8\rho^2z^3}{r^{11}}+\frac{k'_9\rho^2z}{r^{11}}+\frac{k'_{10}z^2}{r^{11}}+\frac{k'_{11}\rho^4}{r^{11}}+\frac{k'_{12}\rho^4z}{r^{11}}\right.$$
$$\left.+\frac{k'_{13}z^3}{r^{11}}+\frac{k'_{14}\rho^2}{r^{11}}+\frac{k'_{15}\rho^2z^2}{r^{13}}+\frac{k'_{16}\rho^2z^3}{r^{13}}+\frac{k'_{17}\rho^2z^4}{r^{13}}+\frac{k'_{18}\rho^4z^2}{r^{13}}+\frac{k'_{19}\rho^4z}{r^{13}}+\frac{k'_{20}\rho^6}{r^{13}}+\frac{k'_{21}z^4}{r^{13}}\right] \tag{45b}$$

where the k_i, k'_i are constants to be determined. Applying the operator ∇^2, Δ to (45a,b) respectively, we get (using (38) or by direct calculation):

$$\nabla^2h^{(5)}=\left[k_1\left(\frac{20}{r^7}\right)+k_2\left(\frac{42}{r^9}\right)+k_3\left(\frac{28z}{r^9}\right)+k_4\left(\frac{72}{r^{11}}\right)+k_5\left(\frac{54z}{r^{11}}\right)+k_6\left(\frac{36z^2}{r^{11}}+\frac{2}{r^9}\right)+k_7\left(\frac{36\rho^2}{r^{11}}+\frac{2}{r^9}\right)+k_8\left(\frac{66z^2}{r^{13}}+\frac{2}{r^{11}}\right)\right.$$
$$+k_9\left(\frac{44z^3}{r^{13}}+\frac{6z}{r^{11}}\right)+k_{10}\left(\frac{44\rho^2z}{r^{13}}+\frac{4z}{r^{11}}\right)+k_{11}\left(\frac{22\rho^2z^2}{r^{13}}+\frac{2z^2}{r^{11}}+\frac{2}{r^9}\right)+k_{12}\left(\frac{22\rho^4}{r^{13}}+\frac{16\rho^2}{r^{11}}\right)+k_{13}\left(\frac{22z^4}{r^{13}}+\frac{12z^2}{r^{11}}\right)$$
$$+k_{14}\left(\frac{88z}{r^{13}}\right)+k_{15}\left(\frac{66\rho^2}{r^{13}}+\frac{4}{r^{11}}\right)+k_{16}\left(\frac{78z^3}{r^{15}}+\frac{6z}{r^{13}}\right)+k_{17}\left(\frac{52z^4}{r^{15}}+\frac{12z^2}{r^{13}}\right)+k_{18}\left(\frac{26\rho^2z^3}{r^{15}}+\frac{4z^3}{r^{13}}+\frac{6\rho^2z}{r^{13}}\right)$$
$$+k_{19}\left(\frac{26\rho^4z}{r^{15}}+\frac{16\rho^2z}{r^{13}}\right)+k_{20}\left(\frac{52\rho^2z^2}{r^{15}}+\frac{2z^2}{r^{13}}+\frac{2}{r^{11}}\right)+k_{21}\left(\frac{26z^5}{r^{15}}+\frac{20z^3}{r^{13}}\right)$$
$$\left.+k_{22}\left(\frac{78\rho^2z}{r^{15}}+\frac{4z}{r^{13}}\right)+k_{23}\left(\frac{60z^5}{r^{17}}+\frac{20z^3}{r^{15}}\right)\right], \tag{46a}$$

$$\Delta w^{(5)}=\left[k'_1\left(\frac{28\rho^2}{r^9}\right)+k'_2\left(\frac{36\rho^2z}{r^{11}}\right)+k'_3\left(\frac{72z}{r^{11}}\right)+k'_4\left(\frac{54\rho^2}{r^{11}}\right)+k'_5\left(\frac{90}{r^{11}}\right)+k'_6\left(\frac{66\rho^2z}{r^{13}}\right)\right.$$

$$+k'_7\left(\frac{44\rho^2z^2}{r^{13}}\right)+k'_8\left(\frac{22\rho^2z^3}{r^{13}}+\frac{6\rho^2z}{r^{11}}\right)+k'_9\left(\frac{66\rho^2z}{r^{13}}\right)+k'_{10}\left(\frac{88z^2}{r^{13}}+\frac{2}{r^{11}}\right)+k'_{11}\left(\frac{44\rho^4}{r^{13}}+\frac{8\rho^2}{r^{11}}\right)$$
$$+k'_{12}\left(\frac{22\rho^4z}{r^{13}}+\frac{8\rho^2z}{r^{11}}\right)+k'_{13}\left(\frac{66z^3}{r^{13}}+\frac{6z}{r^{11}}\right)+k'_{14}\left(\frac{88\rho^2}{r^{13}}\right)+k'_{15}\left(\frac{78\rho^2z^2}{r^{15}}+\frac{2\rho^2}{r^{13}}\right)$$
$$+k'_{16}\left(\frac{52\rho^2z^3}{r^{15}}+\frac{6\rho^2z}{r^{13}}\right)+k'_{17}\left(\frac{26\rho^2z^4}{r^{15}}+\frac{12\rho^2z^2}{r^{13}}\right)+k'_{18}\left(\frac{26\rho^4z^2}{r^{15}}+\frac{8\rho^2z^2}{r^{13}}+\frac{2\rho^4}{r^{13}}\right)$$
$$+k'_{19}\left(\frac{52\rho^4z}{r^{15}}+\frac{8\rho^2z}{r^{13}}\right)+k'_{20}\left(\frac{26\rho^6}{r^{15}}+\frac{24\rho^4}{r^{13}}\right)+k'_{21}\left(\frac{78z^4}{r^{15}}+\frac{12z^2}{r^{13}}\right)\Bigg]. \quad (46b)$$

Rearranging terms and comparing the various coefficients in (44a), (46a) and in (44b), (46b) respectively and then by solving, we get the following values of different constants:

$$k_1=\frac{1}{120}a^5;\ k_2=\frac{1}{21}\left[\frac{1}{36}\left(\frac{4546+16720\beta^2}{525}\right)+\frac{1}{36}\left(\frac{6018+24362\beta^2}{1155}\right)-\left(\frac{431+1739\beta^2}{1155}\right)+\left(\frac{16+61\beta^2}{105}\right)\right]a^3b^2$$

$$k_3=-\frac{1}{24}\frac{\beta\,a^4b\,z}{r^9};$$

$$k_4=\frac{1}{72}\left[2\times\frac{1}{66}\left(\frac{73-178\beta^2}{210}+\frac{91+560\beta^2}{182}+\frac{6(1-\beta^2)}{182}\right)ab^4+\frac{4(1-\beta^2)ab^4}{385}-\frac{6(1-\beta^2)ab^4}{182}+\frac{(1-\beta^2)ab^4}{105}\right]$$

$$k_5=\frac{1}{54}\Bigg[-6\times\frac{1}{44}\left\{\frac{4(1753+1677\beta^2)}{1365}+\frac{20(422+264\beta^2)}{273}-\left(\frac{5354-3730\beta^2}{525}\right)\right\}\beta\,a^2b^3$$
$$-4\times\frac{1}{44}\left\{\frac{6(1753+1677\beta^2)}{1365}+\frac{16(1-\beta^2)}{35}-\left(\frac{2499-6195\beta^2}{2100}\right)\right\}\beta\,a^2b^3+\left(\frac{522-1950\beta^2}{525}\right)\beta\,a^2b^3\Bigg]$$

$$k_6=-\frac{1}{36}\left[\frac{4658+16300\beta^2}{525}+\frac{862+3478\beta^2}{1155}+\frac{1044+4164\beta^2}{231}\right]a^3b^2;$$

$$k_7=\frac{1}{36}\left[\frac{112-420\beta^2}{525}+\frac{64(1-\beta^2)}{1155}\right]a^3b^2;$$

$$k_8=-\frac{1}{66}\left[\frac{73-178\beta^2}{210}+\frac{91+560\beta^2}{182}+\frac{6(1-\beta^2)}{182}\right]a\ b^4$$

$$k_9=\frac{1}{44}\left[\frac{4(1753+1677\beta^2)}{1365}+\frac{20(422+264\beta^2)}{273}-\left(\frac{5354-3730\beta^2)}{525}\right)\right]\beta\ a^2b^3;$$

$$k_{10}=\frac{1}{44}\left[\frac{6(1753+1677\beta^2)}{1365}+\frac{16(1-\beta^2)}{35}-\left(\frac{2499-6195\beta^2)}{2100}\right)\right]\beta\ a^2b^3;\ k_{11}=\left(\frac{431+1739\,\beta^2}{1155}\right)a^3b^2;$$

$$k_{12}=-\frac{4(1-\beta^2)}{1155}a^3b^2;\ k_{13}=\left(\frac{87+347\,\beta^2}{231}\right)a^3b^2;\ k_{14}=\frac{1}{88}\left[\frac{13(1-\beta^2)}{5}-\frac{4(1-\beta^2)}{195}\right]\beta\ a^2b^3;$$

$$k_{15}=-\frac{(1-\beta^2)}{385}a\ b^4;\ k_{16}=0;\ k_{17}=\left(\frac{91+560\,\beta^2}{2184}\right)a\ b^4;\ k_{18}=-\left(\frac{1753+1677\beta^2}{1365}\right)\beta\ a^2b^3;$$

$$k_{19}=-\frac{(1-\beta^2)}{35}\beta\ a^2b^3;\ k_{20}=\frac{3(1-\beta^2)}{182}a\ b^4;\ \ k_{21}=-\left(\frac{422+264\beta^2}{273}\right)\beta\ a^2b^3;$$

$$k_{22}=\frac{(1-\beta^2)}{195}\beta\ a^2b^3;\ k_{23}=-\frac{1}{120}\beta\ b^5; \quad (47a)$$

and

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$$k'_1 = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{28}\left(-\frac{61}{60}a^4 b\right)\right];$$

$$k'_2 = 2(1-\beta^2)^{\frac{1}{2}}\left[-\frac{1}{36}\left(\frac{23123}{1848}\right)\right]\beta\ a^3b^2 ; k'_3 = 2(1-\beta^2)^{\frac{1}{2}}\frac{1}{72}\left(\frac{239}{110}\right)\beta\ a^3b^2 ;$$

$$k'_4 = 2(1-\beta^2)^{\frac{1}{2}}\frac{1}{54}\left[-\left(\frac{624+1885\beta^2}{4200}\right)-\frac{2}{11}\left[\left(\frac{87+355\beta^2}{525}\right)+\frac{1}{13}\left(\frac{569+1107\beta^2}{105}\right)\right]\ \right]a^2b^3$$

$$k'_5 = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{31}{15600}\beta^2a^2b^3\right]; k'_6 = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{7}{165}\beta\ a^3b^2\right];$$

$$k'_7 = 2(1-\beta^2)^{\frac{1}{2}}\frac{1}{44}\left[\left(\frac{27556+38845\beta^2}{4200}\right)+\frac{1}{13}\left(\frac{7202+9558\beta^2}{105}\right)\right]\ a^2b^3 ; k'_8 = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{22}\left(\frac{225}{42}\beta\ a^3b^2\right)\right];$$

$$k'_9 = 2(1-\beta^2)^{\frac{1}{2}}\left[-\frac{1}{1430}\beta\, ab^4\right]\ ;\ k'_{10} = 2(1-\beta^2)^{\frac{1}{2}}\left[-\frac{1}{88}\left(\frac{451}{130}\right)\right]\beta^2a^2b^3 ;$$

$$k'_{11} = 2(1-\beta^2)^{\frac{1}{2}}\frac{1}{44}\left[\left(\frac{87+355\beta^2}{525}\right)+\frac{1}{13}\left(\frac{569+1107\beta^2}{105}\right)\right]\ a^2b^3 ; k'_{12} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{22}\left(\frac{305}{42}\beta\ a^3b^2\right)\right];$$

$$k'_{13} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{66}\left(-\frac{4}{5}\beta\ a^3b^2\right)\right]; k'_{14} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{41}{17160}\right]\beta^2a^2b^3 ;$$

$$k'_{15} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{78}\left(-\frac{2}{5}\beta^2a^2b^3\right)\right];$$

$$k'_{16} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{105}\beta\ a\ b^4\right];\quad k'_{17} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{26}\left(-\frac{725+951\beta^2}{105}\right)a^2b^3\ \right];$$

$$k'_{18} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{26}\left(-\frac{713+963\beta^2}{105}\right)a^2b^3\ \right];\ k'_{19} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{52}\left(\frac{4}{35}\beta\ a\ b^4\right)\right];$$

$$k'_{20} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{26}\left\{\frac{4}{35}(1-\beta^2)a^2b^3\right\}\ \right]; k'_{21} = 2(1-\beta^2)^{\frac{1}{2}}\left[\frac{1}{78}\left(\frac{12}{5}\beta^2a^2b^3\right)\right]. \tag{47b}$$

Making use the values of (47a) into (46a) and then by some manipulations, we see that the resultant form of equation is the same as (44a). Hence (45a) is the solution of (44a). Similarly by using the values of (47b) into (46b) and then by some simplifications, we find that resultant form of the equation is the same as (44b). Therefore (45b) is the solution of (44b).

IV. Physical Interpretation of the Approximate Solutions

In this section we analyze the approximate solutions obtained earlier with the use of the metric:

$$ds^2 = (1-\frac{2M}{r}+A_o)dt^2 - (\frac{4S\rho^2}{r^3}+A')dt\,d\phi - \rho^2A''d\phi^2 - (1+\frac{2M}{r})(d\rho^2+dz^2+\rho^2d\phi^2)$$
$$-Ad\rho^2 - 2Bd\rho dz - Cdz^2 , \tag{48}$$

with a view to determine some properties of the sources that correspond to the solutions. Here M and S are respectively the total mass-energy and angular momentum of the rotating source and $r^2 = (\rho^2+z^2)$. The additional terms represent the structure of the source in various cases [7,13]. The metric that we have already used for the solutions can also be written as follows [7,9,10]:

$$ds^2 = f(dt-wd\phi)^2 - \rho^2 f^{-1}d\phi^2 - e^{\mu}(d\rho^2+dz^2) = h^{-1}(dt^2 - 2wdtd\phi) - ld\phi^2 - e^{\mu}(d\rho^2+dz^2) \tag{49}$$

where the various relations connecting f,k,l,w,h,ρ^2 are as:

$$k = wf = wh^{-1} ; f\,l + k^2 = \rho^2 = f\,l + w^2 f^2 ; l = \rho^2 f^{-1} - w^2 f = \rho^2 h - w^2h^{-1} \text{,etc.} \tag{50}$$

To compare the approximate solutions with the asymptotic form (48), using (48), (49) and (50) we have

$$f = h^{-1} = (1 - \frac{2M}{r} + A_o)\ ;\ wh^{-1} = k = \frac{2S\rho^2}{r^3} + \frac{1}{2}A'\ . \tag{51}$$

For physical interpretation, that is to identify M, S and higher order terms of (48), using (16a,b) into $f = h^{-1}$ and $k = f\,w$ and by some manipulations, we get respectively

$$f = 1 + \lambda f^{(1)} + \lambda^2 f^{(2)} + \lambda^3 f^{(3)} + \lambda^4 f^{(4)} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots , \tag{52a}$$

where $f^{(1)} = -h^{(1)}$, $f^{(2)} = (-h^{(2)} + h^{(1)2})$, $f^{(3)} = (-h^{(3)} + 2h^{(1)}h^{(2)} - h^{(1)2})$

$$f^{(4)} = (-h^{(4)} + 2h^{(1)}h^{(3)} + h^{(2)2} - 3h^{(1)2}h^{(2)} + h^{(1)4})\ ; \tag{52b}$$

and

$$k = f\,w = (1 + \lambda\ f^{(1)} + \lambda^2 f^{(2)} + \lambda^3 f^{(3)} + \lambda^4 f^{(4)})(\lambda\ w^{(1)} + \lambda^2 w^{(2)} + \lambda^3 w^{(3)} + \lambda^4 w^{(4)})$$
$$= \lambda\ w^{(1)} + \lambda^2(w^{(2)} + f^{(1)}w^{(1)}) + \lambda^3(w^{(3)} + f^{(1)}w^{(2)} + f^{(2)}w^{(1)}) + \lambda^4(w^{(4)} + f^{(1)}w^{(3)} + f^{(2)}w^{(2)} + f^{(3)}w^{(1)}) \tag{53}$$

The above relations have been expressed up to λ^4 terms since (33) and (43a,b) are given explicitly up to λ^4 terms.

Now by some manipulations if we express (39 a,b) in terms of $a^4, a^3b, a^2b^2, ab^3, b^4$ and a^3b, a^2b^2, ab^3, b^4 respectively and comparing with (40a,b), we find:

$$h_1(\rho,z) = \frac{1}{2}r^{-6}\,,\ h_2(\rho,z) = -3\beta\rho^2 zr^{-10}\,,\ h_3(\rho,z) = \left\{\left(\frac{69}{70} - \frac{17}{35}\beta^2\right)r^{-8} + \left(\frac{54}{35} + \frac{156}{35}\beta^2\right)z^2r^{-10}\right\},$$

$$h_4(\rho,z) = \beta\left(-zr^{-10} - 5z^3r^{-12}\right),\ h_5(\rho,z) = \frac{1}{2}\left(z^2r^{-12} + 3z^4r^{-14}\right); \tag{54a}$$

$$g_1(\rho,z) = -\frac{6}{5}(1-\beta^2)^{\frac{1}{2}}\rho^2r^{-8}\,,\ g_2(\rho,z) = \frac{96}{35}(1-\beta^2)^{\frac{1}{2}}\rho^2 zr^{-10}\,,$$

$$g_3(\rho,z) = -\frac{2\left(1-\beta^2\right)^{\frac{1}{2}}\rho^2}{7}(r^{-10} + 5z^2r^{-12})\ ;\ g_4(\rho,z) = 0\ . \tag{54b}$$

Solving the corresponding equations (42a,b) with $h_1, \ldots\ldots\ldots, h_5$; $g_1, \ldots\ldots\ldots, g_4$ given above we obtain

$$h^{(4)} = \left[a^4(\frac{1}{24}r^{-4}) + a^3b\beta(\frac{1}{12}zr^{-6} - \frac{3}{8}\rho^2zr^{-8}) + a^2b^2(\frac{1}{35}[(1-\beta^2)r^{-8}\right.$$
$$\left. + (\frac{9}{4} + \frac{13}{2}\beta^2)z^2r^{-10}]) + ab^3\beta(-\frac{1}{6}z^3r^{-10}) + b^4(\frac{1}{24}z^4r^{-12})\right], \tag{55a}$$

$$w^{(4)} = 2(1-\beta^2)^{\frac{1}{2}}\left[a^3b(-\frac{1}{30}\rho^2r^{-6}) + a^2b^2\beta(\frac{2}{35}\rho^2zr^{-8}) - ab^3[\frac{1}{42}(\frac{1}{10}\rho^2r^{-8} + \rho^2z^2r^{-10})]\right] \tag{55b}$$

Comparing (41a,b) with (55a,b), we have

$$h_1'(\rho,z) = \frac{1}{24}r^{-4}\,,\ \ h_2'(\rho,z) = \beta(\frac{1}{12}zr^{-6} - \frac{3}{8}p^2zr^{-8}),\ h_3'(\rho,z) = \frac{1}{35}\left[(1-\beta^2)r^{-8} + (\frac{9}{4} + \frac{13}{2}\beta^2)z^2r^{-10}\right],$$

$$h_4'(\rho,z) = (-\frac{1}{6}z^9r^{-10})\ ,\ h_5'(\rho,z) = \frac{1}{24}z^4r^{-12}\ . \tag{56a}$$

$$g_1'(\rho,z) = (1-\beta^2)^{\frac{1}{2}}(-\frac{1}{15}\rho^2r^{-6})\ , \quad g_2'(\rho,z) = \beta(1-\beta^2)^{\frac{1}{2}}(\frac{4}{35}\rho^2zr^{-8})\ ,$$

$$g_3'(\rho,z) = (1-\beta^2)^{\frac{1}{2}}\left[-\frac{1}{21}(\frac{1}{10}\rho^2r^{-8} + \rho^2z^2r^{-10})\right],\ g_4'(\rho,z) = 0\ . \tag{56b}$$

In the present context the following simple form of (38) are useful:

$$\nabla^2(r^{-4}) = 12r^{-6};\nabla^2(r^{-6}) = 30r^{-8};\nabla^2(zr^{-6}) = 18zr^{-8};\nabla^2(z^2r^{-8}) = 2r^{-8} + 24z^2r^{-10};$$
$$\nabla^2(\rho^2zr^{-8}) = 4zr^{-8} + 8\rho^2zr^{-10}\ . \tag{57a}$$

$$\Delta(\rho^2r^{-6}) = 18\rho^2r^{-8};\Delta(\rho^2r^{-8}) = 40\rho^2r^{-10};\Delta(\rho^2z^2r^{-10}) = 2\rho^2r^{-10} + 30\rho^2z^2r^{-12} \tag{57b}$$

Making use of the explicit expressions for (33), (55 a,b) into (52 a,b) and (53) we write down in detail (although we may not need here complete expression) the power series for f and k, which are as follows [11]:

$$
\begin{aligned}
f = \Big[& 1-\lambda(ar^{-1}-\beta bzr^{-3})+\lambda^{2}[-\tfrac{1}{2}a^{2}r^{-2}+\beta abzr^{-4}-\tfrac{1}{2}b^{2}z^{2}r^{-6}+(ar^{-1}-\beta bzr^{-3})^{2}] \\
& +\lambda^{3}[-\tfrac{1}{6}a^{3}r^{-3}+\tfrac{1}{2}\beta a^{2}bzr^{-5}-ab^{2}\{\tfrac{1}{35}(1-\beta^{2})r^{-5}+\tfrac{1}{14}(3+4\beta^{2})z^{2}r^{-7}\}+\tfrac{1}{6}\beta b^{3}z^{2}r^{-9} \\
& +2(ar^{-1}-\beta bzr^{-3})(\tfrac{1}{2}a^{2}r^{-2}-\beta a bzr^{-4}+\tfrac{1}{2}b^{2}z^{2}r^{-6})-(ar^{-1}-\beta bzr^{-3})^{3}] \\
& +\lambda^{4}[-a^{4}(\tfrac{1}{24}r^{-4})-a^{3}b\beta(\tfrac{1}{12}zr^{-6}-\tfrac{3}{8}\rho^{2}zr^{-8})-a^{2}b^{2}(\tfrac{1}{35}[(1-\beta^{2})r^{-8}+(\tfrac{9}{4}+\tfrac{13}{2}\beta^{2})z^{2}r^{-10}]) \\
& -ab^{3}\beta(-\tfrac{1}{6}z^{3}r^{-10})-b^{4}(\tfrac{1}{24}z^{4}r^{-12})+2(ar^{-1}-\beta bzr^{-3})(\tfrac{1}{6}a^{3}r^{-3}-\tfrac{1}{2}\beta a^{2}bzr^{-5}+ab^{2}[\tfrac{1}{35}(1-\beta^{2})r^{-5} \\
& +\tfrac{1}{14}(3+4\beta^{2})z^{2}r^{-7}]-\tfrac{1}{6}\beta b^{3}z^{3}r^{-9})+(\tfrac{1}{2}a^{2}r^{-2}-\beta abzr^{-4}+\tfrac{1}{2}b^{2}z^{2}r^{-6})^{2}-3(ar^{-1}-\beta bzr^{-3})^{2}(\tfrac{1}{2}a^{2}r^{-2} \\
& -\beta abzr^{-4}+\tfrac{1}{2}b^{2}z^{2}r^{-6})+(ar^{-1}-\beta bzr^{-3})^{4}]\Big] \qquad (58a)
\end{aligned}
$$

$$
\begin{aligned}
k = b(1-\beta^{2})^{\frac{1}{2}}\Big[& -\lambda\rho^{2}r^{-3}+\lambda^{2}[-\tfrac{1}{2}a\rho^{2}r^{-4}+(ar^{-1}-\beta bzr^{-3})(\rho^{2}r^{-3})]+\lambda^{3}[-\tfrac{a^{2}}{5}\rho^{2}r^{-5} \\
& +\tfrac{1}{7}ab\beta\rho^{2}zr^{-7}+\tfrac{1}{2}(a\ r^{-1}-\beta\ b\ zr^{-3})(a\rho^{2}r^{-4})-(-\tfrac{1}{2}a^{2}r^{-2}+\beta\ abz\ r^{-4}-\tfrac{1}{2}b^{2}z^{2}r^{-6} \\
& +(ar^{-1}-\beta bzr^{-3})^{2})(\rho^{2}r^{-3})]+\lambda^{4}[a^{3}(-\tfrac{1}{15}\rho^{2}r^{-6})+a^{2}b\beta(\tfrac{4}{35}\rho^{2}zr^{-8}) \\
& -ab^{2}[\tfrac{1}{21}(10\rho^{2}r^{-8}+\rho^{2}z^{2}r^{-10})]-(ar^{-1}-\beta\ bzr^{-3})(-\tfrac{1}{5}a^{2}\rho^{2}r^{-5}+\tfrac{1}{7}ab\beta\rho^{2}zr^{-7}) \\
& -\tfrac{1}{2}(-\tfrac{1}{2}a^{2}r^{-2}+\beta\ abzr^{-4}-\tfrac{1}{2}b^{2}z^{2}r^{-6}+(ar^{-1}-\beta bzr^{-4})^{2})(a\rho^{2}r^{-4}) \\
& -\{(-\tfrac{1}{6}a^{3}r^{-3}+\tfrac{1}{2}\beta\ a^{2}b\ zr^{-5}-ab^{2}[\tfrac{1}{35}(1-\beta^{2})r^{-5}+\tfrac{1}{14}(3+4\beta^{2})z^{2}r^{-7}] \\
& +\tfrac{1}{6}\beta b^{3}z^{3}r^{-9}+2(ar^{-1}-\beta\ bz\ r^{-3})(\tfrac{1}{2}a^{2}r^{-2}-\beta abz^{-4}+\tfrac{1}{2}b^{2}z^{2}r^{-6}) \\
& -(a\ r^{-1}-\beta\ bzr^{-3})^{3}\}(\rho^{2}r^{-3})]\Big]. \qquad (58b)
\end{aligned}
$$

We make some general remarks before studying some of the details of (58a,b) [11]. Here the approximate scheme is akin to the post- Newtonian, post- post- Newtonian and higher order approximations studied extensively by Chandrasekhar, Chandrasekhar and Nutku, Bardeen, and others [14,15]. These expansions are in powers of c^{-1}, where c is the velocity of light. If we want to relate the present approximation scheme to the post- Newtonian and higher approximations, we can take some combination of the constants λ, a, b, β to be proportional to c^{-1}. However, at this stage we prefer not to do this; we will discuss this aspect in the forthcoming paper. It is noted that when $b = 0$, we get a particular Weyl solution (the Curzon solution [12]), while $a = 0$ yields a particular class of the Papapetrou solution. Both are asymptotically flat, as the approximate solution obtained here.

For astrophysical bodies, star in particular, the departure from spherical symmetry is caused by rotation, so that in its absence the star becomes spherically symmetric. The Kerr solution has this property, but the Tomimatsu-Sato solutions do not have this property. It implies that the sources which do not tend to spherical symmetry in the absence of rotation have some intrinsic axially symmetric but non-spherical structure. For example, a rigid spheroid would have such a structure. As is well-known, the Earth is slightly flattened at poles, so that it is not quite spherically symmetric, but is in fact a spheroid. Although the Earth's departure from spherical symmetry may have been caused by rotation in its early evolution, this departure at present is no longer due to rotation and were the Earth to stop rotating, it would continue to be a spheroid, albeit with near spherical symmetry. Henec the gravitation field of the present rotating Earth would not be described by the

Schwazschild solution (if one were to determine it precisely), but by an axially symmetric stationary metric which does not tend to spherical symmetry in the absence of rotation. There is no compelling reason for considering the general relativistic exterior field of the Earth precisely (there is a marginal reason such as determining the motion of a gyroscope in a satellite circling the Earth, but this can be taken care of by approximate solutions). However the important point here is that although such exterior metrics may not be astrophysically interesting, they do represent physically well-defined situations [7, 11]. In terms of moments, these sources may have dipole, quadrupole or higher order multiple moments, which is discussed by Forrester (1975) [16] and by Shabuddin (1995) [17].

Now if we return to the expansion (58a,b) and compare O(r^{-1}) of (58a) with that of (51), we have $\frac{1}{2}\lambda a$ is the mass of the source. The higher order terms (those proportional to b, b^2, etc.) possibly give contributions to the mass- energy from the rotation, and give other effects of the higher moments of the source. The angular momentum is proportional to b, because $w=0$ when $b=0$ (i.e., no rotation). The leading term (coefficients of $\rho^2 r^{-3}$) of k given by (58b) has the correct behaviour depicted in (51), with $S=\frac{1}{2}\lambda b(1-\beta^2)^{\frac{1}{2}}$. These two important aspects can be summarized in the following equation:

$$M=\frac{1}{2}\lambda a\,;\; S=\frac{1}{2}\lambda b(1-\beta^2)^{\frac{1}{2}}, \tag{59}$$

where M and S are respectively the mass and angular momentum of the source. The effects of the multiple moments of the source are given by the higher order terms of k.

Asyamptotically flat rotating exact solutions are notoriously difficult solutions to find ; the first genuine such solution was found by Kerr in 1963, decades after the advent of general relativity in 1915. In fact, as is well known, physically meaningful solutions of Einstein's equations are very rare. Even if an exact solution is not astrophysically interesting, as indicated, if the solution describes a physically well-defined situation, it can be of considerable interest in elucidating the physical meaning of general relativity. This is an important problem which is still far from a complete solution. The explorations of this paper and the forthcoming paper may be a step towards finding such solutions, or at least getting to know some properties of these solutions.

Returning to the explicit of the approximate solution, several remarks are in order before we attempt to elicit some information from the power series (58a,b). We saw in (20) that $h^{(1)}$ is a harmonic function. Why did we choose it, as in (21), to be $\sigma+\beta\zeta_z$, rather than σ, which is also harmonic. Part of the reason is, we wanted the approximate solution to reduce to the Weyl and Papapetrou solutions, in suitable limits, as stated before. However, the form (21) for $h^{(1)}$, $w^{(1)}$, as is clear from the explicit approximation, gives a more general asymptotic behaviour. An important point is to take the approximation scheme to a certain stage of completion, so that some insight can be gained into the exact solution, of which the power series to a given order are an approximation. By some manipulations, we have from (58a,b)

$$f=1-\lambda ar^{-1}+\frac{1}{2}\lambda^2a^2r^{-2}-\frac{1}{6}\lambda^3a^3r^{-3}+\frac{1}{24}\lambda^4a^4r^{-4}+........ \\ +b\left[\lambda\beta\; zr^{-3}-\lambda^2\beta azr^{-4}+\frac{1}{2}\lambda^3\beta a^2zr^{-5}-\frac{1}{6}\lambda^4\beta a^3zr^{-6}+\frac{1}{8}\lambda^4\beta a^3(zr^{-6}-3\rho^2zr^{-8})+....\right], \tag{60a}$$

$$k=b(1-\beta^2)^{\frac{1}{2}}\left[-\lambda\rho^2r^{-3}+\frac{1}{2}\lambda^2a\rho^2r^{-4}-\frac{1}{5}\lambda^3a^2\rho^2r^{-5}+\frac{1}{20}\lambda^4a^3\rho^2r^{-6}+.....\right]. \tag{60b}$$

The partial series for f and k given in (60a,b) can be written as follows :

$$f=\; e^{-\lambda\, a\, r^{-1}}+.....+b\lambda\beta zr^{-3}(e^{-\lambda\, ar^{-1}}-F(\rho,z;\lambda,a)), \tag{61a}$$

$$k=b(1-\beta^2)^{\frac{1}{2}}(\lambda\rho^2r^{-3})g(\rho,z\,;\lambda,a),\; g(\rho,z\,;\lambda,a)=-1+\frac{1}{2}\lambda ar^{-1}-\frac{1}{5}(\lambda ar^{-1})^2+\frac{1}{20}(\lambda ar^{-1})^3+...... \tag{61b}$$

The structure of the functions f, k is likely to become more clear with terms proportional to b^2, b^3 etc., for which one will need higher order terms. We can compare (60a,b) with a different kind of expansion that in powers of b, as follows:

$$f=f_0+bf_1+b^2f_2+....., \tag{62a}$$

$$w=bw_1+b^2w_2+........ \;. \tag{62b}$$

We use (6a,b) for f , w rather than (15a,b) for h , w and remembering $k = fw$. It is readily seen that substituting (62a,b) into (6a,b) and equating powers of b we get

$$f_0\nabla^2 f_0 - f_{0\rho}^2 - f_{0z}^2 = 0\,;\, f_1\nabla^2 f_0 + f_0\nabla^2 f_1 - 2f_{0\rho}f_{1\rho} - 2f_{0z}f_{1z} = 0\,;\; f_0\Delta w_1 - 2f_{0\rho}w_{1\rho} - 2f_{0z}w_{1z} = 0\,;\text{ etc.,} \quad (63)$$

with $f_{0\rho} = \dfrac{\partial f_0}{\partial \rho}$, etc. Clearly $f_0 = e^{-\lambda a r^{-1}}$, while f_1, w_1 are related to the coefficients of b in (62a,b) respectively. One can go back and forth between (61a,b) and (62a,b) (suitably converting (f, k) to (f, w) and vice versa), for higher and higher orders to gain more insight into the structure of the functions f, k, possibly leading to an exact solution. In any case we can get more information about the asymptotic behaviour of f and k, with interplay of mass-energy and rotation effects [7,10,11].

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References

[1]. Lewis, T., Proc. Roy. Soc. Lond., **A136**, 176 (1932)
[2]. Papapetrou, A., Ann. Physik, **12**, 309 (1953)
[3]. Kerr, R.P., Phys. Rev. Lett., **11**, 237 (1963)
[4]. Tomimatsu, A., and Sato, H., Progr. Theor. Phys., **50**, 95 (1973)
[5]. Herlt, E., Gen. Rel. Grav., **9**, 711 (1978)
[6]. Weyl, H., Ann. Physik, **54**, 117 (1917)
[7]. Islam, J.N., Rotating Fields in General Relativity (Cambridge University Press) (1985)
[8]. Islam, J. N., Math. Proc. Camb. Phil. Soc., **79**, 161 (1976a)
[9]. Salam, M.A., Studies of An Approximation Scheme for Rotating Solutions of Einstein’s Equations, M. Phil. Thesis. University of Chittagong (1988)
[10]. Salam, M.A., Ganit: J. Bangladesh Math. Soc. **17**, 37 (1997)
[11]. Salam, M.A., Studies in the solution of Rotating Fields in General Relativity, Ph.D. Thesis, University of Chittagong (2010)
[12]. Curzon, H.E.J., Proc. Lond. Math. Soc., **23**, 477 (1924)
[13]. Misner, C.W., Thorne, K.S., and Wheeler, J.A., Gravitation, W.H. Freeman, San Francisco (1973)
[14]. Chandrasekhar, S., Astrophys. J., **167**, 447 (1971)
[15]. Bardeen, J.M., Astrophys. J., **167**, 425 (1971)
[16]. Forrester, Ph.D. Thesis, Chapter-6 (1975)
[17]. Shahabuddin, M., Studies on Stationary Axisymmetric Einstein-Maxwell Equations, Ph.D. Thesis, University of Chittagong(1995).