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A Class of Block Multi-Step Methods for the Solutions of Ordinary Differential Equation (Ode)

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Abstract: *In this research, an attempt is made to derive a self starting block procedure for some K-step linear multi-step methods (for K=1, 2 and 3), using Chebyshev polynomial as the basis function. The continuous interpolant were derived and collocated at grid and off-grid points to give the discrete methods used in block and applied simultaneously for the solution of non stiff initial value problem.The regions of absolute stability of the methods are plotted and are shown to be A (α) stable. The methods for K=2 and K=3 were experimented on initial value problems and the results reveal that the newly constructed block methods have good error stability and are efficient.*

Keywords: *Collocation methods, Initial Value Problems, Chebyshev Polynomials, Perturbation Function, Convergence and Stability.*

I. Introduction

We consider the general first order initial value problem

$$y' = f(x, y)$$
$$y(x_0) = y_0 \qquad (1.1)$$

We then seek discrete methods to solve (1.1) at a sequence of nodal points

$$x_n = x_0 + nh \qquad (1.2)$$

where h>0 is the step-length or grid-size defined by

$$h = x_{n+1} - x_n \qquad (1.3)$$

and y(x) denotes the true solution to (1.1) while the approximate solution is denoted by

$$\bar{y}(x) = \{y_n, y_{n+1}, ..., y_N\} \qquad (1.4)$$

1.1 Important Definitions

1.11 Linear Multi Step Methods

Consider the initial value problem for a single first order ordinary differential equation;

$$y^1 = f(x, y); \; y(a) = \eta \qquad (1.5)$$

We seek for solution in the range $a \le x \le b$, where a and b are finite, and we assume that f satisfies a theorem which guarantees that the problem has unique continuously differentiable solution, which is indicated as y(x). Consider the sequence of points $\{x_n\}$ defined by $x_n = x_0 + nh$, $n = 1,2,\ldots$ The parameter h, which will always be recognized as constant, is called the step length. An essential property of the majority of computational methods for the solution of (5) is that of discretization; that is we seek for an approximate solution, not on the continuous interval $a \le x \le b$, but on the discrete point set $\{x_n / n = 0,1,.....;(b-a)/h\}$. Let y_n be an approximation to the theoretical solution at x_n, that is $y(x_n)$, and let $f \equiv f(x_n, y_n)$. If a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between $y_{n+j}, f_{n+j}, j = 0,1,......,k,$ we call it a linear k-step. The general linear multi-step method may thus be written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.6)$$

Where α_j and β_j are constants; we assume that $\alpha_k \equiv 0$ and that both α_0 and β_0 are zero. Since (6) can be multiplied on both sides by the same constant without altering the relationship, the coefficients α_j and β_j are arbitrary to the extent of a constant multiplier. We remove this arbitrariness by assuming throughout that $\alpha_k = 1$,Williams (1972).

1.12. Order And Error Constants

Given a general linear multi-step methods

$$\sum_{J=0}^{k} \propto_J \; y_{n+j} = h\sum_{J=0}^{k} \beta_j f_{n+j}; \alpha_k \mp 0 \qquad (1.7)$$

The order of (7) defined as p can be determined if and only if.

$$C_0 = C_1 = C_2 = ---- = C \propto = 0 \qquad (1.8)$$

And $Cp + 1 \mp 0$ where

$$C_0 = \propto_0 + \propto_1 + --- + \propto_k$$

$$C_1 = \left(\propto_1 + 2 \propto_2 + ----- + k \propto_k\right) - \left(\beta_0 + \beta_1 + --- + \beta_k\right)$$

$$\vdots$$

$$\vdots$$

$$Cp = \frac{1}{q}\left(\propto_1 + 2^q \propto_2 + ----- + k^q \propto_k\right) - \frac{1}{(q-1)}\left(\beta_1 + 2^{q-1}\beta_2 \; + --- + k^{q-1}\beta_k\right)$$

For $q \geq 2$. It follows that $C_{p+1} \neq 0$ is the error constant.

1.13. Theorem (Due Dahlquist)

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero stable, Dahlquist, G and A. B Jorch,(1974). Consistency controls the magnitude of the local truncation error committed at each stage of the calculation whole zero stability controls the manner in which this error is propagated.

1.14. Region Of Absolute Stability

The stability polynomial of the methods is defined by

$$\Pi(r,h) = p(r) - h\partial(r); h = \lambda h \qquad (1.9)$$

To obtain the boundary focus curve of these methods, we get

$$h = \frac{\ell(r)}{\partial(r)} jr = e^{i0}\theta \leq \Pi \qquad (1.10)$$

Lambert J.D (1973).

1.15 Hybrid Schemes

Linear multi-step methods though generally effective for a given function evaluations per steps have poor stability property as the step number increases. The desire to increase the order without increasing the step number of the linear multistep methods and thus without reducing the stability interval led to hybrid schemes because they posses some properties of linear multi-step methods and Runge-Kutta methods. Jain (1979).

1.16 Block Schemes

To ease computational efforts and to avoid the use of starting values while solving initial value problems, a set of discrete methods are used simultaneously on problems which gives solutions of more than one step per computation. These set of discrete schemes are known as block schemes or methods.

II. The Derivation Of The New Methods

In this research work, an attempt is made based on a perturbed collocation method. The power series method is used as the basis for collocation approximation with the Chebyshev polynomials as the perturbation term.

Consider the problem

$$\left.\begin{aligned} &y^1 = f(x,y) \\ &y(x_0) = y_0; x_0 \le x \le x_{n+k} \end{aligned}\right\} \tag{2.1}$$

The exact solution of the perturbed form of (2.1) is given by

$$y_k(x) = \sum_{j=0}^{k} a_j Q_j(x)$$

(2.2a)

$$x_n \le x \le x_{n+k}$$

Where

$$Q_j(x) = x^j, j \ge 0$$

(2.2b)

Is the power series

From (2.1) and (2.2)

$$\sum_{j=0}^{k} a_j Q_j'(x) = f(x,y) + \tau T_k(\bar{x}) \tag{2.3}$$

where $T_k(\bar{x})$ is the Chebyshev polynomial of degree k, valid in $x_n \le x \le x_{n+k}$ and τ is a parameter Fox and Parker(1972).

In particular, we shall be dealing with cases k=1, 2, 3 and 4 in (2.1) and (2.2).

2.1 The Chebyshev Polynomial And Transformation

The Chebyshev polynomials denoted by $T_k(\bar{x})$ obtained by the recurrence relation

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \tag{2.4}$$

are obtained as follows:

$$\left.\begin{aligned} &T_0(\bar{x}) = 1 \\ &T_1(\bar{x}) = x \\ &T_2(\bar{x}) = 2x^2 - 1 \\ &T_3(\bar{x}) = 4x^3 - 3x \\ &T_4(\bar{x}) = 8x^4 - 8x^2 + 1 \end{aligned}\right\} \tag{2.5}$$

While the transformation is given as

$$x = \frac{2\bar{x} - (x_{n+K} + x_n)}{x_{n+K} - x_n}; K = 1,2,3,4. \tag{2.6}$$

2.2 Case K=1

Taking the polynomial $T_k(\bar{x}) = x$

We use (2.6) in $T_1(\bar{x}) = x$ and collocating at x_n and x_{n+1}, we have

$$T_1(x_n) = \frac{2x_n - x_{n+1} - x_n}{x_{n+1} - x_n} = -1$$

$$T_1(x_{n+1}) = \frac{2x_{n+1} - x_{n+1} - x_n}{x_{n+1} - x_n} = 1$$

From equation 2.2b

$$\left.\begin{aligned} Q_0(x) &= 1 \Rightarrow Q_0' = 0 \\ Q_1(x) &= x \Rightarrow Q_1' = 1 \end{aligned}\right\} \tag{2.7}$$

Putting equation (2.7) into (2.3), we obtain

$$a_1 = f(x, y) + \tau T_1(x) \tag{2.8}$$

Now collocating (2.8) at x_{n+j}, j=0, 1 and interpolate (2.2) at $x=x_n$, we get a system of three equations with a_j (j=0, 1) and parameter τ

$$a_0 + a_1 x_n = y_n$$

$$a_1 + \tau = f_n$$

$$a_1 - \tau = f_{n+1}$$

Which gives the matrix

$$\begin{bmatrix} 1 & x_n & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \tau \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+1} \end{bmatrix}$$

Solving the matrix above gives the value

$$\tau = \frac{1}{2}(f_n - f_{n+1})$$

$$a_1 = f_n - \tau$$

$$a_0 = y_n - a_1 x_n$$

From (2.2), we have

$$\bar{y}(x) = a_0 + a_1 x \tag{2.9}$$

Collocating (2.9) at $x=x_{n+1}$ gives

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n) \tag{2.10}$$

This is the well known trapezoidal rule.

2.3 Case K=2

Following the same procedure as in case k=1, we collocate the continuous scheme

$$\bar{y}(x) = a_0 + a_1 x + a_2 x^2 \tag{2.11}$$

at grid and off grid points $x = x_n, x = x_{n+1/2}$ and $x = x_{n+2}$ and this gives the block scheme below;

$$\left.\begin{aligned} y_{n+2} &= y_{n+1} + \frac{h}{2}(f_{n+2} + f_{n+1}) \\ y_n &= y_{n+1} - \frac{h}{2}(f_{n+1} + f_n) \\ y_{n+1/2} &= y_{n+1} - \frac{h}{16}(3f_n + 4f_{n+1} + f_{n+2}) \end{aligned}\right\} \tag{2.12}$$

2.4 **Case K=3**

For case k=3, we collocate the continuous scheme

$$\bar{y}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{2.13}$$

at $x = x_n, x = x_{n+1/2}, x = x_{n+1}$ and $x = x_{n+3}$ yields

$$\left.\begin{aligned}
y_{n+3} &= y_{n+2} + \frac{h}{96}\left(43 f_{n+3} + 55 f_{n+2} + f_{n+1} - 3 f_n\right) \\
y_{n+1} &= y_{n+2} + \frac{h}{24}\left(f_n - 13 f_{n+1} - 13 f_{n+2} + f_{n+3}\right) \\
y_n &= y_{n+2} - \frac{h}{96}\left(39 f_n + 107 f_{n+1} + 54 f_{n+2} - 7 f_{n+3}\right) \\
y_{n+1/2} &= y_{n+2} + \frac{h}{1536}\begin{pmatrix} 4545 f_{n+3} - 14211 f_{n+2} \\ +11907 f_{n+1} - 4545 f_n \end{pmatrix}
\end{aligned}\right\} \qquad (2.14)$$

Mathematical Analysis Of The Block Schemes

The order, error constant, convergence and Region of Absolute Stability analysis of the new block schemes where examined and the summary of the results is given in the table 1.1 below.

Table 1.1: Summary of Mathematical Analysis

STEP	METHOD	ORDER	ERRORCONSTANT	CONVERGENCE	RAS
K=1	$y_{n+1} = y_n + \frac{h}{2}\left(f_{n+1} + f_n\right)$	2	-1/12	Convergent	[0,0]
K=2	$y_{n+1} = y_n + \frac{h}{2}\left(f_{n+1} + f_n\right)$ $y_{n+2} = y_{n+1} + \frac{h}{2}\left(f_{n+2} + f_{n+1}\right)$	2 2	-1/12 -1/12	Convergent Convergent	[0,0] [0,0]
K=3	$y_n = y_{n+2} - \frac{h}{96}\begin{pmatrix} 39 f_n + 107 f_{n+1} \\ +54 f_{n+2} - 7 f_{n+3} \end{pmatrix}$ $y_{n+1} = y_{n+2} + \frac{h}{24}\begin{pmatrix} f_n - 13 f_{n+1} \\ -13 f_{n+2} + f_{n+3} \end{pmatrix}$ $y_{n+3} = y_{n+2} + \frac{h}{96}\begin{pmatrix} 43 f_{n+3} + 55 f_{n+2} \\ + f_{n+1} - 3 f_n \end{pmatrix}$	0 4 3	1/96 11/720 -7/96	Not convergent Convergent Convergent	[0,0] [0,7.384] [-0.563,-57]

Fig 1, Fig 2 and Fig 3 below shows the plotted regions of the three new block schemes of Case K=1, Case K=2 and Case K=3 respectively, plotted with the aid of Maple and Mathlabsoftwares.

Fig 1: Region of Absolute Stability for Block-Scheme K=1

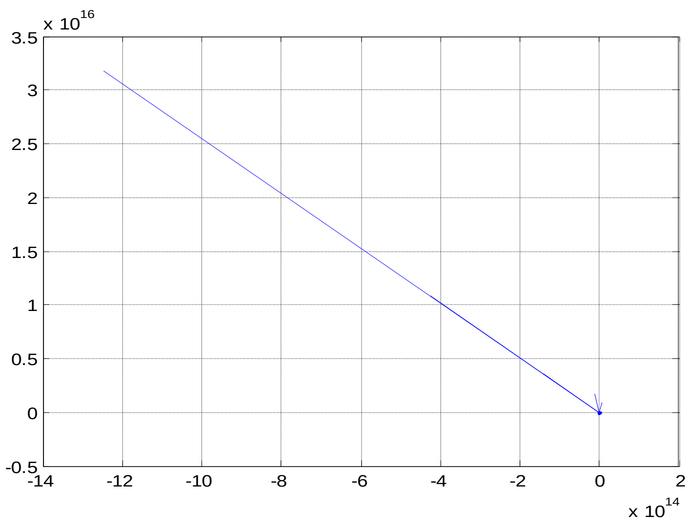


Fig 2: Region of Absolute Stability for Block-Scheme K=2

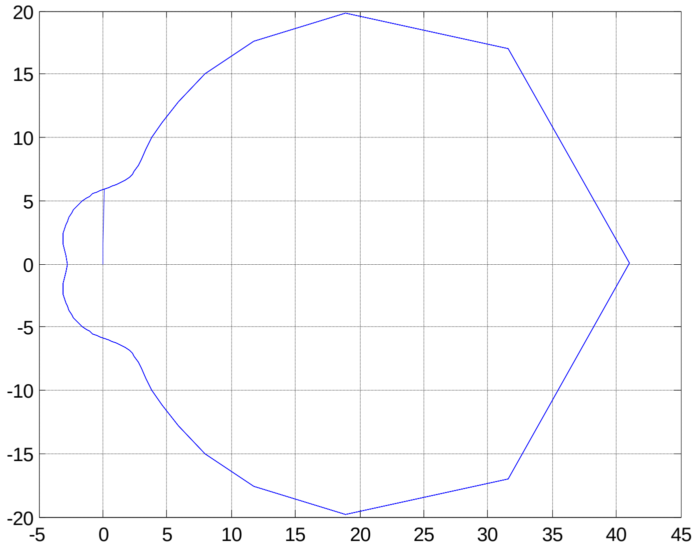
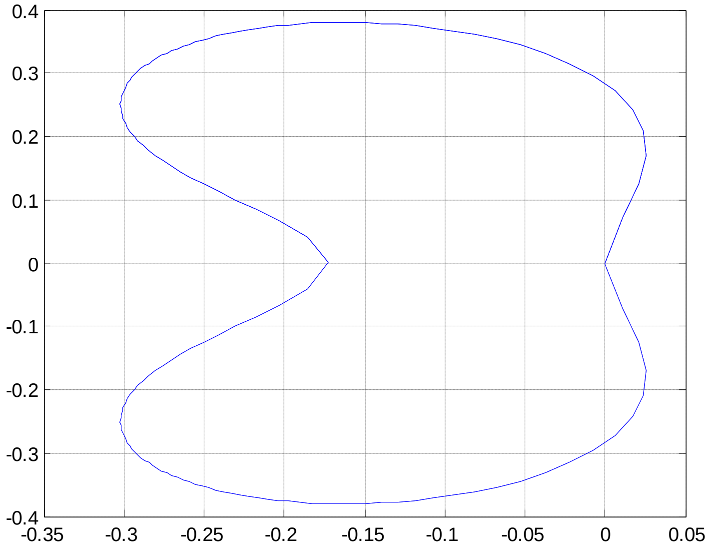


Fig 3: Region of Absolute Stabilty for Block-Scheme K=3



The block schemes k=2 and k=3 are seen to be A-Stable.

III. Numerical Experiment

The newly derived methods K=2 and K=3 are applied to two first order initial value problems

1. $y' = -y, h = 0.1, y_0 = 1$ with the theoretical solution $y(x) = e^{-x}$, see table 1 and table 2 for absolute errors. (Non-Stiff Problem)
2. $y' = 1000(\sin x - y); y(0) = 0; h = 0.1$ with the exact solution $y(x) = \frac{1}{1.001}\left(\sin x - 0.001\cos x + e^{-1000x}\right)$ see table 3 and table 4 for absolute errors (Non-Stiff Problem).

Table 1: Errors Of Example 1 Using The Method K=2

X	Y(x)	Block LMM K=2	Exact Solution	Absolute Error
0.0	Y_0	1.00000000000	1.000000000	0.000000000
0.1	Y_1	0.90583599177	0.904837418	$9.521759*10^{-4}$
0.2	Y_2	0.8195658302	0.8187307531	$8.350771*10^{-4}$
0.3	Y_3	0.7423921660	0.7408182207	$1.5739453*10^{-3}$
0.4	Y_4	0.6716881501	0.670320046	$1.3681041*10^{-3}$
0.5	Y_5	0.6084392519	0.6065306597	$1.908592*10^{-3}$
0.6	Y_6	0.5504926565	0.5488116361	$1.6810204*10^{-3}$
0.7	Y_7	0.4986560206	0.4965853038	$2.0707168*10^{-3}$
0.8	Y_8	0.4511649711	0.4493289641	$1.836007*10^{-3}$
0.9	Y_9	0.4086814356	0.4065696597	$2.1117759*10^{-3}$
1.0	Y_{10}	0.3697593941	0.3678794412	$1.8799529*10^{-3}$
1.1	Y_{11}	0.3349413401	0.3328710837	$2.0702564*10^{-3}$
1.2	Y_{12}	0.3030421648	0.3011942119	$1.8479529*10^{-3}$
1.3	Y_{13}	0.2745064775	0.272531793	$1.974684*10^{-3}$
1.4	Y_{14}	0.2483630034	0.2465969639	$1.7660395*10^{-3}$
1.5	Y_{15}	0.2249761291	0.2231301601	$1.845969*10^{-3}$
1.6	Y_{16}	0.2035498311	0.201896518	$1.6533131*10^{-3}$
1.7	Y_{17}	0.1843827481	0.1826835241	$1.699224*10^{-3}$
1.8	Y_{18}	0.1668224863	0.1652988882	$1.5235981*10^{-3}$
1.9	Y_{19}	0.1511138000	0.1495686192	$1.5451808*10^{-3}$
2.0	Y_{20}	0.1367220095	0.1353352832	$1.3867263*10^{-3}$

Table 2: Errors Of Example 1 Using The Method K=3

X	Y(x)	Block LMM K=3	Exact Solution	Absolute Error
0.0	Y_0	1.00000000000	1.000000000	0.000000000
0.1	Y_1	0.9040064249	0.904837418	$8.309931*10^{-4}$
0.2	Y_2	0.8179589124	0.8187307531	$7.718407*10^{-4}$
0.3	Y_3	0.7341463701	0.7408182207	$6.6718506*10^{-3}$
0.4	Y_4	0.6636730354	0.670320046	$0.6470106*10^{-3}$
0.5	Y_5	0.6005015664	0.6065306597	$6.0290933*10^{-3}$
0.6	Y_6	0.5389708928	0.5488116361	$9.8407433*10^{-3}$
0.7	Y_7	0.4872331499	0.4965853038	$9.3521539*10^{-3}$
0.8	Y_8	0.4408560452	0.4493289641	$8.7429189*10^{-3}$
0.9	Y_9	0.3956835245	0.4065696597	$1.08861352*10^{-2}$
1.0	Y_{10}	0.3577004484	0.3678794412	$1.01789928*10^{-2}$
1.1	Y_{11}	0.3236528654	0.3328710837	$9.2182183*10^{-3}$
1.2	Y_{12}	0.2904896233	0.3011942119	$1.07045886*10^{-3}$
1.3	Y_{13}	0.2626044858	0.272531793	$9.9273072*10^{-3}$
1.4	Y_{14}	0.2376085763	0.2465969639	$8.9883876*10^{-3}$
1.5	Y_{15}	0.2132619025	0.2231301601	$9.8682576*10^{-3}$
1.6	Y_{16}	0.192790130	0.201896518	$9.106388*10^{-3}$
1.7	Y_{17}	0.1744394738	0.1826835241	$8.2440503*10^{-3}$
1.8	Y_{18}	0.1565654516	0.1652988882	$8.7334366*10^{-3}$
1.9	Y_{19}	0.1415361742	0.1495686192	$8.032445*10^{-3}$
2.0	Y_{20}	0.1280641065	0.1353352832	$7.27142218*10^{-3}$

Table 3: Errors Of Example 2 Using The Method K=2

X	Y(x)	Block LMM K=2	Exact Solution	Absolute Error
0.0	Y_0	0.00000000000	0.000000000	0.000000000
0.1	Y_1	0.0978775889867	0.09873967281	$8.6377414*10^{-4}$
0.2	Y_2	0.19861212422	0.1974917724	$1.1203518*10^{-3}$
0.3	Y_3	0.29367612312	0.2942705996	$5.944765*10^{-4}$
0.4	Y_4	0.38934896897	0.3881091721	$1.2397968*10^{-3}$
0.5	Y_5	0.47772734451	0.4780698862	$3.425417*10^{-4}$
0.6	Y_6	0.56460315135	0.5632538839	$1.3492674*10^{-3}$
0.7	Y_7	0.64269516396	0.642810035	$5.405187*10^{-4}$
0.8	Y_8	0.71738482110	0.7159434407	$1.4413804*10^{-3}$
0.9	Y_9	0.78200576059	0.7819233763	$8.23842*10^{-4}$
1.0	Y_{10}	0.84160024414	0.8400905919	$1.5096522*10^{-3}$
1.1	Y_{11}	0.89010794667	0.8898639	$2.440466*10^{-4}$
1.2	Y_{12}	0.93229476301	0.9307459822	$1.5487808*10^{-3}$
1.3	Y_{13}	0.96269451333	0.9623283582	$3.661551*10^{-4}$
1.4	Y_{14}	0.9858502866	0.9842954674	$1.5548192*10^{-3}$
1.5	Y_{15}	0.99687395667	0.9964278216	$4.46135*10^{-3}$
1.6	Y_{16}	1.00012952163	0.9986041984	$1.5253226*10^{-3}$
1.7	Y_{17}	0.99128576765	0.9908028521	$4.829155*10^{-4}$
1.8	Y_{18}	0.97456117814	0.9731017312	$1.4594469*10^{-3}$

1.9	Y_{19}	0.94615467059	0.9456776996	$4.769709*10^{-4}$
2.0	Y_{20}	0.91013061087	0.9088047689	$1.3258419*10^{-3}$

Table 4: Errors Of Example 2 Using The Method K=3

X	Y(x)	Block LMM K=3	Exact Solution	Absolute Error
0.0	Y_0	0.00000000000	0.000000000	0.000000000
0.1	Y_1	0.09802901344	0.09873967281	$7.1056937*10^{-4}$
0.2	Y_2	0.19847783725	0.1974917724	$9.860648*10^{-4}$
0.3	Y_3	0.29368162270	0.2942705996	$5.899769*10^{-4}$
0.4	Y_4	0.38371780158	0.3881091721	$4.391370*10^{-3}$
0.5	Y_5	0.48270433091	0.4780698862	$4.6344447*10^{-3}$
0.6	Y_6	0.55875508031	0.5632538839	$4.4987946*10^{-3}$
0.7	Y_7	0.6371420762	0.642810035	$5.6679634*10^{-3}$
0.8	Y_8	0.72185960939	0.7159434407	$5.9161686*10^{-3}$
0.9	Y_9	0.77610924220	0.7819233763	$5.8141341*10^{-3}$
1.0	Y_{10}	0.83179160723	0.8400905919	$8.2989847*10^{-3}$
1.1	Y_{11}	0.89833508817	0.8898639	$8.4711881*10^{-3}$
1.2	Y_{12}	0.92205574559	0.9307459822	$8.6902367*10^{-3}$
1.3	Y_{13}	0.95390775209	0.9623283582	$8.4206062*10^{-3}$
1.4	Y_{14}	0.99285695777	0.9842954674	$8.5614903*10^{-3}$
1.5	Y_{15}	0.98766446748	0.9964278216	$8.7633542*10^{-3}$
1.6	Y_{16}	98909399908	0.9986041984	$9.5101994*10^{-3}$
1.7	Y_{17}	0.98421678021	0.9908028521	$6.5860719*10^{-3}$
1.8	Y_{18}	0.963121140957	0.9731017312	$9.9805903*10^{-3}$
1.9	Y_{19}	0.93756011157	0.9456776996	$8.1175881*10^{-3}$
2.0	Y_{20}	0.91691006316	0.9088047689	$8.1052942*10^{-3}$

IV. Discussion Of Results And Conclusion

We have presented three new Block-Schemes (K=1, K=2 and K=3) that are convergent, absolutely stable, two (K=2 and K=3) of which were tested on non-stiff initial value problems.The computational results reveal that the new block schemes work well on non-stiff problems, with good error stability and as such are seen to also be efficient for solving initial value problems.

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