IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. I (Mar. - Apr. 2016), PP 161-164
www.iosrjournals.org

Locally Consecutive and Semi-Consecutive Graphs

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Abstract: *A locally consecutive edge-labeling of a graph G is defined as an assignment of distinct positive integers from the set {1, 2, ...,q } to the edges of G where q is the number of edges of G integers so that all the edges incident at each vertex receive consecutive integers and is said to be locally semi consecutive edge labeling, so that at each vertex of G having degree at least two, at least two of the incident edges receive consecutive. In this paper, we prove that a connected locally finite graph G = (V, E) is a locally consecutive graph if and only if G is either a finite or a one-way infinite locally finite caterpillar, a star graph* $K_{1,n}$ *is locally consecutive,* $P_n \bullet C_3$ *and* $C_n \bullet K_1$ *are locally semi-consecutive graphs.By a graph we mean a finite, undirected graph without multiple edges or loops. For graph theoretic terminology, we refer to Harary [2] and Bondy and Murty [4]. For number theoretic terminology, we refer to M. Apostal [1] and Niven and Herbert S. Zuckerman [5].*

Keywords: *Locally consecutive, locally semi - consecutive, caterpillar, tree, star.*

Definition 1.1: An edge labeling of a graph G = (V, E) is an injective function $g : E$ ® $\{1,2,\ldots,|E|\}$ where $|E|$ denotes the cardinality of the set E. It is said to be locally consecutive, if for each vertex v with $d(v) \geq 2$, the set $g(v) = g(E_V) = \{g(e) : e \in E_V\}$, where $E_V = \{e$ Î $E : e \text{ is incident with } v\}$ consists of consecutive integers and it is said to be locally semi-consecutive graph if the set $g(v)$, $d(v) \geq 2$, consists of at least two consecutive integers.

Definition 1.2: A graph which admits locally semi-consecutive edge labeling is called locally semi-consecutive graph and a graph which admits locally consecutive edge labeling is called locally consecutive graph.

An edge labeling [3] of a graph G = (V, E) is an injective function $g : E$ ® $\{1,2,\ldots,|E|\}$ where $|E|$ denotes the cardinality of the set E. It is said to be locally consecutive, if for each vertex v with $d(v) \geq 2$, the set $g(v) = g(E_V) = \{g(e) : e \in E_V\}$, where $E_V = \{e$ Î $E : e \text{ is incident with } v\}$ consists of consecutive integers and it is said to be locally semi-consecutive graph if the set g(v), $d(v) \geq 2$, consists of at least two consecutive integers.

Definition 1.3: A (one-way infinite) caterpillar is a connected graph in which the set of vertices of degree exceeding one induce a (one-way infinite) path. A graph is said to be locally finite if the degree of each vertex in the graph is finite.

Theorem 2.1: A connected locally finite graph G = (V, E) is a locally consecutive graph if and only if G is either a finite or a one-way infinite locally finite caterpillar.

Proof: Let G have a locally consecutive edge labeling $g : E \to N$. Let v_1 be the vertex for which $g(v_1) = \{1, 2, \ldots, a_1\}$. Let $v_2, v_3, \ldots, v_{a_1+1}$ be the vertices adjacent to v_1. We claim that no two of them are adjacent and $\deg(v_i) = 1$ for $i = 2, \ldots, a_1$. Suppose $v_r v_s \in E(G)$ for $1 < r < s \leq a_1 + 1$. Then $r \in g(v_r)$ and $r+1, r+2, \ldots, s \in g(v_1)$ and hence $r+1, r+2, \ldots, s \notin g(v_r)$ a contradiction to the local consecutiveness of g. This also implies that $\deg(v_i) = 1$ for each $i = 2, 3, \ldots, a_1$.

Next, because of the local consecutiveness of the edge labeling g we see that $g(v_{a_1+1}) = \{a_1, a_1+1, \ldots, a_2\}$. Let $v_{a_1+2}, v_{a_1+3}, \ldots, v_{a_1+a_2-1}$ be the vertices adjacent to v_{a_1+1}. Again, we can show similarly as before that no two of these (new) vertices are adjacent in G as also $\deg(v_{a_1+j}) = 1$

for j = 2, 3, …, a_1+a_2-2, except possibly $v_{a_1+a_2-1}$. We can continue this process indefinitely until, of course, all the vertices would have been exhausted.

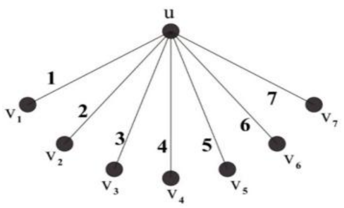
The above procedure renders the graph G with the property that the set of vertices of degree exceeding one induces a (possibly one-way infinite) path in G. This implies that G must be either a finite or a one-way infinite locally finite caterpillar.

Conversely, if G is either a finite, or a one-way infinite locally finite caterpillar then it can be represented on the plane as a bipartite graph with a bipartition {A, B} such that no two vertices within the set A or the set B are adjacent in the graph and also such that no two edges of G cross each other. Then one labels the edges of G with natural numbers starting from the 'top' of the plane representation of the caterpillar going down sequentially, possibly indefinitely, towards its 'bottom'. The resulting labeling of the edges of G is obviously a locally consecutive edge-labeling of G, and the proof is complete.

Theorem 2.2: A star graph $K_{1,n}$ is locally consecutive.

Proof: Let G (V, E) = $K_{1,n}$, the star graph. Let V = {u, v_1, v_2, …, v_n} be the vertex set where u is the centre vertex and v_i's are pendant vertices and
E = {uv_i: $1 \leq i \leq n$} be the edge set of G.
Define $f : E \rightarrow \{1, 2, ..., n\}$ by f(uv) = i

The vertex u is the only one with degree > 1 and the other n vertices $\{v_i : 1 \leq i \leq n\}$ are pendant vertices. The vertices adjacent with u are $v_1, v_2, ..., v_n$ and the labels of the corresponding edges $uv_1, uv_2, ..., uv_n$ are the first n natural numbers 1, 2, …, n which are consecutive integers. Hence a star graph is a locally consecutive graph.



The locally consecutive numbering of $K_{1,7}$.

Theorem 2.3: The graph $P_n \bullet C_3$ is a locally semi-consecutive graph.

Proof: Let G (V, E) = $P_n \bullet C_3$.

Let $V = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ be the vertex set and $E = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i, v_iw_i, w_iu_i : 1 \leq i \leq n\}$ the edge set of G. G has 3n vertices and 3n+3 edges.

Define a function $f : E \rightarrow \{1, 2, ..., 3n+3\}$ by

$$f(u_iv_i) = 6i-5 \quad 1 \leq i \leq n$$
$$f(v_i w_i) = 6i-4 \quad 1 \leq i \leq n$$
$$f(w_i u_i) = 6i-3 \quad 1 \leq i \leq n-1$$
$$f(w_n u_n) = 6n-7$$
$$f(u_i u_{i+1}) = 6i-2 \quad 1 \leq i \leq n-1$$
$$f(v_i v_{i+1}) = 6i-1 \quad 1 \leq i \leq n-2$$
$$f(v_{n-1} v_n) = 6n-3$$
$$f(w_iw_{i+1}) = 6i \quad 1 \leq i \leq n-1$$

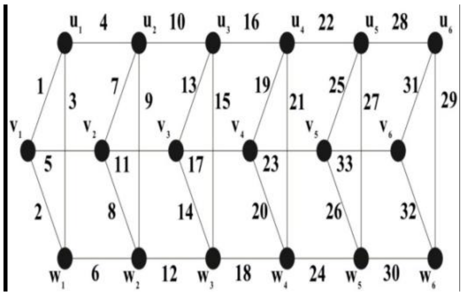
The edges incident at the vertex u_1 are u_1v_1, u_1w_1 and u_1u_2 and the labels given are respectively 1, 3 and 4. (3 and 4 are consecutive integers). The edges incident at the vertex u_i for $2 \leq i \leq n-1$ are

$u_i v_i, u_i w_i, u_{i-1} u_i$ and $u_i u_{i+1}$. The labels given are respectively 6i-5, 6i-3, 6(i-1)-2 = 6i - 8 and 6i-2 and of them
6i-3 and 6i-2 are consecutive integers. The edges incident at the vertex u_n are $u_n v_n$, $u_n w_n$ and $u_{n-1} u_n$ and the labels given are respectively 6n-5, 6n-7 and 6n-8. (6n-7 and 6n-8 are consecutive integers).

The edges incident at the vertex v_1 are $u_1 v_1$, $v_1 w_1$ and $v_1 v_2$ and the labels given are respectively 1, 2 and 5. (1 and 2 are consecutive integers). The edges incident at the vertex v_i for $2 \le i \le n-1$ are $u_i v_i, v_i w_i, v_{i-1} v_i$ and $v_i v_{i+1}$. The labels given are respectively 6i-5, 6i-4, 6(i-1) -1 = 6i-7 and 6i-1 and of them 6i-5 and 6i-4 are consecutive integers. The edges incident at the vertex v_n are $u_n v_n$, $v_n w_n$ and $v_{n-1} v_n$ and the labels given are respectively 6n-5, 6n-4 and 6n-3. (6n-5, 6n-4 and 6n-3 are consecutive integers).

The edges incident at the vertex w_1 are $u_1 w_1$, $v_1 w_1$ and $w_1 w_2$ and the labels given are respectively 3, 2 and 6. (3 and 2 are consecutive integers). The edges incident at the vertex w_i for $2 \le i \le n-1$ are $u_i w_i, v_i w_i, w_{i-1} w_i$ and $w_i w_{i+1}$. The labels given are respectively 6i-3, 6i-4, 6(i-1) = 6i-6 and 6i and of them 6i-3 and 6i-4 are consecutive integers. The edges incident at the vertex w_n are $u_n w_n$, $v_n w_n$ and $w_{n-1} w_n$ and the labels given are respectively 6n-7, 6n-4 and 6n-6. (6n-7 and 6n-6 are consecutive integers).

Hence $P_n \bullet C_3$ is a locally semi-consecutive graph.



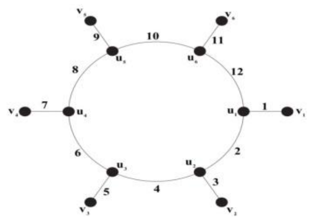
Locally semi-consecutive labeling of $P_6 \bullet C_3$

Theorem 2.4: The graph $C_n \bullet K_1$ is a locally semi-consecutive graph.

Proof: Let G(V, E) = $C_n \bullet K_1$ be the graph with a cycle C_n and to each vertex of the cycle C_n, a pendant vertex is attached. G (V, E) is a graph with 2n vertices and 2n edges. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_n and $\{v_1, v_2, \ldots, v_n\}$ be the set of pendent vertices. That is, $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and let $E = \{u_1 u_n\} \cup \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_j v_j : 1 \le j \le n\}$.

Define $f : E \rightarrow \{1, 2, \ldots, 2n\}$ by $f(u_i v_{i+1}) = 2i, \quad 1 \le i \le n-1, f(u_n v_1) = 2n$
$f(u_j v_j) = 2j - 1, 1 \le j \le n$

The labels of those edges incident on each cycle vertex u_i, $i \neq 1$ deg(u_i)=3, are 2i-2, 2i-1 and 2i and the labels on the edges incident on u_1 are 1, 2 and 2n.Therefore any vertex of degree greater than or equal to two atleast two edge labels are consecutive integers. Hence $C_n \bullet K_1$ is a locally semi-consecutive graph.



Locally semi-consecutive labeling of $C_6 \bullet K_1$.

Remark: Cycle is an example of a graph which is not locally semi consecutive.

Theorem 2.5: Let G be a unit cyclic graph consisting of a unique triangle with vertices v_1, v_2, v_3 with deg v_2 = deg v_3 = 2, a path p = ($v_1, u_1, u_2, \ldots, u_n$) of length n and k pendant vertices ($w_1, w_2, \ldots, w_k$) adjacent to v_1. Then G is locally semi-consecutive graph.

Proof: Let G (V, E) be the given graph. Let V= $\{v_1, v_2, v_3\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{w_1, w_2, \ldots, w_k\}$ be the vertex set and $E = \{v_1v_2, v_2v_3, v_3v_1, v_1u_1\} \cup \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{v_1w_j : 1 \le j \le k\}$ be the edge set of G.

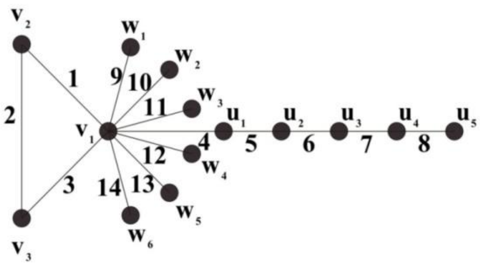
Then $|V| = n + k + 3$ and $|E| = n + k + 3$

Define $f : E \to \{1, 2, \ldots, n + k + 3\}$ by

$f(v_1v_2) = 1$; $f(v_2v_3) = 2$; $f(v_3v_1) = 3$; $f(v_1u_1) = 4$ and

$f(u_iu_{i+1}) = 4+i$, $1 \le i \le n-1$; $f(v_1w_i) = n+3+j$, $1 \le j \le k$

By the above function at least two edges incident on any vertex get consecutive integers as labels. Hence G (V, E) is a locally semi-consecutive graph.



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