*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. III (Mar. - Apr. 2016), PP 17-25*
*www.iosrjournals.org*

# A Transformation Formula for Composite Appell's Hypergeometric Functions

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***Abstract:*** *This paper gives a transformation formula for composite Appell's hypergeometric functions of two complex variables which satisfy an integer Pfaffian system. These functional equations for* $\gamma = 1, c = 1$ *are established.*
***Keywords:*** *arithmetic–geometric mean, hypergeometric function- - Pfaffian system -composite.*

## I. Introduction

We first recall the basic Eular integrals which define the beta function:

$$\beta(a,b) = \int_0^1 u^{a-1}\left(1-u\right)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

For a and b with positive real parts, and Gauss hypergeometric function :

$${}_2F_1(a;b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n = \frac{1}{\beta(b,c-b)} \int_0^1 \frac{t^{b-1}\left(1-t\right)^{c-b-1}}{(1-tz)^a} dt,$$

where in the integral it is assumed that $z \notin [1,\infty)$ and that the real parts of b and c-b are positive, also, the hypergeometric function satisfies the Gauss quadratic transformation formula:

$$\left(1+z\right)^{2a} F\left(a; a-b+\frac{1}{2}, b+\frac{1}{2}; z^2\right) = F\left(a; b, 2b; \frac{4z}{(1+z)^2}\right)$$

which implies that the reciprocal of the arithmetic – geometric mean of 1 and $x \in (0.1)$ coincides with $F\left(\frac{1}{2};\frac{1}{2},1;1-x^2\right)$, refer [12] .

when $\alpha = \beta = \frac{1}{2}$ ; this equality reduces to

$$\frac{1+z}{2} F\left(\frac{1}{2};\frac{1}{2},1;1-z^2\right) = F\left(\frac{1}{2};\frac{1}{2},1;1-\left(\frac{2\sqrt{z}}{1+z}\right)^2\right),$$

It is shown in [5], [6] and [13] that transformation formulas of hypergeometric functions of several variables imply expressions of common limits of multiple sequences.
And these transformation formulas are extended by introducing a parameter in [8].
Makky has related the study of the composite hypergeometric function in one complex variable in [2] as follows:

$$F_A(a,b;\alpha,\beta;c,\gamma;z,w) = f(a,b;c;z)\, g(\alpha,\beta;\gamma;w) \text{ , since}$$

$$f\left(a,b;c;z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n \quad \text{and} \quad g\left(\alpha,\beta;\gamma;z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} w^n ,$$

$$\mathrm{D}_{1,2}\mathrm{F}_{\mathrm{A}}(a,b;\alpha,\beta;c;\gamma;z,w) = (zd_z + wd_w)F_A + [(a+b-c)z + (\alpha+\beta-\gamma)w]F_A +$$

$$\frac{(c-a)(c-b)}{c} zF_A(c+) + \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma} wF_A(\gamma+),$$

$(D_{1,2}+a+\alpha)F_A = aF_A(a+)+\alpha F_A(\alpha+)$ and

$D_{1,2}=d_1+d_2$, $d_1=z_1\dfrac{\partial}{\partial z_1}$, $d_2=z_2\dfrac{\partial}{\partial z_2}$.

Also, the composite Appell's hypergeometric function of two variables which given in the form, refer [1] ; [3] and [10]

$$\Psi(a,\alpha;b,b';\beta,\beta';c;\gamma;z_1,z_2,z_3,z_4)=F(a;b,b';c;z_1,z_2)G(\alpha;\beta,\beta';\gamma;z_3,z_4)$$
$$=\sum_{n_1,n_2,n_3,n_4}\frac{(a)_{n_1+n_2}(\alpha)_{n_3+n_4}(b)_{n_1}(b')_{n_2}(\beta)_{n_3}(\beta')_{n_4}}{n_1!n_2!n_3!n_4!(c)_{n_1+n_2}(\gamma)_{n_3+n_4}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}(z_4)^{n_4},$$

since

$$F(a;b,b';c;z_1,z_2)=\sum_{n_1,n_2}\frac{(a)_{n_1+n_2}(b)_{n_1}(b')_{n_2}}{n_1!n_2!(c)_{n_1+n_2}}(z_1)^{n_1}(z_2)^{n_2},$$

$$G(\alpha;\beta,\beta';\gamma;z_3,z_4)=\sum_{n_3,n_4}\frac{(\alpha)_{n_3+n_4}(\beta)_{n_3}(\beta')_{n_4}}{n_3!n_4!(\gamma)_{n_3+n_4}}(z_3)^{n_3}(z_4)^{n_4}$$

$$D_{1,2,3,4}\Psi(a,\alpha;b,b';\beta,\beta';c;\gamma;z_1,z_2,z_3,z_4)=$$
$$G(\alpha;\beta,\beta';\gamma;z_3,z_4)D_{1,2}F(a;b,b';c;z_1,z_2)+F(a;b,b';c;z_1,z_2)D_{3,4}G(\alpha;\beta,\beta';\gamma;z_3,z_4)$$
$$=z_1b\ \Psi(b+)+z_2b'\ \Psi(b'+)+z_3\beta\ \Psi(\beta+)+z_4\beta'\ \Psi(\beta'+)+$$
$$+\frac{a-c}{c}\left[z_1(b+d_1)+z_2(b'+d_2)\right]\Psi(c+)+\frac{\alpha-\gamma}{\gamma}\left[z_3(\beta+d_3)+z_4(\beta'+d_4)\right]\Psi(\gamma+)$$

$D_{1,2}=d_1+d_2$, $D_{3,4}=d_3+d_4$, $d_1=z_1\dfrac{\partial}{\partial z_1}$, $d_2=z_2\dfrac{\partial}{\partial z_2}$, $d_3=z_3\dfrac{\partial}{\partial z_3}$, $d_4=z_4\dfrac{\partial}{\partial z_4}$.

## II. Transformation Formula

Suppose that

$$F(\alpha;\beta,\beta';\gamma;z_1,z_2)=\sum_{n_1,n_2}\frac{(\alpha)_{n_1+n_2}(\beta)_{n_1}(\beta')_{n_2}}{n_1!n_2!(\gamma)_{n_1+n_2}}(z_1)^{n_1}(z_2)^{n_2}$$
$$G(a;b,b';c;z_3,z_4)=\sum_{n_3,n_4}\frac{(a)_{n_3+n_4}(b)_{n_3}(b')_{n_4}}{n_3!n_4!(c)_{n_3+n_4}}(z_3)^{n_3}(z_4)^{n_4},$$

are two Appell's hypergeometric functions of two variables, then the composite of these function of two variables gives as follows :

$$\Psi(\alpha;a,\beta,\beta';b,b';\gamma;c;z_1,z_2,z_3,z_4)=F(\alpha;\beta,\beta';\gamma;z_1,z_2)G(a;b,b';c;z_3,z_4)$$
$$=\sum_{n_1,n_2,n_3,n_4}\frac{(\alpha)_{n_1+n_2}(\beta)_{n_1}(\beta')_{n_2}}{n_1!n_2!(\gamma)_{n_1+n_2}}\frac{(a)_{n_3+n_4}(b)_{n_3}(b')_{n_4}}{n_3!n_4!(c)_{n_3+n_4}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}(z_4)^{n_4},$$

where $z=(z_1,z_2,z_3,z_4)$ satisfies $|z_i|<1$ ; $j=1,2,3,4$; $c\neq 0,-1,-2,-3,....$ ; $\gamma\neq 0,-1,-2,-3,...........$

and

$$(\alpha,n)=\alpha(\alpha+1)......(\alpha+n-1)=\frac{\Gamma(\alpha,n)}{\Gamma(\alpha)}.$$

This function admits the integral representation of Euler type :

$$F(\alpha;\beta,\beta';\gamma;z_1,z_2)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha}\left(1-t\right)^{\gamma-\alpha}\left(1-z_1t\right)^{-\beta_1}\left(1-z_2t\right)^{-\beta_2}\frac{dt}{t(1-t)}$$

$$G(a;b,b';c;z_3,z_4)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a}\left(1-t\right)^{c-a}\left(1-z_3t\right)^{-b_1}\left(1-z_4t\right)^{-b_2}\frac{dt}{t(1-t)}\ .$$

For properties of Appell's hypergeometric function $F\ ;G$ ; refer to [4] and [11].

Also, transformation formulas for $F(\alpha;\beta,\beta';\gamma;z_1,z_2)$ and $G(a;b,b';c;z_3,z_4)$ are written in the form refer [7]:

$$\left(z_1z_2\right)^{\frac{1-\gamma}{2}}\left(\frac{z_1z_2}{2}\right)^{\gamma}F\left(\frac{3+\gamma}{4},\frac{1+\gamma}{4},\frac{1+\gamma}{4},\frac{3+3\gamma}{4};1-z_1^2,1-z_2^2\right)$$

$$=F\left(\gamma,\frac{1+\gamma}{4},\frac{1+\gamma}{4},\frac{3+3\gamma}{4};1-\frac{z_1\left(1+z_2\right)}{z_1+z_2},1-\frac{z_2\left(1+z_1\right)}{z_1+z_2}\right) \qquad (2.1)$$

and

$$\left(z_3z_4\right)^{\frac{1-c}{2}}\left(\frac{z_3z_4}{2}\right)^{c}G\left(\frac{3+c}{4},\frac{1+c}{4},\frac{1+c}{4},\frac{3+3c}{4};1-z_3^2,1-z_4^2\right)$$

$$=G\left(c,\frac{1+c}{4},\frac{1+c}{4},\frac{3+3c}{4};1-\frac{z_3\left(1+z_4\right)}{z_3+z_4},1-\frac{z_4\left(1+z_3\right)}{z_3+z_4}\right) \qquad (2.2)$$

where $\left(z_1,z_2\right)$ and $\left(z_3,z_4\right)$ are in a small neighborhood of $\left(1,1\right)$ and the values of :

$\left(z_1z_2\right)^{\frac{1-\gamma}{2}}$ ; $\left(\frac{z_1z_2}{2}\right)^{\gamma}$ ; $\left(z_3z_4\right)^{\frac{1-c}{2}}$ ; $\left(\frac{z_3z_4}{2}\right)^{c}$ at $\left(z_1,z_2\right)=\left(1,1\right)$ and $\left(z_3,z_4\right)=\left(1,1\right)$ are in 1 .

Also, the following vector valued functions:

$${}^{t}\left(F_0,\frac{\partial F_0}{\partial z_1},\frac{\partial F_0}{\partial z_2}\right) \quad ; \quad {}^{t}\left(F_1,\frac{\partial F_1}{\partial z_1},\frac{\partial F_1}{\partial z_2}\right)$$

where $F_0\left(z_1,z_2\right)$ and $F_1\left(z_3,z_4\right)$ are left and right hand sides of (2.1) ; respectively.

And

$${}^{t}\left(G_0,\frac{\partial G_0}{\partial z_3},\frac{\partial G_0}{\partial z_4}\right) \quad ; \quad {}^{t}\left(G_1,\frac{\partial G_1}{\partial z_3},\frac{\partial G_1}{\partial z_4}\right)$$

where $G_0\left(z_1,z_2\right)$ and $G_1\left(z_3,z_4\right)$ are left and right hand sides of (2.2) ; respectively.

Thus, the transformation formulas of composite hypergeometric functions is formulated as follows:

**Theorem (2.1)**

The transformation formulas of composite hypergeometric functions for

$$\left(z_1z_2\right)^{\frac{1-\gamma}{2}}\left(z_3z_4\right)^{\frac{1-c}{2}}\left(\frac{z_1z_2}{2}\right)^{\gamma}\left(\frac{z_3z_4}{2}\right)^{c}\Psi\left(\begin{matrix}\frac{3+\gamma}{4},\frac{3+c}{4},\frac{1+\gamma}{4},\frac{1+\gamma}{4},\frac{1+c}{4},\frac{1+c}{4},\frac{3+3\gamma}{4},\frac{3+3c}{4};\\ 1-z_1^2,1-z_2^2,1-z_3^2,1-z_4^2\end{matrix}\right)$$

$$=\Psi\left(\begin{array}{c}\gamma,c,\frac{1+\gamma}{4},\frac{1+\gamma}{4},\frac{1+c}{4},\frac{1+c}{4},\frac{3+3\gamma}{4},\frac{3+3c}{4};1-\frac{z_1(1+z_2)}{z_1+z_2},\\ 1-\frac{z_2(1+z_1)}{z_1+z_2},1-\frac{z_3(1+z_4)}{z_3+z_4},1-\frac{z_4(1+z_3)}{z_3+z_4}\end{array}\right) \qquad (2.3)$$

where $(z_1,z_2,z_3,z_4)$ are in a small neighborhood of $(1,1,1,1)$ and the values of :

$(z_1z_2)^{\frac{1-\gamma}{2}}$ , $(z_3z_4)^{\frac{1-c}{2}}$ ; $\left(\frac{z_1z_2}{2}\right)^{\gamma}$ and $\left(\frac{z_3z_4}{2}\right)^{c}$ at $(z_1,z_2,z_3,z_4)=(1,1,1,1)$ are in 1.

**Proof :**

Consider the following vector valued functions:

$$^t\left(\Psi_0,\frac{\partial\Psi_0}{\partial z_1},\frac{\partial\Psi_0}{\partial z_2},\frac{\partial\Psi_0}{\partial z_3},\frac{\partial\Psi_0}{\partial z_4}\right);\quad {}^t\left(\Psi_1,\frac{\partial\Psi_1}{\partial z_1},\frac{\partial\Psi_1}{\partial z_2},\frac{\partial\Psi_1}{\partial z_3},\frac{\partial\Psi_1}{\partial z_4}\right) \qquad (2.4)$$

where $\Psi_0(z_1,z_2,z_3,z_4)$ and $\Psi_1(z_1,z_2,z_3,z_4)$ are left and right hand sides of (2.3) ; respectively . Each of them takes the value :

$$^t\left(1,1,\frac{-\gamma}{6},\frac{-\gamma}{6},\frac{-c}{6},\frac{-c}{6}\right)$$

at the $(z_1,z_2,z_3,z_4)=(1,1,1,1)$ and satisfies an integer Pfaffian system :

$$\begin{aligned} d\Psi(z) &= dF(z_1,z_2).G(z_3,z_4)=G(z_3,z_4)dF(z_1,z_2)+F(z_1,z_2)dG(z_3,z_4)\\ &=G(z_3,z_4)(\Omega_1dz_1+\Omega_2dz_2)F(z_1,z_2)+F(z_1,z_2)(\Omega_3dz_3+\Omega_4dz_4)G(z_3,z_4)\end{aligned}$$

where $\Omega_1$ ; $\Omega_2$ ; $\Omega_3$ *and* $\Omega_4$ are respectively as follows:

$$\begin{pmatrix} 0 & 1 & 0\\ \frac{\gamma(1+\gamma)z_2(1+z_1z_2)}{2z_1(1-z_1^2)(z_1+z_2)^2} & \frac{(1+\gamma)\left((2z_1^2-1)(2z_1^2-z_2^2)-z_1^2z_2^2\right)}{2z_1(1-z_1^2)(z_1+z_2)^2}+\frac{2\gamma}{z_1+z_2} & \frac{(1+\gamma)z_2(1-z_2^2)}{2(1-z_1^2)(z_1^2-z_2^2)}\\ \frac{-\gamma(\gamma+1)}{2(z_1+z_2)^2} & \frac{z_1((1-\gamma)z_1+2\gamma z_2)}{2z_2(z_1^2-z_2^2)} & \frac{-z_2((1-\gamma)z_2+2\gamma z_1)}{2z_1(z_1^2-z_2^2)} \end{pmatrix};$$

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{-\gamma(1+\gamma)}{2(z_1+z_2)^2} & \frac{z_1((1-\gamma)z_1+2\gamma z_2)}{2z_2(z_1^2-z_2^2)} & \frac{-z_2((1-\gamma)z_2+2\gamma z_1)}{2z_1(z_1^2-z_2^2)} \\ \frac{\gamma(1+\gamma)z_1(1+z_1z_2)}{2z_2(1-z_2^2)(z_1+z_2)^2} & \frac{-(1+\gamma)z_1(1-z_1^2)}{2(1-z_2^2)(z_1^2-z_2^2)} & \frac{-(1+\gamma)((2z_2^2-1)(2z_2^2-z_1^2)-z_1^2z_2^2)}{2z_2(1-z_2^2)(z_1+z_2)^2} \end{pmatrix};$$

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{c(c+1)z_4(1+z_3z_4)}{2z_3(1-z_3^2)(z_3+z_4)^2} & \frac{(1+c)((2z_3^2-1)(2z_3^2-z_4^2)-z_3^2z_4^2)}{2z_3(1-z_3^2)(z_3+z_4)^2}+\frac{2c}{z_3+z_4} & \frac{(1+c)z_4(1-z_4^2)}{2(1-z_3^2)(z_3^2-z_4^2)} \\ \frac{-c(c+1)}{2(z_3+z_4)^2} & \frac{z_3((1-c)z_3+2cz_4)}{2z_4(z_3^2-z_4^2)} & \frac{-z_4((1-c)z_4+2cz_3)}{2z_3(z_3^2-z_4^2)} \end{pmatrix};$$

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{-c(1+c)}{2(z_3+z_4)^2} & \frac{z_3((1-c)z_3+2cz_4)}{2z_4(z_3^2-z_4^2)} & \frac{-z_4((1-c)z_4+2cz_3)}{2z_3(z_3^2-z_4^2)} \\ \frac{c(1+c)z_3(1+z_3z_4)}{2z_4(1-z_4^2)(z_3+z_4)^2} & \frac{-(1+c)z_3(1-z_3^2)}{2(1-z_4^2)(z_3^2-z_4^2)} & \frac{-(1+c)((2z_4^2-1)(2z_4^2-z_3^2)-z_3^2z_4^2)}{2z_4(1-z_4^2)(z_3+z_4)^2} \end{pmatrix}.$$

Thus, $\Psi_0(z_1,z_2,z_3,z_4)=\Psi_1(z_1,z_2,z_3,z_4)$.

By putting $c=1$ and $\gamma=1$ for the equality (2.3) in the above theorem , the following is obtained:

**Corollary**

For $(z_1,z_2,z_3,z_4)$ are in a small neighborhood of $(1,1,1,1)$, it is found that:

$$\left(\frac{z_1z_2}{2}\right)\left(\frac{z_3z_4}{2}\right)\Psi\left(1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2};1-z_1^2,1-z_2^2,1-z_3^2,1-z_4^2\right)$$
$$=\Psi\left(1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2};1-\frac{z_1(1+z_1)}{z_1+z_2},1-\frac{z_2(1+z_1)}{z_1+z_2},1-\frac{z_3(1+z_4)}{z_3+z_4},1-\frac{z_4(1+z_3)}{z_3+z_4}\right).$$

Also, the transformation formulas of the composite hypergeometric function $\Psi(\alpha;a,\beta,\beta';b,b';\gamma;c;z_1,z_2,z_3,z_4)$ is formulated in the following:

**Theorem (2.2)**

The composite hypergeometric function $\Psi(\alpha;a,\beta,\beta';b,b';\gamma;c;z_1,z_2,z_3,z_4)$ satisfy the transformation formula :

$$\left(\frac{1+z_1+z_2}{3}\right)^{\gamma}\left(\frac{1+z_3+z_4}{3}\right)^{c}\Psi\left(\begin{array}{l}\frac{\gamma}{3},\frac{c}{3},\frac{\gamma+1}{6},\frac{\gamma+1}{6},\frac{c+1}{6},\frac{c+1}{6},\frac{\gamma+1}{2},\frac{c+1}{2};\\ 1-z_1^3,1-z_2^3,1-z_3^3,1-z_4^3\end{array}\right)$$

$$=\Psi\left(\frac{\gamma}{3},\frac{c}{3},\frac{\gamma+1}{6},\frac{\gamma+1}{6},\frac{c+1}{6},\frac{c+1}{6},\frac{\gamma+5}{6},\frac{c+5}{6};z_1',z_2',z_3',z_4'\right)$$

where $z=(z_1,z_2,z_3,z_4)$ are in a small neighborhood U of $(1,1,1,1)$, a branch of $\left(\frac{1+z_1+z_2}{3}\right)^{\gamma}$ and $\left(\frac{1+z_3+z_4}{3}\right)^{c}$ on U is given by assigning 1 to its value at $(z_1,z_2,z_3,z_4)=(1,1,1,1)$, and

$$z_1'=\left(\frac{1+\omega z_1+\omega^2 z_2}{1+z_1+z_2}\right)^3 \quad ; \quad z_2'=\left(\frac{1+\omega^2 z_1+\omega z_2}{1+z_1+z_2}\right)^3 \quad \text{and} \quad \omega=\frac{-1+\sqrt{3}}{2}.$$

**Proof :**

Let $\hat{f}(z)$ and $\hat{g}(z)$ be the connection 1- forms in Fact 1 for

$$\hat{f}(z)={}^t\left(\begin{array}{l}f_0(z_1,z_2,z_3,z_4),z_1\frac{\partial f_0}{\partial z_1}(z_1,z_2,z_3,z_4),z_2\frac{\partial f_0}{\partial z_2}(z_1,z_2,z_3,z_4),\\ z_3\frac{\partial f_0}{\partial z_3}(z_1,z_2,z_3,z_4),z_4\frac{\partial f_0}{\partial z_4}(z_1,z_2,z_3,z_4)\end{array}\right);$$

$$f_0(z_1,z_2,z_3,z_4)=\Psi(\alpha;a,\beta_1,\beta_2;b_1,b_2;\gamma;c;z_1,z_2,z_3,z_4)$$

and

$$\hat{g}(z)={}^t\left(\begin{array}{l}g_0(z_1,z_2,z_3,z_4),z_1\frac{\partial g_0}{\partial z_1}(z_1,z_2,z_3,z_4),z_2\frac{\partial g_0}{\partial z_2}(z_1,z_2,z_3,z_4),\\ z_3\frac{\partial g_0}{\partial z_3}(z_1,z_2,z_3,z_4),z_4\frac{\partial g_0}{\partial z_4}(z_1,z_2,z_3,z_4)\end{array}\right);$$

$$g_0(z_1,z_2,z_3,z_4)=\Psi(\alpha';a',\beta_1',\beta_2';b_1',b_2';\gamma';c';z_1,z_2,z_3,z_4)$$

respectively. It is easy to see that the vector valued function

$$f(z)={}^t\left(f_0,\frac{\partial f_0}{\partial z_1},\frac{\partial f_0}{\partial z_2},\frac{\partial f_0}{\partial z_3},\frac{\partial f_0}{\partial z_4}\right) \text{ and } g(z)={}^t\left(g_0,\frac{\partial g_0}{\partial z_1},\frac{\partial g_0}{\partial z_2},\frac{\partial g_0}{\partial z_3},\frac{\partial g_0}{\partial z_4}\right)$$

satisfy the Pfaffian systems :

$$df=\Omega_f(z)f \quad ; \quad dg=\Omega_g(z)g$$

respectively, where

$$\Omega_f(z)=P\,\Omega_{\hat{f}}(z)P^{-1}+d\,P\,P^{-1};$$

$$\Omega_g(z)=P\,\Omega_{\hat{g}}(z)P^{-1}+d\,P\,P^{-1} \text{ and}$$

$$P = diag\left(1, \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}\right) = \begin{pmatrix} 1 & & & & \\ & \frac{1}{z_1} & & & \\ & & \frac{1}{z_2} & & \\ & & & \frac{1}{z_3} & \\ & & & & \frac{1}{z_4} \end{pmatrix}.$$

Consider the vector valued function

$$F(x) = {}^t\left(F, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}, \frac{\partial F}{\partial x_4}\right)$$

for

$$F(x_1, x_2, x_3, x_4) = \left(\frac{1+x_1+x_2}{3}\right)^{\gamma} \left(\frac{1+x_3+x_4}{3}\right)^{c} F\left(1-x_1^3, 1-x_2^3, 1-x_3^3, 1-x_4^3\right)$$

$$F(1,1,1,1) = \left(1, 1, \frac{\gamma}{3} - \frac{3\alpha\beta_1}{\gamma}, \frac{\gamma}{3} - \frac{3\alpha\beta_2}{\gamma}, \frac{c}{3} - \frac{3ab_1}{c}, \frac{c}{3} - \frac{3ab_2}{c}\right) \tag{2.5}$$

and the Pfaffian system: $dF = \Omega_F(x)F$ where

$$\Omega_F(x) = Q\left[J_1\,\Omega_f(x)J_1^{-1} + dJ_1J_1^{-1}\right]Q^{-1} + d\,Q\,Q^{-1};$$

since:

$$J_1 = \begin{pmatrix} 1 & & & & \\ & -3x_1^3 & & & \\ & & -3x_2^3 & & \\ & & & -3x_3^3 & \\ & & & & -3x_4^3 \end{pmatrix}$$

$$Q=\begin{pmatrix} \xi & & & & \\ \frac{d\xi}{dx_1} & \xi & & & \\ \frac{d\xi}{dx_2} & & \xi & & \\ \frac{d\xi}{dx_3} & & & \xi & \\ \frac{d\xi}{dx_4} & & & & \xi \end{pmatrix}=\xi\begin{pmatrix} 1 & & & & \\ \frac{\gamma}{1+x_1+x_2} & 1 & & & \\ \frac{\gamma}{1+x_1+x_2} & & 1 & & \\ \frac{\gamma}{1+x_3+x_4} & & & 1 & \\ \frac{\gamma}{1+x_3+x_4} & & & & 1 \end{pmatrix}$$

since

$$\xi=\left(\frac{1+x_1+x_2}{3}\right)^{\gamma}\left(\frac{1+x_3+x_4}{3}\right)^{c}$$

and $\Omega_f(x)$ is the pull – pack of $\Omega_f(z)$ under the map

$$(x_1,......,x_4)\to(z_1,......,z_4)=\left(1-x_1^3,......,1-x_4^3\right).$$

Consider the vector valued function:

$$G(z)={}^t\left(G_0,\frac{\partial G_0}{\partial x_1},\frac{\partial G_0}{\partial x_2},\frac{\partial G_0}{\partial x_3},\frac{\partial G_0}{\partial x_4}\right)$$

for the pull – pack $G_0(x_1,......,x_4)$ of $g_0(z_1,......,z_4)$ under the map

$\xi:\ (x_1,......,x_4)\to(z_1,......,z_4)$

$$=\left(\left(\frac{1+\omega z_1+\omega^2 z_2}{1+z_1+z_2}\right)^3,\ \left(\frac{1+\omega^2 z_1+\omega z_2}{1+z_1+z_2}\right)^3,\ \left(\frac{1+\omega z_3+\omega^2 z_4}{1+z_3+z_4}\right)^3,\ \left(\frac{1+\omega^2 z_3+\omega z_4}{1+z_3+z_4}\right)^3\right).$$

It satisfies :

$$G(1,1,1,1)={}^t\left(1,0,0,0,0\right) \qquad (2.6)$$

and the Pfaffian system: $dG=\Omega_G(x)G$ where

$$\Omega_G(x)=J_2\,\Omega_g(x)J_2^{-1}+d\,J_2\,J_2^{-1}, \qquad J_2=\begin{pmatrix}1 & \\ & t\,J\end{pmatrix},$$

$\Omega_g(x)$ is the pull – pack of $\Omega_g(z)$ under the map $\xi$ and J is the Jacobi matrix of the map $\xi$.

Note that $F_0(x)=G_0(x)$ on U if and only if

$$F(1,1,1,1)=G(1,1,1,1) \text{ and } \Omega_F(x)=\Omega_G(x).$$

By (2.5) and (2.6) , it is established that:

$$\frac{\gamma}{3}-\frac{3\alpha\beta_1}{\gamma}=\frac{\gamma}{3}-\frac{3\alpha\beta_2}{\gamma}=\frac{c}{3}-\frac{3ab_1}{c}=\frac{c}{3}-\frac{3ab_2}{c}=0\ .$$

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