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Linear and Nonlinear State-Feedback Controls of HOPF Bifurcation

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Abstract : *Bifurcation control has gained increasing attention in the last decade. This paper makes an attempt to highlight a simple and unified state-feedback methodology for developing Hopf bifurcation linear or nonlinear controls for continuous-time systems. The control task can be either shifting an existing bifurcation or creating a new one. Some numerical simulations are included to illustrate the methodology and verify the theoretical results.*

Keywords: *Eigenvalues, Hopf bifurcation, Control.*

I. Introduction

Bifurcation control refers to the control of bifurcative properties of a nonlinear dynamical system, thereby resulting in some desired system output behaviors. Typical examples of bifurcation control include delaying the onset of an inherent bifurcation, relocating an existing bifurcation point, modifying the shape or type of a bifurcation chain, introducing a new bifurcation at a preferable parameter value, stabilizing a bifurcated periodic trajectory, changing the multiplicity, amplitude, and/or frequency of some limit cycles emerging from bifurcation, optimizing the system performance near a bifurcation point, or a certain combination of some of these **[3,5,9,10]**. Bifurcation control is also important **as** an effective strategy for chaos control since period-doubling bifurcation is a typical route to chaos in many nonlinear dynamical systems.

Bifurcation control has great potential in many engineering applications. In some physical systems such **as** the stressed system, delay of bifurcations provides **an** opportunity for obtaining stable operating conditions for the machine beyond the margin of operability at the normal situation **.** It is often desirable to be able to modify the stability of bifurcated system limit cycles, in applications of some conventional control problems such **as** thermal convection experiments **[11].** Other examples include stabilization via bifurcation control in tethered satellites **[6],** magnetic bearing systems **[2],** voltage dynamics of electric power systems **[4];** delay of bifurcation in compressor stall in gas turbine jet engines**,** and in rotating chains via external periodic forcing **[11];** and in various mechanical systems such **as** robotics and electronic systems such **as** laser machines and nonlinear circuits **[7].** Bifurcations can be controlled by means of linear delayed-feedback **[7, 8]** or nonlinear feedback **[1],** employing harmonic balance approximation **[6],** and applying the quadratic invariants in the normal form **[8].** In this paper, a unified nonlinear ***as*** well **as** simple linear state feedback technique is developed for Hopf bifurcation control, which by nature is different from the **afore**mentioned existing approaches. Both problems of shifting and creating a Hopf bifurcation are discussed, and Computer simulations are included to illustrate the methodology and to verify the theoretical results.

II. Hopf Bifurcation: Fundamentals

In this section, the classical fundamentals of continuous-time Hopf bifurcation are briefly reviewed. These background concepts are needed in the development of the bifurcation control technique given in the subsequent sections.

First, consider a two-dimensional continuous –time autonomous parameterized system

$$\dot{x} = f\,(x, y\,; \mu)$$
$$\dot{y} = g\,(x, y; \mu) \qquad (1)$$

Where $\mu \in R$ is a real variable parameter, and both $f\,, g \in C^1(R^2)$**.** Assume that the system has an equilibrium point (x^*, y^*), satisfying $f(x^*, y^*; \mu) = g(x^*, y^*; \mu) = 0$ for **all** $\mu \in R$**.** Let $J(\mu)$ be its Jacobian at this equilibrium, and suppose that J has a pair of complex conjugate eigenvalues, $\lambda_{1,2}(\mu)$ with $\lambda_1 = \bar{\lambda}_2$. Assume that λ_1 moves from the left-half plane to the right **as** μ varies, so does λ_2, and they cross the imaginary axis at the moment $\mu = \mu^*$ in such a way that

$$\mathrm{Re}\{\lambda_1(\mu^*)\} = 0 \qquad and \qquad \frac{\partial \mathrm{Re}\{\lambda_1(\mu)\}}{\partial \mu}\Big|_{\mu=\mu^*} \succ 0 \qquad (2)$$

Where, Re {.} denotes the real part of a complex number. Here, the second condition in **(2)** refers to **as** the ***transversality*** condition for the crossing of the eigenlocus at the imaginary **axis,** which means that the eigenlocus is not tangent to the imaginary axis. Then, the system undergoes a ***Hopf bifurcation*** at the bifurcation point (x^*, y^*, μ^*) **[4,18].** More precisely, in any small left-neighborhood of μ^* **(i.e.,** $\mu \prec \mu^*$ ***)*** **,** (x^*, y^*) is a stable focus; and in any small right-neighborhood of μ^* **(i.e.,** $\mu \succ \mu^*$ ***)*** **,** this focus changes to be unstable, surrounded by a limit cycle of amplitude $O(\sqrt{|\mu - \mu^*|})$.

III. Controlling Hopf Bifurcation

The Hopf bifurcation control problem is the following: Design **a** (simple) controller,

$$u(t;\mu) = u(x, y; \mu) \qquad (3)$$

That can move the existing Hopf bifurcation point (x^*, y^*, μ^*) to a new position $(x^\circ, y^\circ, \mu^\circ)$, which does not change the original equilibrium point at (x^*, y^*) . It is clear that this controller must satisfy

$$u(x^*, y^*; \mu) = 0 \qquad (4)$$

We Can Add The Controller (3) To Any Of Two Equations Of System (1).

(a) For the sake of calculation let us add the controller (3) to the ***first*** equation, namely:

$$\dot{x} = f(x, y; \mu) + u(x, y; \mu)$$
$$\dot{y} = g(x, y; \mu) \qquad (5)$$

This controlled system has a Jacobian at (x°, y°) given by

$$J(\mu) = \begin{bmatrix} f_x + u_x & f_y + u_y \\ g_x & g_y \end{bmatrix} \qquad (6)$$

With eigenvalues

$$\lambda^c_{1,2}(\mu) = \frac{1}{2}\left[(f_x + u_x + g_y) \pm \sqrt{(f_x + u_x + g_y)^2 - 4[g_y(f_x + u_x) - g_x(f_y + u_y)]} \ \right] \qquad (7)$$

To have a Hopf bifurcation at $(x^\circ, y^\circ, \mu^\circ)$ **as** required, the classical Hopf bifurcation theory says that the following conditions must be hold:

(i) (x°, y°) is an equilibrium point of the controlled system (5) **,** namely,

$$f(x^\circ, y^\circ; \mu) + u(x^\circ, y^\circ; \mu) = 0$$
$$g(x^\circ, y^\circ; \mu) = 0 \qquad (8)$$

(ii) The eigenvalues $\lambda^c_{1,2}(\mu)$ of the controlled system (5) are purely imaginary at the point $(x^\circ, y^\circ, \mu^\circ)$ and are complex conjugate:

$$(f_x + u_x + g_y)\Big|_{\mu=\mu^\circ} = 0 \qquad (9)$$

$$g_y(f_x + u_x) - g_x(f_y + u_y)\Big|_{\mu=\mu^\circ} \succ 0 \qquad (10)$$

$$(f_x + u_x + g_y)^2 - 4[g_y(f_x + u_x) - g_x(f_y + u_y)]\Big|_{\mu\neq\mu^\circ} \succ 0 \qquad (11)$$

(iii) The crossing of the eigenlocus at the imaginary axis is not tangential (transversal), namely

$$\frac{\partial \mathrm{Re}\{\lambda_1^c(\mu)\}}{\partial \mu}\Big|_{\mu=\mu^*} = \frac{\partial (f_x + u_x + g_y)}{\partial \mu}\Big|_{\mu=\mu^*} \succ 0 \qquad (12)$$

(b) Now let us add the controller (3) to the ***second*** equation, namely:

$$\dot{x} = f(x, y; \mu)$$
$$\dot{y} = g(x, y; \mu) + u(x, y; \mu) \qquad (13)$$

This controlled system has a Jacobian at (x°, y°) given by

$$J(\mu)=\begin{bmatrix} f_x & f_y \\ g_x+u_x & g_y+u_y \end{bmatrix} \qquad (14)$$

With eigenvalues

$$\lambda_{1,2}^{c}(\mu)=\frac{1}{2}\left[(f_x+u_y+g_y)\pm\sqrt{(f_x+u_y+g_y)^2-4[f_x(g_y+u_y)-f_y(g_x+u_x)]}\ \right] \qquad (15)$$

To have a Hopf bifurcation at $(x^{\circ}, y^{\circ}, \mu^{\circ})$ **as** required, the classical Hopf bifurcation theory says that the following conditions must be hold:

(i) (x°, y°) is an equilibrium point of the controlled system (13), namely,

$$f(x^{\circ}, y^{\circ}; \mu)=0$$
$$g(x^{\circ}, y^{\circ}; \mu)+u(x^{\circ}, y^{\circ}; \mu)=0 \qquad (16)$$

(ii) The eigenvalues $\lambda_{1,2}^{c}(\mu)$ of the controlled system (13) are purely imaginary at the point $(x^{\circ}, y^{\circ}, \mu^{\circ})$ and are complex conjugate:

$$(f_x+u_y+g_y)\Big|_{\mu=\mu^{\circ}}=0 \qquad (17)$$

$$f_x(g_y+u_y)-f_y(g_x+u_x)\Big|_{\mu=\mu^{\circ}}\succ 0 \qquad (18)$$

$$(f_x+u_y+g_y)^2-4[f_x(g_y+u_y)-f_y(g_x+u_x)]\Big|_{\mu\neq\mu^{\circ}}\succ 0 \qquad (19)$$

(iii) The crossing of the eigenlocus at the imaginary axis is not tangential (transversal), namely

$$\frac{\partial\,\mathrm{Re}\{\lambda_1^{c}(\mu)\}}{\partial\mu}\Big|_{\mu=\mu^{*}} = \frac{\partial(f_x+u_y+g_y)}{\partial\mu}\Big|_{\mu=\mu^{*}}\succ 0 \qquad (20)$$

Now we present some examples to illustrate these results:

Example I: Van Der Pol Oscillator

$$\dot{x}=y$$
$$\dot{y}=\mu(1-x^2)y-x \qquad (21)$$

(a) (Nonlinear controller)

Clearly the system has a Hopf Bifurcation at the point $(x^{*}, y^{*}, \mu^{*})=(0,0,0)$. If the aim of control is to move the bifurcation value of μ from $\mu^{*}=0$ *to* $\mu^{\circ}\succ 0$ while preserving the equilibrium point of the system to be unaltered: $(x^{\circ}, y^{\circ})=(x^{*}, y^{*})=(0,0)$, then the condition (9) yields a nonlinear controller of the form

$$u=-\mu^{\circ}\,x\,(1-\frac{x^2}{3}) \qquad (22)$$

Which satisfied all conditions required by equations (4) and (8) – (12) stated in (i) – (iii) above. More precisely, since $(x^{\circ}, y^{\circ})=(0,0)$ *and* $u(x^{*}, y^{*}; \mu)=-\mu^{\circ}x^{*}(1-\frac{(x^{*})^2}{3})=0$, conditions (4) and (8) are satisfied.

For condition (10) to be satisfied we must show that $g_y(f_x+u_x)\Big|_{\mu=\mu^{\circ}}\succ g_x(f_y+u_y)\Big|_{\mu=\mu^{\circ}}$.

Now $g_y(f_x+u_x)\Big|_{\mu=\mu^{\circ}}=-\mu^2$ and $g_x(f_y+u_y)\Big|_{\mu=\mu^{\circ}}=-1$, so condition (10) says that $-\mu^2\succ -1$ or $\mu^2\prec 1$, so condition (10) must be satisfied merely for $-1\prec\mu\prec 1$, but as mentioned before we choose μ only for values $\mu\succ 0$. Thus the nonlinear controller (14) is valid only for $0\prec\mu\prec 1$.

Similarly condition (11) is satisfied.

And finally, since $\frac{\partial\,\mathrm{Re}\{\lambda_1^{c}(\mu)\}}{\partial\mu}\Big|_{\mu=\mu^{*}} = \frac{\partial(f_x+u_x+g_y)}{\partial\mu}\Big|_{\mu=\mu^{*}}=1\succ 0$ condition (12) is satisfied.

(b) (Linear controller)

Again the system has a Hopf Bifurcation at the point $(x^*, y^*, \mu^*) = (0,0,0)$. If the aim of control is to move the bifurcation value of μ from $\mu^* = 0$ *to* $\mu^\circ \succ 0$ while preserving the equilibrium point of the system to be unaltered: $(x^\circ, y^\circ) = (x^*, y^*) = (0,0)$, then the condition (17) yields a linear controller of the form

$$u = -\mu^\circ (1-(x^\circ)^2) y \qquad (23)$$

Which satisfied all conditions required by equations (4) and (16) – (20) stated in (i) – (iii) above. We can check these results by the same manner as we did in the nonlinear controller.

Note that this linear feedback controller may significantly extend the operational range of the key system parameter, μ , of the system **as** desired. Such an extension can be very important in some real applications: it may actually enhance the stability and/or performance robustness of the dynamical systems.

It is very important to mention that different bifurcation control tasks can be performed by means of different controllers for the Van Der Pol oscillator. For example if the goal of control is to shift the original equilibrium point of the system from (x^*, y^*) =(0,0) to , say $(x^\circ, y^\circ) == (x^\circ, 0)$, with $0 \prec x^\circ \prec 1$, while maintaining the bifurcation parameter value at $\mu^\circ = \mu^* = 0$, then another linear feedback controller can be designed. Actually the simple constant controller $u = x^\circ$ can perform the task. This fact can be easily checked by following the same steps discussed above.

Example II: the above control procedures can be applied to other dynamical systems, for which different controllers can be designed with satisfaction of conditions (8) – (12) or (16) – (20). The following two systems give typical examples [1]:
the first one is

$$\dot{x} = \mu x - y + xy^2$$

$$\dot{y} = x + \mu y + y^3 \qquad (24)$$

For which linear controller $u = -2(\mu^\circ + 2(y^\circ)^2) y$ changes the original Hopf bifurcation point (0, 0, 0) to (0.1, 0.1, 0.9) and the nonlinear controller $u = -2\mu^\circ - \frac{4y^3}{3}$ moves the original bifurcation point to (0.1, 0.1, 0.99) , respectively ; and the second system is

$$\dot{x} = \mu x - y + xy^2$$

$$\dot{y} = x + \mu y - x^2 \qquad (25)$$

For which linear controller $u = -2(\mu^\circ + 2(y^\circ)^2) y$ changes the original Hopf bifurcation point (0, 0, 0) to $(-\frac{1}{2}, \frac{\sqrt{6}}{4}, \frac{-(4\sqrt{6}+3)}{8})$ and the nonlinear controller $u = x^2 - 2\mu^\circ - \frac{y^3}{3}$ moves the original bifurcation point to (0.1, 0.1, 0.99), respectively.

IV. Conclusion

In this paper, a simple and unified state-feedback control strategy **has** been developed for Hopf bifurcations, for continuous-time systems and for both problems of shifting and creating a Hopf bifurcation point in the controlled system. Other types of bifurcations, for instance, pitchfork, saddle node, transcritical, and period-doubling bifurcations can **also** be controlled by similar linear or simple nonlinear controllers, which are investigated in other topics and publications.

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