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The Approximate Solution for Solving Linear Fredholm Weakly Singular Integro-Differential Equations by Using Chebyshev Polynomials of the First kind

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Abstract: *In this paper, we use Chebyshev polynomials method of the first kind of degree n to solve linear Fredholm weakly singular integro- differential equations (LFWSIDE$_s$) of the second kind.This techniques transform the linear Fredholm weakly singular integro-differential equations to a system of a linear algebraic equations .Application are presented to illustrate the efficiency and accuracy of this method.*
Keywords: *Linear Fredholm integro-differential equations, Weakly singular kernel, Chebyshev polynomials, Trapezodial rule.*

I. Introduction

The weakly singular integral equations have many applications in mathematical physics. These equations arise in the heat conduction problem posed by mixed boundary conditions,the Dirichlet problem, and radiative equilibrium[1].It is difficult to solve these equations analytically and analytical solutions. Hence, numerical schemes are required for dealing with these equations in a proper manner. Therefore many researchers used several numerical methods to solve weakly singular integral equations (WSIEs) including the sinc-collocation method [2], Spectral methods[3]. A collocation method with cubic splines[4].As well as several numerical methods were used to solve weakly singular integro- differential equations (WSIDEs) including piecewise polynomial collocation methods [5], A spline collocation method [6], Legender multiwavelets method [7].In this paper Chebyshev polynomials of the first kind of degree n are defined in section two. In the section three the proposed method for solving linear Fredholm weakly singular integro-differential equations is used. Application is given in section four for confirming the efficiency of the proposed method .Section five contains conclusions of the paper.

II. Chebyshev Polynomials Of The First Kind $T_n(x)$, [9]

The Chebyshev polynomials of the first kind of degree n is asset of orthogonal polynomials and it is defined by the recurrence relation

$T_0(x) = 1$

$T_1(x) = x$

$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, for each $n \geq 1$. (1)

2.1 Properties of Chebyshev polynomials $T_n(x)$

1. The Chebyshev polynomials of the first kind $T_n(x), n = 0,1, \ldots$ are aset of orthogonal polynomials over the interval [-1,1] with respect to the weight function $W(x) = (1 - x^2)^{-1/2}$, that is :

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \neq 0 \\ \pi & n = m = 0 \end{cases} \quad (2)$$

The Chebyshev polynomials of the first kind can be defined by the trigonometric identity $T_n(\cos(\theta)) = \cos(n\theta)$ for n = 0,1,2,3,…

2. $T_n(x)$ has n distinct real roots x_i on the interval [-1,1], these roots are defined by :

$x_i = \cos\left(\frac{(2i+1)\pi}{2N}\right), i = 0,1,2, \ldots, N-1$ (3)

are called Chebyshev nodes.$T_n(x)$ assumes its absolute extrema at

$x_j = \cos\left(\frac{j\pi}{N}\right)$for $j = 0,1,2, \ldots, N$ (4)

3. A polynomial of degree N in Chebyshev form is a polynomial

$p(x) = \sum_{n=0}^{N} a_n T_n(x)$ (5)

Where T_n is the n^{th}Chebyshev form

The first few chebyshev polynomials of the first kind for N=0,1,2,3,4,5 are given in figure(1)

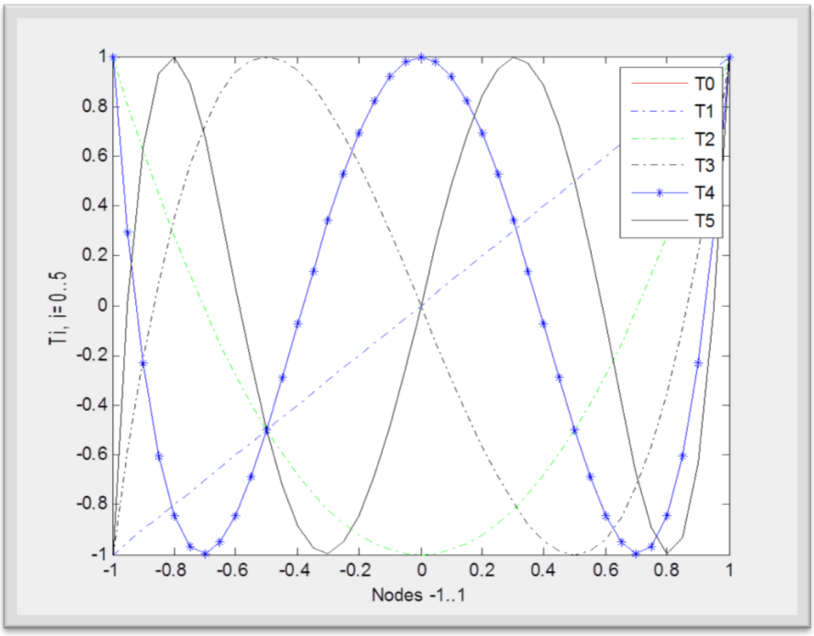


Figure (1): The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

2.2 Shifted Chebyshev Polynomials

Shifted Chebyshev polynomials are also of interest when the range of the independent variable is [0,1] instead of [-1,1]. The shifted Chebyshev polynomials of the first kind are defined as

$T_n^*(x) = T_n(2x-1), \ \ 0 \le x \le 1$ (6)

Similarly, one can also build shifted polynomials for a generic interval $[a,b]$ where $\bar{x}_i = \frac{b-a}{2}x_i + \frac{b+a}{2}$,(7)

The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5 for interval [0,1] are given in figure(2).

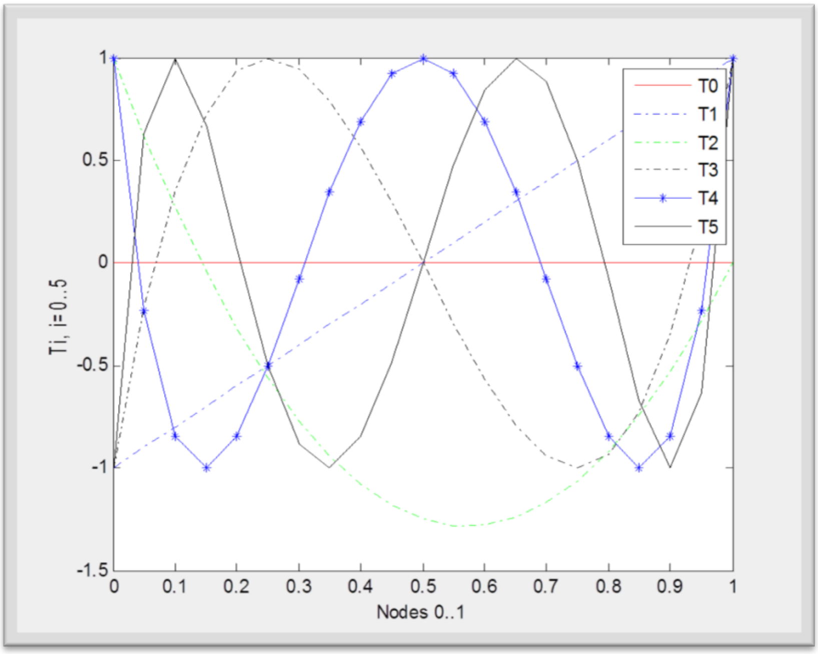


Figure (2):The first few shifted Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

III. The Approximate Solution Of Linear Fredholm Weakly Singular Integro-Differential Equations:

We consider the LFWSIDE$_s$of the second kind given by:

$\mu_1 y'(x) + \mu_2 y(x) + \int_a^b k(x,t)y(t)dt = f(x) \quad x \in [a,b]$ (8)

withinitialcondition $y(a) = \beta$

where $\beta, \mu_1$and μ_2 are a constantsand $k$and $f$are given functions and y isthe solution to bedetermined.Moreoverassume that the kernel$k(x,t) = \frac{H(x,t)}{|x-t|^\alpha}$ $\forall\, x,t \in [a,b]$ with$x \neq t$ where$0 < \alpha < 1$.As well as we assume that the kernel$H$ is in $L^2[a,b]^2$and the unknown y and theright hand side f are in$L^2[a,b]$.Also we suppose that $H(x,t)$ satisfies in the Lipschitz condition,

$|H(x_1,t) - H(x_2,t)| \leq L_x|x_1 - x_2|$(9)

todetermine anapproximate solution of(8). Firstly if the function $y(x)$defined in[-1,1].Suppose this functionmay be represented by first kind Chebyshev polynomials,[9]:

$y(\bar{x}) \cong \sum_{i=0}^{\infty} T_i(\bar{x})b_i$(10)

If the series (10) is truncated,then its written as follows:

$y(\bar{x}) \cong \sum_{i=0}^{N} T_i(\bar{x})b_i \cong T(\bar{x})B$(11)

$y'(\bar{x}) \cong (\sum_{i=0}^{N} T_i(\bar{x})b_i)' \cong \ (T(\bar{x})B)'$(12)

where $T(\bar{x}) = [T_0(\bar{x}), T_1(\bar{x}), T_2(\bar{x}), \ldots, T_N(\bar{x})]$,

$B = [b_0, b_1, b_2, \ldots, b_N]^T$clearly $T$is $1 \times (N+1)$ vectors and B is$(N+1) \times 1$ vectors. then theaimistofind Chebyshevcoefficients,that isthematrix B. Wefirst substitute the Chebyshev nodes, which aredefined by

$\bar{x}_i = \cos\left(\frac{(2i+1)\pi}{2N}\right), i = 0,1,2, \ldots, N-1$ into (11) and (12)and then rearrange a new matrix form to determine B:

$\mu_1 y' + \mu_2 y + k = f$ (13)

In which $k$is the integral part of (8) and

$$y' = \begin{pmatrix} y'(\bar{x}_0) \\ y'(\bar{x}_1) \\ . \\ . \\ . \\ y'(\bar{x}_N) \end{pmatrix}, y = \begin{pmatrix} y(\bar{x}_0) \\ y(\bar{x}_1) \\ . \\ . \\ . \\ y(\bar{x}_N) \end{pmatrix}, k = \begin{pmatrix} k(\bar{x}_0) \\ k(\bar{x}_1) \\ . \\ . \\ . \\ k(\bar{x}_N) \end{pmatrix}, f = \begin{pmatrix} f(\bar{x}_0) \\ f(\bar{x}_1) \\ . \\ . \\ . \\ f(\bar{x}_N) \end{pmatrix} \quad (14)$$

by substituting (11) and(12) into(13) gives (N+1) linear algebraicequations in (N+1)unknown coefficients .Theseequationsare solvedby using(Matlab R2010b) to obtain the unknowncoefficients B which arethen substituteinto (11) toget the approximatesolution of(8).Ifthefunction $y(x)$definedin [0,1] .The transformation $\tilde{x} = \frac{1}{2}[(b-a)\bar{x} + (a+b)]$is used to transformthenodes$\bar{x}_i$ in $[-1,1]$ into the correspondingnodes $\tilde{x}_i$in $[0,1]$.

Two algorithm for solving (LFWSIDE$_s$) on interval [-1,1] and [0,1] are given as follows:

3.1 The Algorithm for Solving (LFWSIDE$_s$) into [-1,1]:

Input**:** $a, b, \alpha, N, M, y(x), f(x), \mu_1, \mu_2, \epsilon$.

Output:The approximate solution of the LFWSIDE$_s$.

Step 1: process: Find $T_i(\bar{x}), (T_i(\bar{x}))'$. (Chebyshev polynomials)

Step 2: Find roots $\bar{x}_i, i = 0,1, \ldots, N$ (roots of Chebyshev polynomials)

Step 3: Find roots

$$t_{ij}{}^* = a + j + k \pm \epsilon, k = \frac{b-a}{M}, i = 0, \ldots, N, j = 0, \ldots, M, \bar{x} = t.$$

Step 4: Calculate $R_i = \frac{k}{2}[Y_{i0} + 2\sum_{k=1}^{M-1} Y_{ik} + Y_{iM}]$ (Trapezoidal rule)

where $Y_{ij} = \frac{T_{ij}(t)}{|\bar{x}_i - t_{ij}|^\alpha}, i = 0,1, \ldots, N, j = 0,1, \ldots, M$

Step 5: Construct the system:

$$A_{ik} = b_k\left(\mu_1 * \left(T_k(\bar{x}_i)\right)' + \mu_2 * T_k(\bar{x}_i)\right) + b_k * R_i, i, k = 0, \ldots, N,$$

(where b_k are unknown values) , $B_i = f(\bar{x}_i), i = 0, \ldots, N$.

Step 6: Solve the linear system $A_{ik} = B_i$ by using $X_i = inv(A_{ik}) * (B_i)^t$ and find the unknowns b_k.

Step 7: Calculate the approximate function $y_N(\bar{x}) = \sum_{i=0}^{N} T_i(\bar{x})b_i$.

Step 8: Calculate absolute error is the comparison between the exact and the approximate solutions.

Step 9: END of the process.

3.2TheAlgorithm for Solving($LFWSIDE_s$)into[0,1]:
Input: $a, b, \alpha, N, M, y(x), f(x), \mu_1, \mu_2, \epsilon$.
Output: The approximate solution of the $LFWSIDE_s$.
Step1:process : Find $T_i(\bar{x})$. (Chebyshev polynomials)
Step 2 :Find $T_i^*(\tilde{x}), (T_i^*(\tilde{x}))'$. (shifted Chebyshev polynomials)
Step 3 : Find roots $\bar{x}_i, i = 0, \dots, N$.(roots of Chebyshev polynomials)
Step 4: Find roots $\tilde{x}_i$ by using the transformation
$\tilde{x}_i = \frac{1}{2}[(b-a)\bar{x}_i + (a+b)], i = 0, \dots, N$ (roots of shifted Chebyshev polynomials)
Step 5: Find roots

$$tij^* = a + j + k \pm \epsilon, \qquad k = \frac{b-a}{M}, \qquad i = 0, \dots, N, j = 0, \dots, M, \tilde{x} = t.$$

Step 6:Calculate $R_i = \frac{k}{2}[Y_{i0} + 2\sum_{k=1}^{M-1} Y_{ik} + Y_{iM}]$ (Trapezoidal rule)
where $Y_{ij} = \frac{\tilde{t}ij^*(t)}{|\tilde{x}_i - \tilde{t}ij^*|^{\alpha}}$, $i = 0, \dots, N, j, 0, \dots, M$.
Step 7: Construct the system :

$$A_{ik} = b_k(\mu_1 * (T_k^*(\tilde{x}_i))' + \mu_2 * T_k^*(\tilde{x}_i)) + b_k * R_i \text{ , } i, k = 0, \dots, N,$$

(where b_k are unknown values), $B_i = f(\tilde{x}_i), i = 0, \dots, N$.
Step 8: Solve the linear system $A_{ik} = B_i$ by using $X_i = inv(A_{ik}) * (B_i)^t$
and find the unknowns b_k.
Step 9: Calculate the approximate function $y_N(\tilde{x}) = \sum_{i=0}^{N} T_i^*(\tilde{x}) b_i$.
Step 10 : Calculate absolute error is the comparison between the exact and the approximate solutions.
Step 11: END of the process.

IV. Application

Inthis section,numerical example is given to explain the applicability and effectiveness of the proposed method.TheComputationshavebeen performedby usingMatlab R2010b.Consider the following $LFWSIDE_s$,[7]:

$$y'(\tilde{x}) + \int_0^1 |\tilde{x} - \tilde{t}|^{-1/2} y(\tilde{t}) dt = f(\tilde{x}) \qquad (15)$$

with initial condition $y(0) = 1$
here the forcing function f is selected such that the exact solution is

$$y(\tilde{x}) = \tilde{x}^2 + \frac{1}{\tilde{x} + 1}$$

Tables(1),(2) and (3) illustratethe comparisonbetweentheexactandthe approximate solutiondepending on Mean Square Error(MSE)and Elapsed Time(ET).

Table(1): Results obtained and errors for example(1): N= 10

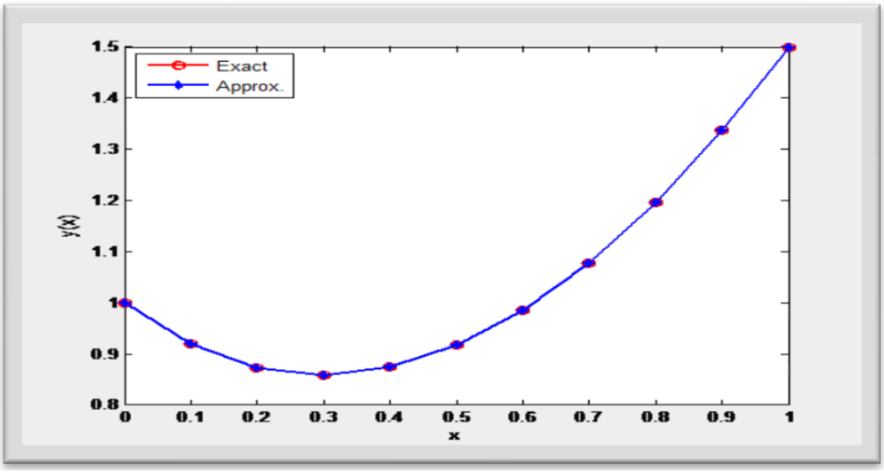
$\tilde{x}$-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0051	1.000000	1.000001	0.000001
0.0452	0.919091	0.919092	0.000001
0.1221	0.873333	0.873334	0.000001
0.2297	0.859231	0.859231	0.000001
0.3591	0.874286	0.874286	0.000001
0.5	0.916667	0.916667	0.000000
0.6409	0.985000	0.985000	0.000000
0.7703	1.078235	1.078235	0.000000
0.8779	1.195556	1.195556	0.000000
0.9548	1.336316	1.336316	0.000000
0.9949	1.500000	1.500000	0.000000
MSE	**2.5217e - 013**	**ET**	**63.371781Sec.**

Table(2): Results obtained and errors for example(1): N=12

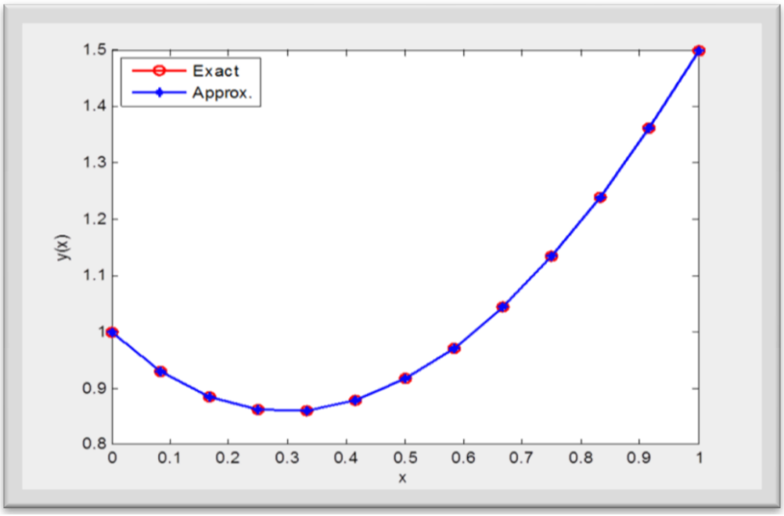
$\tilde{x}$-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0037	1.000000	1.000000	0.000000
0.0325	0.930021	0.930021	0.000000
0.0885	0.884921	0.884921	0.000000
0.1684	0.862500	0.862500	0.000000
0.2676	0.861111	0.861111	0.000000
0.3803	0.879493	0.879493	0.000000
0.5	0.916667	0.916667	0.000000
0.6197	0.971857	0.971857	0.000000
0.7324	1.044444	1.044444	0.000000
0.8316	1.133929	1.133929	0.000000
0.9115	1.239899	1.239899	0.000000
0.9675	1.362017	1.362017	0.000000
0.9964	1.500000	1.500000	0.000000
MSE	**3.6854e-016**	**ET**	**216.990689Sec.**

Table(3): Results obtained and errors for example(1): N=14

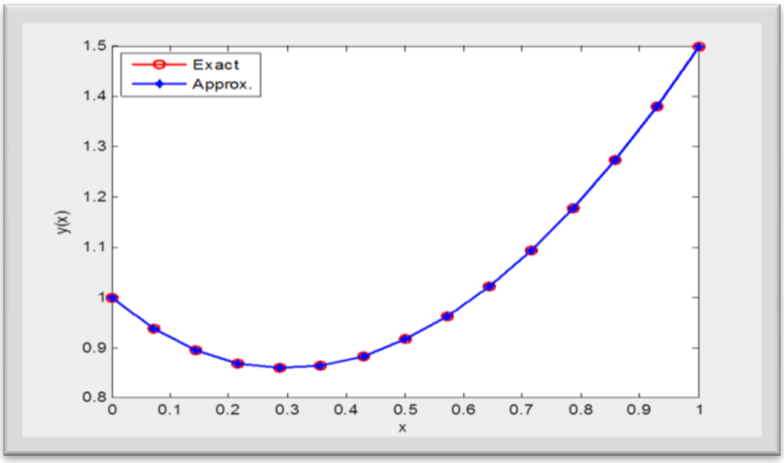
x̃-values	**Exact solution**	**Approximate solution**	**Absolute error**
0.0027	1.000000	1.000000	0.000000
0.0245	0.938435	0.938435	0.000000
0.067	0.895408	0.895408	0.000000
0.1284	0.869448	0.869448	0.000000
0.2061	0.859410	0.859410	0.000000
0.2966	0.864393	0.864393	0.000000
0.3960	0.883673	0.883673	0.000000
0.5	0.916667	0.916667	0.000000
0.604	0.962894	0.962894	0.000000
0.7034	1.021961	1.021961	0.000000
0.7939	1.093537	1.093537	0.000000
0.8716	1.177347	1.177347	0.000000
0.933	1.273155	1.273155	0.000000
0.9755	1.380763	1.380763	0.000000
0.9973	1.500000	1.500000	0.000000
MSE	**3.4425e-018**	**ET**	**899.911312Sec**.



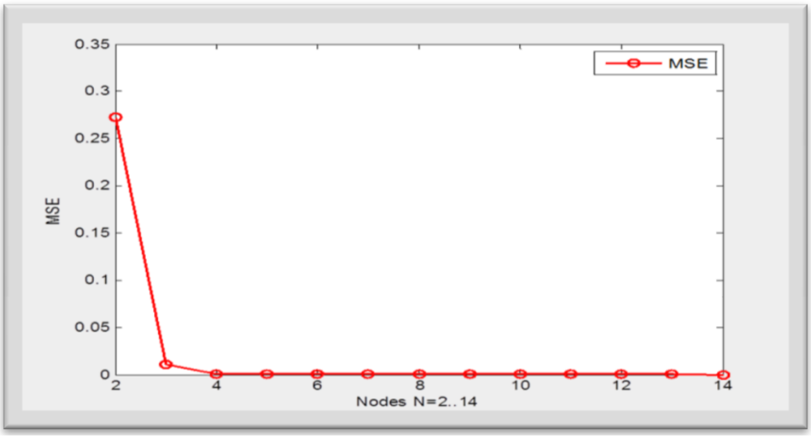
Figure(3): A Comparison between the exact and the approximate solution usingexpansion method of Chebyshev polynomials ofthe first kind of application for N=10.



Figure(4): A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomialsof the first kind ofapplication for N=12.



Figure(5): A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application for N=14.



Figure(6): A Comparison between the ShiftedChebyshev nodes and the MSE of application when N=2….14.

V. Conclusions

In this paper,we have submittingexpansion method usingChebyshev polynomials of thefirst kind of degree n as basis function for approximating the solution of oneweakly singular integro-differential equation : whicharethe LFWSIDE's. In the application we have reduced the solution of LFWSIDE's to the system oflinear equationsbyremoving the singularity using an approximate point t, and we have the following results:

- whenthe degreeofexpansion method of Chebyshev polynomials ofthefirst kind is increases the error is decreases.Which is shown in Tables(1),(2), and(3).
- The proposed method is a precise and activeto solve LFWSIDE's.
- This method can be extended and applied to the system of LFWSIDE's.

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