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Bezier Curve Method for Solving Linear and Nonlinear Fredholm-Volterra Integral Equations

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Abstract: *This work explores the solution of Fredholm-Volterra integral equations using the Bezier curve method. Both linear and nonlinear equations of the Fredholm-Volterra integral equations are considered. The convergence of the Bezier curve method for integral equations is analysed. The high accuracy of the results of the approximation shows that the approximate solutions obtained by the Bezier curve method are very good and efficient approximations of the exact solution of these equations.*
Keywords: *Bezier Curves, Control Points, Fredholm-Volterra Integral Equations, Residual Function, Function Approximation.*

I. Introduction

Frequently, the mathematical modelling of problems arising from real world deals with Fredholm-Volterra integral and integro-differential equations [1]. These are usually difficult to solve analytically and in many cases, the solution must be approximated. Several approximation methods have been proposed including [2] who used the asymptotic methods on the second kind of Fredholm-Volterra integral equations and [3] applied the finite difference approach to obtain solution of Fredholm-Volterra integral equations. Also, the discrete Adomian decomposition method (DADM) which arises when the quadrature rules are used to approximate the integrals which cannot be computed analytically was used by [4] to solve two dimensional Fredholm-Volterra integral equations. These numerical methods usually transform the Fredholm-Volterra integral equations into a system of equations that can be solved by direct or iterative methods.

[5] used the Bezier control points approximating data and functions. [6] proposed the use of the control point of the Bernstein-Bezier form for solving differential equations numerically and [7] used this approach for solving singular-perturbed two-point boundary value problems. The Bezier curves are used in solving partial differential equations including the wave and heat equations [8,9]. [10] used triangular Bezier patches of degree n with C^k continuity to approximate the exact solution of partial differential equations. Bezier curves are also used for solving dynamical systems [11].

Some other applications of Bezier functions and control points are found in the works of [12] and [13] where they are used in computer-aided geometric design and image compression. [14] successfully used the Bezier control point method to solve delay differential equations. Bezier control points method is also used to solve constrained quadratic optimal control of time varying linear system systems [12].

Our focus in this work however, is to use the Bezier curve technique to solve the Fredholm-Volterra integral equations which is a novel approach.

II. Bezier Curve Method

We will consider a Fredhom-Volterra integral equation of the form:

$$y(t) = x(t) + \lambda_1 \int_{t_0}^{t_f} k_1(t,s,y(s))ds + \lambda_2 \int_{t_0}^{t} k_2(t,s,y(s))ds, \quad t \in [t_0, t_f] \tag{1}$$

with $y(t_0) = y_0 = a$.

We choose a degree of n and symbolically express the solution $y(t)$ in the degree n $(n \geq m)$ Bezier form

$$y(t) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{t-t_0}{h}\right) \tag{2}$$

where $h = t_f - t_0$ and

$$B_{r,n}\left(\frac{t-t_0}{h}\right) = \binom{n}{r} \frac{1}{h^n} (t_f - t)^{n-r} (t-t_0)^r$$

and the control points $a_0, a_1, ..., a_n$ are to be determined.
Substitute the approximate solution $y = y(t)$ into eq. (1) to obtain the residual function

$$R(t) = y(t) - \left(x(t) + \lambda_1 \int_{t_0}^{t_f} k_1(t, s, y(s))ds + \lambda_2 \int_{t_0}^{t_f} k_2(t, s, y(s))ds \right)$$

To find the approximate solution $y(t)$ using the Bezier curves, we choose the sum of squares or the Euclidean norm of the Bezier control points of the residual to be the measure quantity. This quantity is minimized to get the approximate solution. This is the Bezier curve method. The computations are done using the well known symbolic software Maple 13.

III. Convergence Analysis

We now extend the concept used by [14] to analyse the convergence of the Bezier curve method for Fredholm-Volterra integral equations. The following Lemma 1 and theorem 2 will help us to prove the convergence of the approximate solutions.

Lemma 1

For a polynomial in Bezier form

$$y(t) = \sum_{r=0}^{n_1} a_{r,n_1} B_{r,n_1}(t) \qquad (3)$$

we have

$$\frac{\sum_{r=0}^{n_1} a_{r,n_1}^2}{n_1+1} \geq \frac{\sum_{r=0}^{n_1+1} a_{r,n_1+1}^2}{n_1+2} \geq ... \geq \frac{\sum_{r=0}^{n_1+m_1} a_{r,n_1+m_1}^2}{n_1+m_1+1} \to \int_0^1 y^2(t)dt, \qquad m_1 \to +\infty$$

where a_{r,n_1+m_1} is the Bezier coefficient of $y(t)$ after been degree elevated to degree $n_1 + m_1$ [6,15]

Theorem 2

If f is a continuous complex function on $[a,b]$, there exist a sequence of polynomial p_n such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$. If f is real, then P_n may be taken real [16].

The following problem is considered based on Fredholm-Volterra integral equations.

$$L_1(t, y(t)) = y(t) - \int_0^1 k_1(t, s, y(s))ds - \int_0^t k_2(t, s, y(s))ds = x(t), t \in [0,1] \qquad (4)$$

$$y(0) = y_0 = a$$

where a is a given real number and $k_{1,2}(t,s) \in L^2$ and $x(t) \in L^2$ are known functions for $t \in [t_0, t_f]$, in particular $[0,1]$. Covergence of the approximate solution is done in degree raising of the Bezier polynomial approximation.

Theorem 3

If the integral equation (4) has a unique C^1 continous solution $\bar{y}$, then the approximate solution y obtained by the control-point-based method converges to the exact solution $\bar{y}$ as the degree of the approximate solution tends to infinity.

Proof

For an abitrary small positive number $\varepsilon > 0$, by the Weierstrass theorem 2 one can easily find polynomial $Q_{1,N_1}(t)$ of degree N such that

$$\| Q_{1,N_1}(t) - \bar{y}(t) \|_\infty \leq \frac{\varepsilon}{16}$$

Where ||.|| stands for the L_∞ -norm over $[0,1]$. In particular, we have

$$\| a-Q_{1,N_1}(0)\|_\infty \le \frac{\varepsilon}{16} \tag{5}$$

Generally, $Q_{1,N_1}(t)$ does not satisfy the boundary conditions. With a small pertubation with a constant polynomial α, for $P_{1,N_1}(t)$, we can get the polynomial $P_{1,N_1}(t)=Q_{1,N_1}(t)+\alpha$ such that $P_{1,N_1}(t)$ satisfies the boundary condition $P_{1,N_1}(0)=a$. Thus $F_{1,N_1}(0)+\alpha=a \;\; \Rightarrow Q_{1,N_1}(0)=a-\alpha$. From equation (5),

$$\| a-Q_{1,N_1}(0)\|_\infty = \| a-(a-\alpha)\|_\infty = \| a-a+\alpha \|_\infty$$

$$\Rightarrow \| a-Q_{1,N_1}(0)\| = \| \alpha \|_\infty \le \frac{\varepsilon}{16} \tag{6}$$

Thus

$$\begin{aligned} &\| P_{1,N_1} - \bar{y}(t)\|_\infty = \| Q_{1,N_1}(t)+\alpha-\bar{y}(t)\|_\infty \\ &= \| Q_{1,N_1}(t)-\bar{y}(t)+\alpha \|_\infty \\ &\le \| Q_{1,N_1}(t)-\bar{y}(t)\|_\infty + \|\alpha\|_\infty \\ &\le \frac{\varepsilon}{16}+\frac{\varepsilon}{16} \\ &= \frac{\varepsilon}{8} < \frac{\varepsilon}{6}. \end{aligned} \tag{7}$$

Let

$$\begin{aligned} &LP_N(t) = L(t, P_{1,N_1}(t)) \\ &= P_{1,N_1}(t) - \int_0^1 k_1(t,s,y(s))ds - \int_0^t k_2(t,s,y(s))ds = x(t) \end{aligned}$$

for every $t \in [0,1]$. For $N \ge N_1$, the upper bound of the residual may be found

$$\begin{aligned} &\| LP_N(t)-y(t)\|_\infty = \| L(t,P_{1,N_1}(t))-y(t)\|_\infty \\ &\le \| P_{1,N_1}(t)-\bar{y}(t)\|_\infty + \int_0^1 \| k_1(t,s,P_{1,N_1}(s))ds \|_\infty + \int_0^t \| k_2(t,s,P_{1,N_1}(s))ds\|_\infty \\ &\le c_1\left(\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}\right) = c_1\frac{\varepsilon}{2} \\ &\le c_1\varepsilon \end{aligned} \tag{8}$$

where $c_1 = 1+\| k_1(t,s)\|_\infty + \| k_2(t,s)\|_\infty$ is a constant.

The resisdual $R(P_N) = LP_N(t) - y(t)$ is considered as a polynomial; if not so, we can make use of the Taylor series to express it. Representing the residual $R(P_N)$ in Bezier form, we have

$$R(P_N) = \sum_{r=0}^{m_1} d_{r,m_1} B_{r,m_1}(t) \tag{9}$$

By Lemma 1, there exist an integer $M \ge N$ such that

$$\left| \frac{1}{m_1+1}\sum_{r=0}^{m_1} d_{r,m_1}^2 - \int_0^1 (R(P_N))^2 dt \right| < \varepsilon \tag{10}$$

$$\frac{1}{m_1+1}\sum_{r=0}^{m_1} d_{r,m_1}^2 < \varepsilon + \int_0^1 (R(P_N))^2 dt \le \varepsilon + (c_1\varepsilon) \tag{11}$$

If $y(t)$ is an approximate solution of (4) gotten from the Bezier curve method of degree $m_2 (m_2 \ge m_1 \ge M)$.

Let

$$R(t, y(t)) = L(t, y(t)) - y(t)$$

$$= \sum_{r=0}^{m_2} c_{r,m_2} B_{r,m_2}(t), \quad m_2 \geq m_1 \geq M \tag{12}$$

The norm for the difference-approximated solution $y(t)$ and the exact solution $\bar{y}(t)$ is

$$\| y(t) - \bar{y}(t) \| = \int_0^1 \left| y(t) - \bar{y}(t) \right| dt \tag{13}$$

It can be shown that

$$\| y(t) - \bar{y}(t) \| \leq c\left(\left| y(0) - \bar{y}(0) \right| + \| R(t, y(t)) - R(t, \bar{y}(t)) \|_2^2 \right)$$

$$= c\int_0^1 \sum_{r=0}^{m_2} \left(c_{r,m_2} B_{r,m_2}(t) \right) dt$$

$$\leq \frac{c}{m_2 + 1} \sum_{r=0}^{m_2} c_{r,m_2}^2$$

This inequality is arrived at using Lemma 1 where c is a constant positive number and equation (9). Thus

$$\| y(t) - \bar{y}(t) \| \leq \frac{c}{m_2 + 1} \sum_{r=0}^{m_2} c_{r,m_2}^2$$

$$\leq \frac{c}{m_2 + 1} \sum_{r=0}^{m_2} d_{r,m_2}^2 \tag{14}$$

$$\leq \frac{c}{m_1 + 1} \sum_{r=0}^{m_2} d_{r,m_1}^2$$

$$\leq c(\varepsilon + c_1^2 \varepsilon^2) = \varepsilon, \qquad m_1 \geq M \tag{15}$$

This inequality is arrived at from (11). Thus

$$\| y(t) - \bar{y}(t) \| \leq \varepsilon_1$$

The infinite norm and the norm in (13) are equivalent, hence the result.

IV. Results

We now test the accuracy of the Bezier curve method on some examples of FVIEs.

Example 1:

Consider the following linear Fredholm-Volterra integral equation [17]

$$y(t) = 1 + \int_0^{\pi} y(s) ds - \int_0^t (t - s) y(s) ds$$

with the initial condition $y(0) = 1$ and the exact solution $y(t) = cos(t)$.

We define the residual function

$$R(t) = y(t) - 1 + \int_0^t (t - s) y(s) ds - \int_0^{\pi} y(s) ds$$

We choose n of degree 8 to express $y(t)$ in (2). Upon expansion we get a system of equations in b_i's

$$\begin{aligned} b_0 = {} & -1 + a_0 - 7\pi^8 a_1 + 7\pi^8 a_6 + 21\pi^8 a_2 + 24\pi^7 a_5 + 80\pi^7 a_3 - a_0 \pi \\ & - 4\pi^7 a_6 + \frac{280}{3}\pi^6 a_2 + \frac{140}{3}\pi^6 a_4 + \frac{28}{3}\pi^6 a_0 - \frac{140}{3}\pi^6 a_1 - \frac{28}{3}\pi^6 a_5 \\ & - 60\pi^7 a_2 - 4\pi^7 a_0 - 60\pi^7 a_4 - 14\pi^4 a_3 - \frac{280}{3}\pi^6 a_3 - 84\pi^5 a_2 \\ & + 56\pi^5 a_1 - 14\pi^5 a_0 - 14\pi^5 a_4 + 56\pi^5 a_3 + 14\pi^4 a_0 - 42\pi^4 a_1 \end{aligned}$$

$$-\frac{70}{9}\pi^9 a_4+\frac{8}{9}\pi^9 a_1+\frac{56}{9}\pi^9 a_5+42\pi^4 a_2-\frac{28}{3}\pi^3 a_0-19\pi^9 a_0$$
$$+35\pi^8 a_4-\frac{28}{9}\pi^9 a_2-\frac{28}{9}\pi^9 a_6+\frac{56}{9}\pi^9 a_3+\frac{8}{9}\pi^9 a_7-\frac{1}{9}\pi^9 a_8$$
$$-\pi^8 a_7+\pi^8 a_0-21\pi^8 a_5-35\pi^8 a_3-\frac{28}{3}\pi^3 a_2+\frac{56}{3}\pi^3 a_1$$
$$+4\pi^2 a_0-4\pi^2 a_1+24\pi^7 a_1,$$

$$b_1=-1+15a_0+45a_1+35\pi^8 a_4-a_0\pi-19\pi^9 a_0-\frac{70}{9}\pi^9 a_4+\frac{8}{9}\pi^9 a_1$$
$$+\frac{56}{9}\pi^9 a_5-\frac{28}{9}\pi^9 a_2-\frac{28}{9}\pi^9 a_6+\frac{56}{9}\pi^9 a_3+\frac{8}{9}\pi^9 a_7-19\pi^9 a_8$$
$$-\pi^8 a_7+\pi^8 a_0-21\pi^8 a_5-35\pi^8 a_3-7\pi^8 a_1+7\pi^8 a_6+21\pi^8 a_2$$
$$-60\pi^7 a_2-4\pi^7 a_0-60\pi^7 a_4+24\pi^7 a_5+80\pi^7 a_3+24\pi^7 a_1$$
$$-4\pi^7 a_6+\frac{280}{3}\pi^6 a_2+\frac{140}{3}\pi^6 a_4+\frac{28}{3}\pi^6 a_0-14\pi^5 a_4-\frac{140}{3}\pi^6 a_1$$
$$-\frac{28}{3}\pi^6 a_5-\frac{280}{3}\pi^6 a_3-84\pi^5 a_2+56\pi^5 a_1-14\pi^5 a_0-\frac{28}{3}\pi^3 a_0$$
$$+56\pi^5 a_3+14\pi^4 a_0-42\pi^4 a_1-14\pi^4 a_3+42\pi^4 a_2-\frac{28}{3}\pi^3 a_2$$
$$+\frac{56}{3}\pi^3 a_1+4\pi^2 a_0-4\pi^2 a_1$$

…

The solution of the b_i's using the least squares optimization tool in Maple yields the Bezier control points
a_0=1.0, a_1=1.00000000196967864, a_2=0.982142851464157029, a_3= 0.946428589303568436, a_4=0.893452352735771283, a_5= 0.824404797923046440, a_6=0.741021787976740232, a_7=0.645486183649343626, a_8=0.540302305868139987
This yields upon substitution into (2) the approximate solution

$$u(t)=1.0+0.00000001600000000t-0.50000027t^2+0.0000022t^3+0.04165669t^4+0.0000237t^5$$
$$-0.0014205t^6+0.0000226t^7+0.000017731t^8$$

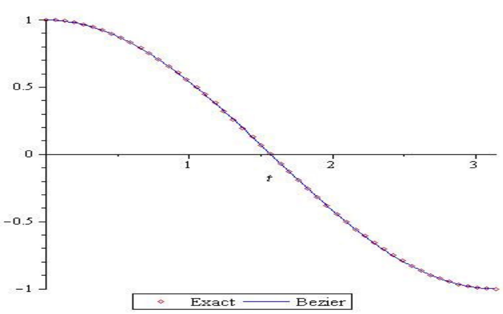

Figure 1: Exact and approximate solutions of Example 1

Example 2:
Consider the following linear Fredholm-Volterra integral equation [17]

$$y(t)=2-t-t^2-6t^3+t^5+\int_0^t sy(s)ds+\int_{-1}^1 (t+s)y(s)ds$$

with the initial condition $y(0) = 2$ and the exact solution $y(t) = 2+3t-5t^3$.
The residual function is

$$R(t) = y(t) - 2 + t + t^2 + 6t^3 - t^5 - \int_0^t sy(s)ds - \int_{-1}^1 (t+s)y(s)ds$$

The control points are obtained
a_0= 2.0, a_1= 2.37499999999886136, a_2=2.74999999999822142, a_3= 3.03571428571221613,
a_4=3.14285714285494811, a_5= 2.98214285714111238, a_6= 2.46428571428545996,
a_7= 1.49999999999958988, a_8= 0.0

Substituting the control points into (2), we get the approximate solution

$$u(t) = 2.0 + 3.0t - 5.0t^3 - 0.0000001000000000t^4 - 0.00000001000000000t^8.$$

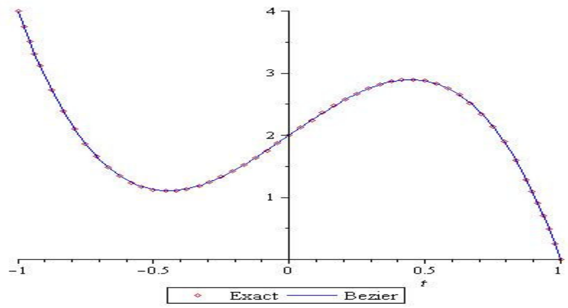

Figure 2: Exact and approximate solutions of Example 2

Example 3:
Consider the following nonlinear Fredholm-Volterra integral equation [17]

$$y(t) = 2t - 112t^4 - 53 + 14\int_0^t (t-s)(y(s))^2 ds + \int_0^1 (1+s)y(s)ds$$

with the intial condition $y(0) = 0$ and the exact solution $y(t) = 2t$.
The residual function

$$R(t) = y(t) - 2t + 112t^4 + 53 - 14\int_0^t (t-s)(y(s))^2 ds - \int_0^1 (1+s)y(s)ds$$

is used to find the following control points
a_0= 0.0, a_1= 0.250000000012537692, a_2=0.499999999998005372, a_3= 0.750000000012363554,
a_4= 0.99999999999651922, a_5= 1.25000000000833356, a_6= 1.49999999999906986,
a_7= 1.75000000000436606, a_8= 2.0
Substituting the control points into (2) gives the approximate solution

$$u(t) = 2.0t$$

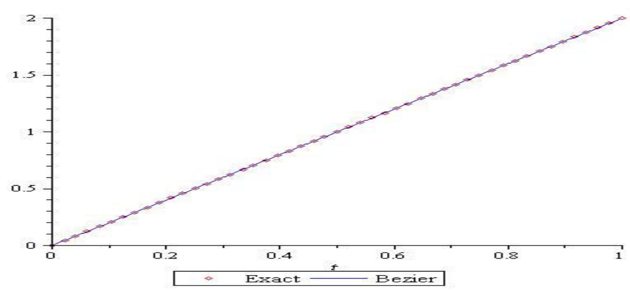

Figure 3: Exact and approximate solutions of Example 3

Fig. 1, 2 and 3 are plots of the of the solutions of Examples 1, 2 and 3 respectively. Worthy of note is Example 3, where the exact and the approximate solution are exactly the same. The results clearly show as can be seen from the graphs that the approximate solution by the Bezier curve method is a very good approximation of the exact solution of Fredholm-Volterra integral equations.

V. Conclusion

In this work, the Bezier curve method has been used successfully to find the approximate solution of linear and nonlinear Fredholm-Volterra integral equations (FVIEs). The solutions clearly agree with the exact solutions of these equations. The results are ploted in Fig. 1 to 3. We conclude here, that the Bezier curves ia a highly accurate and efficient technique for finding approximate solutions of the Fredholm-Volterra integral equations.

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