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Oscillation and Asymptotic Behaviour of Solutions of Second Order Homogeneous Neutral Difference Equations with Positive and Negative Coefficients

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Abstract: *Sufficient conditions in terms of the coefficient sequences for the oscillation and asymptotic behaviour of nonoscillatory solutions of a class of second order nonlinear neutral difference equations have been obtained. The results improve some of the earlier results in the continuous case.*
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I. Introduction

In this paper, we consider the oscillation and asymptotic behaviour of nonoscillatory solutions of the second order neutral difference equations of the form

$$\Delta^2\left[x(n)-\sum_{i=1}^{k}c_i(n)x(n-\tau_i)\right]+\sum_{i=1}^{l}p_i(n)G\big(x(n-\delta_i)\big)-\sum_{i=1}^{m}q_i(n)G\big(x(n-\sigma_i)\big)=0 \qquad (1.1)$$

where $n \in N_0 = \{n_0, n_0+1, n_0+2, \dots\}$ and $n_0 \in Z$ and Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$ and $\Delta^2 x(n) = \Delta\big(\Delta x(n)\big)$.

We assume the following conditions without further mention:

(i) $l \geq m$
(ii) $\tau_1, \tau_2, \dots, \tau_k$, $\delta_1, \delta_2, \dots, \delta_l$ and $\sigma_1, \sigma_2, \dots, \sigma_m$ are non-negative integers
(iii) $\{p_i(n)\}_{i=1}^{l}$ and $\{q_i(n)\}_{i=1}^{m}$ are sequences of non-negative real numbers
(iv) $G: R \to R$ is nondecreasing and $xG(x) > 0$ for $x \neq 0$
(v) $p_i(n) - q_i(n - \delta_i + \sigma_i) \geq 0$ for $i = 1,2,\dots,m$

By a solution of equation (1.1), we mean a real sequence $\{x(n)\}$ which is defined for $n \in n_0 - \rho$ where $\rho = \max\{\tau_1, \tau_2, \dots, \tau_k, \ \delta_1, \delta_2, \dots, \delta_l, \sigma_1, \sigma_2, \dots, \sigma_m\}$ and which satisfies equation (1.1) for all sufficiently large values of n. The solution $\{x(n)\}$ is said to be oscillatory if for every $n \in N_0$, there exists $n_j \in N_0$ such that $x_n x_{n+j} \leq 0$. The solution $\{x(n)\}$ is said to be nonoscillatory if it is eventually of constant sign.

Asymptotic behaviour of solutions of first order and second order neutral difference equations with positive and negative coefficients have been studied by many others. For example see [1,4-6,8-10,15,16,18] and the references cited there in. It is recently that second order neutral difference equations with positive and negative coefficients have been given a series of study. For the general theory of difference equation the reader is refereed to [20,21].

In 1990, Ladas [7] and Chuanxi and Ladas [2] studied the oscillatory behaviour of solutions of the equations with positive and negative coefficients of the form

$$a_{n+1} - a_n + p a_{n-k} - q a_{n-l} = 0. \qquad (1.2)$$

In 1999, Zhou [19] in 2000 Tang, Yu and Peng [13], in 2003 Chuang and Cheng [3] and in 2004 Shan and Kleigao Ge [17] considered the neutral delay difference equation with positive and negative coefficients of the form

$$\Delta(x_n - c_n x_{n-\tau}) + p_n x_{n-k} - q_n x_{n-l} = 0.$$

with and without the condition

$$\sum_{n_0}^{\infty}(p_s - q_{s-k+l}) = \infty \quad (1.3)$$

In the year 2004, Pon.Sundar and V. Sadhasivam [11] considered the equation

$$\Delta[x(n) - c(n)x(n-r)] + p(n)x(n-\tau) - q(n)x(n-\sigma) = 0 \quad (1.4)$$

and obtained some new results by establishing and using some new lemmas which are interesting in their own right and which may have further applications in analysis.

Several authors including Chuanxi and Ladas, Ladas and Zhou have investigated the oscillations of solutions of equations (1.4). They also considered the following equation

$$\Delta\big[x(n) - c(n)f\big(x(n-r)\big)\big] + p(n)g\big(x(n-\tau)\big) - q(n)g\big(x(n-\delta)\big) = 0 \quad (1.5)$$

and established some sufficient conditions for the non-existence of positive solutions of equation (1.4).

In 2002, Thandapani and Mahalingam [14] have considered the following neutral difference equation of the form

$$\Delta\big(c_n\Delta(x_n + cx_{n-k})\big) + p_n x_{n-l} - q_n x_{n-m} = 0 \quad (1.6)$$

and obtained sufficient conditions for the existence of nonoscillatory solution.

Pon. Sundar and V. Sadhasivam [12] established some sufficient conditions for the oscillation of the solutions of second order neutral delay difference equation of the form

$$\Delta^2\left[x(n) \pm \sum_{i=1}^{l} c_i(n)x(n-c_i)\right] + \sum_{i=1}^{m} p_i(n)x(n-\delta_i) - \sum_{i=1}^{r} q_i(n)x(n-\sigma_i) = 0. \quad (1.7)$$

We consider the ranges on $\sum_{i=1}^{k} c_i(n)$

(A_1) $\quad 0 \le \sum_{i=1}^{k} c_i(n) \le c < 1$

(A_2) $\quad -1 \le c_1 \le \sum_{i=1}^{k} c_i(n) \le 0$

(A_3) $\quad -c_3 \le \sum_{i=1}^{k} c_i(n) \le -c_2 < -1$

(A_4) $\quad 1 \le c_4 \le \sum_{i=1}^{k} c_i(n) \le c_5$

(A_5) $\quad -c_6 \le \sum_{i=1}^{k} c_i(n) \le -c_7 \le 0.$

The following assumptions are needed for use in the sequel:

(H_1) $\quad \lim_{|n|\to\infty} \inf \frac{G(n)}{n} \le \beta,$ where $\beta > 0$ is a real number

(H_2) $\quad \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{n_0}^{n-1} [p_i(s) - q_i(s - \delta_i + \sigma_i)] = \infty$

(H_3) $\quad \lim_{n\to\infty} K \sum_{n_0}^{n-1} s \left\{\sum_{i=1}^{m} \big(p_i(s) - q_i(s - \delta_i + \sigma_i)\big)\right\} > n$ for some positive constant K

(H_4) $\quad \beta \sum_{i=1}^{m} \sum_{s-\delta_i+\sigma_i}^{\infty} q_i(\theta) < 1$ for every $i = 1,2,\dots,m$ when $\delta_i \ge \sigma_i$

(H_5) $\quad c + \beta \sum_{i=1}^{m} \sum_{s-\delta_i+\sigma_i}^{\infty} q_i(\theta) < 1$

(H_6) $\quad \delta_i \ge \sigma_i$ for every $i = 1,2,\dots.,m$

(H_7) $\quad \sigma_i \ge \delta_i$ for every $i = 1,2,\dots.,m$

(H_8) $\quad \beta \sum_{i=1}^{m} \sum_{s-\delta_i+\sigma_i}^{\infty} q_i(\theta) < 1 + c_7.$

The following result will be needed for our use [22].

Lemma1.1. *Let* $-\infty < a < 0,\ 0 < \tau < \infty,\ n_0 \in Z$ *and suppose that a real sequence* $\{x(n)\}_{n\geq n_0 - z}$ *satisfies the inequality*

$$x(n) \leq a + \max_{n-\tau\leq s\leq \tau} x(s)$$

for $n \geq n_0$. *Then* $x(n)$ *cannot be a nonnegative sequence.*

II. Main Results

The case when $\delta_i \geq \sigma_i,\ i = 1$ to m:

In this section, we consider equation (1.1) when $\delta_i \geq \sigma_i$, $i = 1$ to m. We shall obtain sufficient conditions under which a solution of the equation is either oscillatory or tends to zero as $n \to \infty$. We observe that the result holds when G is either linear or sublinear. This is mainly due to the assumption (H_1).

Theorem2.1. *Let* $c_i(n),\ i = 1$ *to* k *be as in* (A_1). *If* $(H_1), (H_5)$, (H_6) *and either of* (H_2) *or* (H_3) *are satisfied, then every solution of equation (1.1) is either oscillatory or tends to zero as* $n \to \infty$.

Proof. Let $x(n)$ be a solution of equation (1.1). If $x(n)$ is oscillatory, then there is nothing to prove. Let $x(n)$ be nonoscillatory. Assume that $x(n) > 0$ eventually. There exists a $n_1 \geq n_0 + \rho > 0$ such that $x(n) > 0, x(n-\rho) > 0$ for $n \geq n_1$. Setting

$$\Delta w(n) = x(n) - \sum_{i=1}^{k} c_i(n)\, x(n-\tau_i) - \sum_{i=1}^{m}\sum_{n_0}^{n-1}\sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\, G\big(x(\theta-\sigma_i)\big) \tag{2.1}$$

Hence equation (1.1) can be written as

$$\Delta^2 w(n) + \sum_{i=1}^{m}\{p_i(n) - q_i(n-\delta_i+\sigma_i)\}\, G\big(x(n-\delta_i)\big) \leq 0 \tag{2.2}$$

for $n \geq n_1$. Hence $\Delta^2 w(n) \leq 0$ for $n \geq n_1$. Thus there exists a $n_2 \geq n_1$ such that $\Delta w(n) > 0$ or $\Delta w(n) < 0$ for $n \geq n_2$. Let $\Delta w(n) < 0$ for $n \geq n_2$. This inturn implies that $w(n) < 0$ for $n \geq n_3 \geq n_2$ and $\lim_{n\to\infty} w(n) = -\infty$. Then there exist $n_4 > n_3$ and $\lambda > 0$ such that $w(n) < -\lambda$ for $n - \rho > n_4$. Hence from (2.1)

$$\begin{aligned} x(n) &= w(n) + \sum_{i=1}^{k} c_i(n)\, x(n-\tau_i) + \sum_{i=1}^{m}\sum_{n_0}^{n-1}\sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\, G\big(x(\theta-\sigma_i)\big) \\ &\leq -\lambda + \left[\sum_{i=1}^{k} c_i(n) + \sum_{i=1}^{m}\sum_{n_0}^{n-1}\sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\right] \max_{n-\rho\leq s\leq n} x(s) \\ &\leq -\lambda + \max_{n-\rho\leq s\leq n} x(s). \end{aligned}$$

Then by Lemma 1.1, it follows that $x(n)$ cannot be nonnegative, a contradiction. Hence $\Delta w(n) < 0$ is not possible.

Next, Suppose that $\Delta w(n) > 0$ for $n \geq n_2$. Then $w(n) > 0$ or $w(n) < 0$ for large n, say $n \geq n_5 \geq n_2$. First, suppose that $w(n) < 0$ for $n \geq n_5$. Then $w(n)$ is bounded and

$$x(n) - \sum_{i=1}^{k} c_i(n)\, x(n-\tau_i) - \sum_{i=1}^{m}\sum_{n_0}^{n-1}\sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\, G\big(x(\theta-\sigma_i)\big) < 0. \tag{2.3}$$

We claim that $x(n)$ is bounded. If not, then there exists a sequence $\{T_j\}_{j=1}^{\infty}$, $T_j > n_5$ for every j such that $T_j \to \infty$ and $x(T_j) \to \infty$ as $j \to \infty$. In particular, for $n = T_j$ (2.3) gives

$$x(T_j)\left[1 - c - \beta\sum_{i=1}^{m}\sum_{n_0}^{n-1}\sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\right] < 0.$$

Letting $j \to \infty$, we obtain a contradiction. Hence our claim holds. Further, if $\lim_{n\to\infty}\sup x(n) = \lambda > 0$, then summation of (2.2) from n_5 to $n-1$ yields a contradiction, because G is nondecreasing and (H_2) or (H_3) holds. Hence $x(n) \to 0$ as $n \to \infty$.

Finally, suppose that $w(n) > 0$ for $n \geq n_5$. From the increasingness of $w(n)$, it follows that there exists a real $\beta_0 > 0$ such that $w(n) > \beta_0$ for large n, that is

$$w(n) = x(n) - \sum_{i=1}^{k} c_i(n)\, x(n-\tau_i) - \sum_{i=1}^{m}\sum_{n_0}^{n-1}\sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\, G\big(x(\theta-\sigma_i)\big) > \beta_0 \tag{2.4}$$

for $n \geq n_6 > n_5$. This in turn implies that there exists a positive number β_1 such that

$$w(n) \geq \beta_1 w(n) \tag{2.5}$$

for $\geq n_7 > n_6$. If this is not true, then there exists a sequence $\{T_j'\}$, $T_j' \to \infty$ as $j \to \infty$ such that

$$w(T_j') \leq \frac{1}{j} w(T_j')$$

$$\left(1 - \frac{1}{j}\right) w(T_j') \leq 0$$

a contradiction for large j. Hence (2.5) holds. Consequently $x(n) \geq \beta_1 w(n)$ for $n \geq n_7$. Then from (2.2)

$$\Delta^2 w(n) + G\big(\beta_1 w(n - \delta_i)\big) \leq 0 \tag{2.6}$$

for $n \geq n_7$. Let (H_2) hold. Since $w(n) > \mu$ for some $\mu > 0$. Summing (2.6) from n_7 to $n - 1$ and letting $n \to \infty$, we obtain a contradiction.

Next, suppose that (H_3) holds. Set $r(n) = -\Delta w(n)$. Then $\Delta r(n) = -\Delta^2 w(n)$ and

$$n\Delta r(n) \geq G(\beta, \mu) n \sum_{i=1}^{m} \{p_i(n) - q_i(n - \delta_i + \sigma_i)\}$$

for $n \geq n_7$. Summing the above inequality from n_7 to $n - 1$ gives

$$n \geq \frac{G(\beta, \mu)}{-r(n_7)} \sum_{n_7}^{n-1} \sum_{i=1}^{m} \{p_i(s) - q_i(s - \delta_i + \sigma_i)\}$$

a contradiction. Hence $w(n) > 0$ is not possible for large n. If $x(n) < 0$, for large n, then one may proceed as above to prove the theorem. This completes the proof of the theorem.

Theorem 2.2. *Let $\sum_{i=1}^{k} c_i(n)$ be in the range (A_5). If $(H_1), (H_2), (H_6)$ and (H_8) are satisfied, then every solution of equation (1.1) is either oscillatory or tends to zero as $n \to \infty$.*

Proof. Let $x(n)$ be a nonoscillatory solution of equation (1.1). Assume that $x(n) > 0$ and $x(n - \rho) > 0$ for $n \geq n_1 \geq n_0 + \rho > 0$. Setting $w(n)$ as in (2.1) we obtain (2.2). Hence $\Delta^2 w(n) \leq 0$ for $n \geq n_1$. Then $\Delta w(n) > 0$ or $\Delta w(n) < 0$ for some $n \geq n_2 \geq n_1$.

Let $\Delta w(n) > 0$ for $n \geq n_2$. The summation of (2.2) from n_2 to $n - 1$ gives

$$\Delta w(n_1) \geq \sum_{i=1}^{m} \sum_{n_2}^{n-1} \big(p_i(s) - q_i(s - \delta_i + \sigma_i)\big)\, G\big(x(s - \delta_i)\big).$$

Letting $n \to \infty$, the above inequality, in view of (H_2) yields $G\big(x(n)\big) \to 0$ as $n \to \infty$. Hence $x(n) \to 0$ as $n \to \infty$.

Next, suppose that $\Delta w(n) < 0$ for $n \geq n_2$. Thus there exists a $n_3 \geq n_2$ such that $w(n) < 0$ for $n \geq n_3$ and $\lim_{n\to\infty} w(n) = -\infty$. We claim that $x(n)$ is bounded. If not, there exists a sequence $\{T_j\}_{j=1}^{\infty}$ such that $T_j \geq n_3$ for every j, $T_j \to \infty$ as $j \to \infty$, $w(T_j) \to \infty$ and $x(T_j) \to \infty$ as $n \to \infty$ and $\max_{n_3 \leq n \leq T_j} x(n) = x(T_j)$. Then we have

$$\begin{aligned} w(T_j) &= x(T_j) - \sum_{i=1}^{k} c_i(T_j)\, x(T_j - \tau_i) - \sum_{i=1}^{m} \sum_{n_0}^{T_j - 1} \sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\, G\big(x(\theta - \sigma_i)\big) \\ &\geq x(T_j)\left[1 - \sum_{i=1}^{k} c_i(T_j) - \beta \sum_{i=1}^{m} \sum_{n_0}^{T_j - 1} \sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta)\right]. \end{aligned}$$

Letting $j \to \infty$ in view of (H_8), we obtain $w(T_j) \to \infty$ as $j \to \infty$ a contradiction. Hence our claim holds. That is, $x(n)$ is bounded. Consequently $w(n)$ is bounded, a contradiction.

If $x(n) < 0$ the proof of the theorem may be treated similarly. Thus the theorem is proved.

Theorem 2.3. *Let $\sum_{i=1}^{k} c_i(n)$ be in the range (A_5). If $(H_1), (H_6), (H_8)$ and*

$$(H_9) \qquad \sum_{i=1}^{m} \{p_i(n) - q_i(n - \delta_i + \sigma_i)\} \geq b\,,$$

$b \geq 0$ is a constant, hold, then every solution of equation (1.1) is oscillatory.

Proof. Suppose that $x(n)$ is a nonoscillatory solution of (1.1). Assume that $x(n) > 0$ and $x(n - \rho) > 0$ for $n \geq n_1 \geq n_0 + \rho > 0$. Then from (2.2), we have $\Delta^2 w(n) \leq 0$ for $n \geq n_1$ and hence $\Delta w(n) > 0$ or $\Delta w(n) < 0$

for some $n \geq n_2 \geq n_1$. If $\Delta w(n) < 0$ for $n \geq n_2$, then $\lim_{n\to\infty} w(n) = -\infty$. Proceeding as in Theorem 2.2 one may show that $x(n)$ is bounded. Consequently, $w(n)$ is bounded, a contradiction.

Next, suppose that $\Delta w(n) > 0$ for $n \geq n_2$. Then summing (2.2) from n_2 to $n-1$ we obtain

$$\infty > \Delta w(n_2) \geq b \sum_{n_2}^{\infty} G\big(x(s-\delta_i)\big).$$

Therefore, $G\big(x(n)\big)$ is bounded. Since G is nondecreasing, and since $uG(u) > 0$ for $u \neq 0$ then $x(n)$ is bounded. Hence

$$z(n) = x(n) - \sum_{i=1}^{k} c_i(n)\, x(n-\tau_i)$$

is also bounded. Thus

$$\Delta z(n) = \Delta w(n) + \sum_{i=1}^{m} \sum_{n-\delta_i+\sigma_i}^{n-1} q_i(s)\, G\big(x(s-\sigma_i)\big) \geq 0$$

since $\{q_i(n)\}$ be a sequence of nonnegative real numbers which converges to zero. Then by Dirichlet's test, we have

$$\sum_{n-\delta_i+\sigma_i}^{n-1} q_i(s)\, G\big(x(s-\sigma_i)\big)$$

converges and

$$\lim_{n\to\infty} \sum_{n-\delta_i+\sigma_i}^{n-1} q_i(s)\, G\big(x(s-\sigma_i)\big) = 0\,.$$

Hence $z(n)$ is nondecreasing and $z(n) > 0$, because for all large n, (A_5) holds. Hence $\lim_{n\to\infty} z(n) = \mu$. If $0 < \mu < \infty$, then for $0 < \epsilon < \mu$, there exists a $n_3 \geq n_2$ such that $z(n) > \mu - \epsilon$ for $n \geq n_3$. Hence $z(n)$ is not bounded, a contradiction. Hence $x(n) \nrightarrow 0$ for large n.

In a similar way one may show that $x(n) \nless 0$ for all large n. This completes the proof of the theorem.

Proceeding as in the lines of Theorem 2.1, one may prove the following theorem.

Theorem 2.4. *Let $c_i(n),\ i = 1$ to k be in the range of (A_2) or (A_3). Further assume that $(H_1), (H_4)$ and (H_6) hold. If either (H_2) or (H_3) holds, then every solution of equation (1.1) is oscillatory or tends to zero as $n \to \infty$.*

Theorem 2.5. *Let $c_i(n),\ i = 1$ to k be in the range of (A_4). Let $(H_1), (H_6)$ and*

$$(H_{10}) \qquad \sum_{i=1}^{m} \sum_{n_0}^{\infty} \sum_{s-\delta_i+\sigma_i}^{s-1} q_i(\theta) < \infty$$

hold. If either (H_2) or (H_3) is satisfied, then every bounded solution of equation (1.1) is oscillatory or tends to zero as $n \to \infty$.

Proof. Since *x(n)* is bounded, then $\lim_{n\to\infty} \sup x(n) > 0$ implies that $\Delta w(n) \to -\infty$ as $n \to \infty$ and hence $w(n) \to -\infty$ as $n \to \infty$. On the other hand, since $x(n)$ is bounded and (H_1) and (H_{10}) hold, then (2.1) yields that $w(n) \to \infty$ as $n \to \infty$, a contradiction. Thus the theorem is proved.

Remark 2.1. In the above results, the condition (H_6) forces us to assume (H_1). The above result remain true when G is linear or sublinear. The phototype of G satisfying the hypothesis of the above results is $G(u) = |u|^{\gamma} sgn\, u,\ \ \gamma \leq 1$.

III. Main Results

The Case when $\sigma_i \geq \delta_i\,,\ i = 1$ to k:

In the following theorems, we shall replace the assumption (H_6) by (H_7). Hence the following results remain true for all types of G.

Theorem 3.1. *Let $c_i(n),\ i = 1$ to k be in the range of (A_2) or (A_3) or (A_5). Further assume that (H_2) and (H_6) hold. Then every solution of equation (1.1) is oscillatory or tends to zero as $n \to \infty$.*

Proof. Let $x(n)$ be a eventually positive solution of (1.1) then $w(n) > 0$ or $w(n) < 0$ for all large n. If $w(n) < 0$ for all large n, then $x(n) < 0$ for large n, a contradiction. If $w(n) > 0$ for all large n, then $\Delta w(n) > 0$ for large n, say $n \geq n_2$. Summation (2.2) from n_2 to $n-1$ and using (H_2) and the nondecreasing property of G, we see that $x(n) \to 0$ as $n \to \infty$. The above line holds when $x(n) < 0$ for all large n. The proof is complete.

Theorem 3.2. *Suppose that* $c_i(n),\ i = 1$ *to* k *be in the range of* (A_1). *If* (H_2) *and* (H_6) *hold, then every solution of equation (1.1) is oscillatory or tends to zero as* $n \to \infty$.

Proof. Let $x(n)$ is an eventually positive solution of (1.1), setting $w(n)$ as in (2.1), we obtain (2.2) for all large n. Hence $w(n) > 0$ or $w(n) < 0$ for all large n. If $w(n) > 0$ for large n, then $\Delta w(n) > 0$ eventually. The summation of (2.2) from t_1 to ∞, in view of (H_2) and the nondecreasing property of G, we see that $x(n) \to 0$ as $n \to \infty$. Let $w(n) < 0$ for all large n. Then

$$x(n) < w(n) + \sum_{i=1}^{k} c_i(n)\, x(n - \tau_i) \qquad (3.1)$$

If $\lim_{n\to\infty} w(n) = -\lambda$, $\lambda > 0$, then we obtain, for large n

$$\limsup_{n\to\infty} x(n) \le -\lambda + c \limsup_{n\to\infty} x(n)$$

or

$$(1-c) \limsup_{n\to\infty} x(n) < -\lambda < 0$$

a contradiction to the fact that $x(n) > 0$ eventually. If $\lim_{n\to\infty} w(n) = 0$, then taking lim sup on both sides in (3.1), we have

$$\limsup_{n\to\infty} x(n) < c \limsup_{n\to\infty} x(n)$$

which ultimately yields that $x(n) \to 0$ as $n \to \infty$ eventually. The proof of the theorem is same if $x(n) < 0$ eventually. This completes the proof of the theorem.

Theorem 3.3. *Let* $c_i(n),\ i = 1,2,\dots,m$ *be in the range of* (A_4). *If* $(H_2), (H_6)$ *and*

$$(H_{11}) \qquad \sum_{i=1}^{m} \sum_{n_0}^{\infty} \sum_{s}^{s-1+\delta_i+\sigma_i} q_i(\theta) < 1,$$

then every bounded solution of equation (1.1) is oscillatory or tends to zero as $n \to \infty$.

Proof. If $x(n) > 0$ for large n, and bounded, then (H_{11}) implies that $w(n)$ is bounded. If $\lim_{n\to\infty} \sup x(n) > 0$, then summation of equation (2.2) from n_2 to ∞, n_2 large enough, we have $\Delta w(n) \to -\infty$, a contradiction to the boundedness of $w(n)$. Hence $x(n) \to 0$ as $n \to \infty$. The proof of the theorem may be treated similarly if we assume that $x(n) < 0$ for large n. Thus the proof is complete.

Remark 3.1. From the above results, it follows that when $G(u) = u$, that is in the linear case, the assumption $\delta_i \ge \sigma_i$ or $\delta_i \le \sigma_i$ is not required though many authors have assumed it. It would be interesting if one removes the condition (H_1) on G for $\delta_i \ge \sigma_i$, $i = 1$ *to* k.

IV. References

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