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The Highly Pathogenic Avian Influenza Epidemic Model with Vertical Transmission Function In Poultry

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Abstract. *In the present paper we discuss the highly pathogenic avian influenza epidemic model in the presence of vertical transmission function in poultry. We introduce a basic reproduction number* R_0 *for the model. The stability of the equilibrium points of the system is studied and discussed. Finally a numerical example is also included to illustrate the effectiveness of the proposed model.*
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I. Introduction

Since 2005 highly pathogenic avian influenza A H5N1 viruses have spread from Asia to Africa and Europe infecting poultry, humans and wild birds. The highly pathogenic avian influenza has a high death rate, which is about 100 percent for birds and more than 70 percent for humens [7]. Up to now we found that avian influenza virus subtypes which can directly infect human are: H5N1, H7N1, H7N2, H7N3, H7N7, H9N2, H7N9, subtype. The avian influenza virus not only caused human causalities, but also hit the poultry industry. The highly pathogenic influenza A virus subtype H5N1 has killed millions of poultry in a growing number of countries throughout Asia, Europe and Africa. India's first H5N1 outbreak was reported in January of 2006 in the Navapur District, Maharashta, India, since H5N1 has been reported in 8 different states. In West Bengal, 54 H5N1 outbreaks in poultry were reported between Jan 2006 and Aug 2010 making it the state with highest incidence in India.

By March 2013, the world has reported a total of infection of highly pathogenic H5N1 avian influenza in 622 cases, including 371 deaths. The distribution of cases in 15 countries, including China, is found in 45 cases, 30 cases of death. Most of human being infected with H5N1 avian influenza are young people and children. In March 2013, human infection with H7N9 avian influenza was first found in China. By May 1, 2013 Shanghai, Anhai, Jiangsu, Zhejing, Beijing, Henan, Shandong, Jiangxi, Hunan, Fujian and other 10 city have reported 127 confirmed case, including 26 death cases studied by Che [1]. The number of mathematical modeling studies have been carried out to quantify the potential burden of an influenza pandemic in human being and to assess various control strategies considered by et al. [2, 3, 4, 6, 8, 9, 10, 11,13]. Avian influenza modeling studies involving humans and birds was carried out in Gumel [5] and Iwami [7].

In this paper we consider highly pathogenic avian influenza epidemic model with vertical transmission function in the poultry. In the next section, we present the model. In third section we derive the disease free equilibrium and the endemic equilibrium. In the fourth section, we prove some theorems for the global stability of the disease free and endemic equilibrium. The last section contains a numerical simulation and discussion.

II. Mathematical Model.

2.1 Basic Model.

Shuqin Che et al. [1] has proposed the following four dimensional system of autonomous differential equation model for the avian influenza

$$\frac{dX}{dt}=c-\frac{\omega XY}{1+\delta Y}-dX,$$
$$\frac{dY}{dt}=\frac{\omega XY}{1+\delta Y}-(d+m)Y,$$
$$\frac{dS}{dt}=b-\frac{\beta SY}{1+\delta Y}-\alpha S, \quad \text{... (2.1)}$$
$$\frac{dI}{dt}=\frac{\beta SY}{1+\delta Y}-(\varepsilon+\alpha+\gamma)I,$$
$$\frac{dR}{dt}=\gamma I-\alpha R$$

Here $X(t)$ and $Y(t)$ are the numbers of susceptible poultry and infected poultry of birds respectively, $S(t)$, $I(t)$ and $R(t)$ the number of susceptible, infected and recovered of human being respectively. The parameters c and b are respectively the natural birth rate of Avian and human being. d and α are respectively the natural mortality of poultry and human being. m and ε are respectively the poultry and human mortality due to illness. ω stands for infectious rate of susceptible poultry to infected poultry, β stands for infected poultry of the infection rate of susceptible individuals, γ is the recovery rate that infects individuals through treatment. When Y is small, the contact ratio, infected poultry and susceptible poultry, is appromately proportional to the Y: with the increase of Y, the contact rate gradually reaches saturation. When Y is very largy, it is close to a constant ω/δ. The same way to explain $\beta/(1+\delta Y)$, that is to say, δ is a parameter which is effects of infectious diseases, when the contact rate of the disease is saturated.

2.2 Model with Vertical Transmission Function in Poultry.

The model (2.1) with vertical transmission function in poultry is given by

$$\frac{dX}{dt}=c-\frac{\omega XY}{1+\delta Y}-dX-pcY,$$
$$\frac{dY}{dt}=\frac{\omega XY}{1+\delta Y}-(d+m)Y+pcY,$$
$$\frac{dS}{dt}=b-\frac{\beta SY}{1+\delta Y}-\alpha S, \quad (2.2)$$
$$\frac{dI}{dt}=\frac{\beta SY}{1+\delta Y}-(\varepsilon+\alpha+\gamma)I,$$
$$\frac{dR}{dt}=\gamma I-\alpha R.$$

where p is suitable constant. The rest of the parameters have similar meaning as for as the model (2.1).

3. Equilibria of the system.

The first four equation of system (2.2) do not contain R, by the method of Vanden Driessche and Watmough Diekmann [12]

$$\begin{aligned}
\frac{dX}{dt} &= c - \frac{\omega XY}{1+\delta Y} - dX - pcY, \\
\frac{dY}{dt} &= \frac{\omega XY}{1+\delta Y} - (d+m)Y + pcY, \\
\frac{dS}{dt} &= b - \frac{\beta SY}{1+\delta Y} - \alpha S, \\
\frac{dI}{dt} &= \frac{\beta SY}{1+\delta Y} - (\varepsilon+\alpha+\gamma)I.
\end{aligned} \tag{3.1}$$

It can be checked that the system (3.1) has two non-negative equilibrium and one of them disease free equilibrium $E_0\left(X^o, Y^o, S^o, I^o\right) = \left(\frac{c}{d}, 0, \frac{b}{\alpha}, 0\right)$ we can get the basic reproductive numbers of the system (3.1) $R_0 = \frac{c\omega}{d(d+m-pc)}$

Existence of $E_+\left(X^*, Y^*, S^*, I^*\right)$

Here X^*, Y^*, S^*, I^* are the positive solution of the following algebraic equation,

$$\begin{aligned}
& c - \frac{\omega XY}{1+\delta Y} - dX - pcY = 0, \\
& \frac{\omega XY}{1+\delta Y} - (d+m)Y + pcY = 0, \\
& b - \frac{\beta SY}{1+\delta Y} - \alpha S = 0, \\
& \frac{\beta SY}{1+\delta Y} - (\varepsilon+\alpha+\gamma)I = 0.
\end{aligned} \tag{3.2}$$

Solving (3.2) we get

$$X^* = \frac{(d+m+c\delta)(d+m-pc)}{(d+m)(d\delta+\omega) - pcd\delta}, Y^* = \frac{c\omega - d(d+m-pc)}{(d+m)(d\delta+\omega) - pcd\delta}$$

$$S^* = \frac{b(1+\delta Y^*)}{\beta Y^* + \alpha(1+\delta Y^*)}, I^* = \frac{b\beta Y^*}{(\varepsilon+\alpha+\gamma)[\beta Y^* + \alpha(1+\delta Y^*)]}$$

Where $(d+m) > pc, (d+m)(\omega + d\delta) > pcd\delta.$

Theorem 3.1. If $R_0 \le 1,$ the system (2.2) only exists the disease-free equilibrium $E_0\left(\frac{c}{d}, 0, \frac{b}{\alpha}, 0\right),$ when $(d+m) > pc, (d+m)(\omega + d\delta) > pcd\delta$ and $R_0 > 1,$ there exists only one endemic equilibrium $E_+\left(\frac{(d+m+c\delta)(d+m-pc)}{(d+m)(d\delta+\omega) - pcd\delta}, Y^*, \frac{b(1+\delta Y^*)}{\beta Y^* + \alpha(1+\delta Y^*)}, \frac{b\beta Y^*}{(\varepsilon+\alpha+\gamma)[\beta Y^* + \alpha(1+\delta Y^*)]}\right)$

Where

$$Y^* = \frac{c\omega - d(d+m-pc)}{(d+m)(d\delta+\omega) - pcd\delta}$$

III. Linear stability analysis.

Theorem 4.1 The disease free equilibrium E_0 is locally asymptotically stable if $R_0 \le 1,$ and disease free equilibrium E_0 is unstable if $R_0 > 1.$

Proof. The Jacobian matrix of system (3.1) is

$$J = \begin{bmatrix} -d - \frac{\omega Y}{1+\delta Y} & -\frac{\omega X(1+\delta Y) - \delta\omega XY}{(1+\delta Y)^2} - pc & 0 & 0 \\ \frac{\omega Y}{1+\delta Y} & \frac{\omega X(1+\delta Y) - \delta\omega XY}{(1+\delta Y)^2} - (d+m-pc) & 0 & 0 \\ 0 & -\frac{\beta S(1+\delta Y) - \delta\beta SY}{(1+\delta Y)^2} & -\frac{\beta Y}{1+\delta Y} - \alpha & 0 \\ 0 & \frac{\beta S(1+\delta Y) - \delta\beta SY}{(1+\delta Y)^2} & \frac{\beta Y}{1+\delta Y} & -(\varepsilon+\alpha+\gamma) \end{bmatrix}$$

Now, Jacobian matrix of system (3.1) at $E_0\left(\frac{c}{d}, 0, \frac{b}{\alpha}, 0\right)$, is

$$J_{E_0} = \begin{bmatrix} -d & -\frac{\omega c}{d} - pc & 0 & 0 \\ 0 & \frac{\omega c}{d} - (d+m-pc) & 0 & 0 \\ 0 & -\frac{\beta b}{\alpha} & -\alpha & 0 \\ 0 & \frac{\beta b}{\alpha} & 0 & -(\varepsilon+\alpha+\gamma) \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} -d-\lambda & -\frac{\omega c}{d} & 0 & 0 \\ 0 & \frac{\omega c}{d} - (d+m-pc) - \lambda & 0 & 0 \\ 0 & -\frac{\beta b}{\alpha} & -\alpha-\lambda & 0 \\ 0 & \frac{\beta b}{\alpha} & 0 & -(\varepsilon+\alpha+\gamma)-\lambda \end{vmatrix} = 0$$

$$(d+\lambda)(\alpha+\lambda)(\varepsilon+\alpha+\gamma+\lambda)\left[\left\{\frac{\omega c}{d} - (d+m-pc)\right\} - \lambda\right] = 0 \qquad (4.1)$$

The roots of (4.1) are

$$-d,\ -\alpha,\ -(\varepsilon+\alpha+\gamma),\quad \frac{\omega c}{d} - (d+m-pc)$$

The first three roots having negative real parts and fourth root $\frac{\omega c}{d} - (d+m-pc)$ will have negative real part if $R_0 \le 1$. Thus all roots of (4.1) have negative real parts so E_0 is locally asymptotically stable if $R_0 \le 1$, and the root $\frac{\omega c}{d} - (d+m-pc)$ will have positive real part if $R_0 > 1$, so E_0 is an unstable.

Theorem 4.2 The disease free equilibrium E_0 is globally asymptotically stable if $R_0 \le 1$. .

Proof. Consider the Lyapunov function

$$L_1 = X - X^\circ \ln X + Y$$

$$= X' - \frac{X^\circ}{X} X' + Y'$$

$$= \left(1 - \frac{X^\circ}{X}\right) X' + Y'$$

$$= \left(1 - \frac{X^\circ}{X}\right)\left[c - \frac{\omega XY}{1+\delta Y} - dX - pc\right] + \frac{\omega XY}{1+\delta Y} - (d+m-pc)Y$$

$$= \left(1 - \frac{X^\circ}{X}\right)\left[dX^\circ - dX - \frac{\omega XY}{1+\delta Y} - pcY\right] + \frac{\omega XY}{1+\delta Y} - (d+m-pc)Y$$

$$\le -\frac{d(X - X^\circ)^2}{X} + (d+m-pc)Y\left(\frac{\omega X^\circ}{(d+m-pc)} - 1\right)$$

$$= -\frac{d(X - X^\circ)^2}{X} + (d+m-pc)Y\left(R_0 - 1\right).$$

When $R_0 \le 1,$ we can get $L_1' \le 0$ and $L_1' = 0$ has no other closed trajectory in addition to E_0 is globally asymptotically stable if and only if $R_0 \le 1.$

Theorem 4.3 The endemic equilibrium E_+ is locally asymptotically stable if $R_0 > 1.$

Proof. The Jacobian matrix of system (3.1) at $E_+(X^*, Y^*, S^*, I^*)$ is

$$J_{E_+} = \begin{bmatrix} -d - \frac{\omega Y^*}{1+\delta Y^*} & -\frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} - pc & 0 & 0 \\ \frac{\omega Y^*}{1+\delta Y^*} & \frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} - (d+m-pc) & 0 & 0 \\ 0 & -\frac{\beta S^*(1+\delta Y^*) - \delta\beta S^* Y^*}{(1+\delta Y^*)^2} & -\frac{\beta Y^*}{1+\delta Y^*} - \alpha & 0 \\ 0 & \frac{\beta S^*(1+\delta Y^*) - \delta\beta S^* Y^*}{(1+\delta Y^*)^2} & \frac{\beta Y^*}{1+\delta Y^*} & -(\varepsilon+\alpha+\gamma) \end{bmatrix}$$

$$J_{E_+} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

Where

$$A = \begin{bmatrix} -d - \frac{\omega Y^*}{1+\delta Y^*} & -\frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} - pc \\ \frac{\omega Y^*}{1+\delta Y^*} & \frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} - (d+m-pc) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -\frac{\beta S^*(1+\delta Y^*) - \delta\beta S^* Y^*}{(1+\delta Y^*)^2} \\ 0 & \frac{\beta S^*(1+\delta Y^*) - \delta\beta S^* Y^*}{(1+\delta Y^*)^2} \end{bmatrix}$$

$$C=\begin{bmatrix} -\frac{\beta Y^*}{1+\delta Y^*}-\alpha & 0 \\ \frac{\beta Y^*}{1+\delta Y^*} & -(\varepsilon+\alpha+\gamma) \end{bmatrix}$$

$$O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus J_{E_+} evaluated is stable if and only if so are A and C. The characteristic equations of the matrix A is

$$\lambda^2+h_1\lambda+h_2=0,$$

Where $h_1=d+(d+m-pc)\frac{\delta Y^*}{1+\delta Y^*}+\frac{\omega Y^*}{1+\delta Y^*}$, $h_2=[(d+m)(\omega+d\delta)-pcd\delta]\frac{Y^*}{1+\delta Y^*}$

Here $1+\delta Y^*>0$ when $R_0>1$, $(d+m)(\omega+d\delta)>pcd\delta$, so when $(d+m)>pc$, $(d+m)(\omega+d\delta)>pcd\delta$ and $R_0>1$ then $h_1,h_2>0$ by the Hurwitz criterion.

The characteristic roots of matrix A have negative real parts.

The characteristic equation of the matrix C

$$\left(\lambda+\frac{\beta Y^*}{1+\delta Y^*}+\alpha\right)\left(\lambda+\varepsilon+\alpha+\gamma\right)=0.$$

The characteristic roots of C are, $-\frac{\beta Y^*}{1+\delta Y^*}-\alpha,-(\varepsilon+\alpha+\gamma)$

When $R>1$, The characteristic roots of C have negative real parts.

So all characteristic roots of the Jacobian matrix J_{E_+} have negative real parts if and only if $R_0>1$. Thus the endemic equilibrium E_+ is locally asymptotically stable if $R_0>1$.

Theorem 4.4 The endemic equilibrium E_+ is globally asymptotically stable if $R_0>1$..

Proof: Consider the Lyapunov function

$$L_2=X^*\left(\frac{X}{X^*}-1-\ln\frac{X}{X^*}\right)+Y^*\left(\frac{Y}{Y^*}-1-\ln\frac{Y}{Y^*}\right)$$

Then

$$L_2^{'}=X^*\left(\frac{X^{'}}{X^*}-\frac{X^*}{X}.\frac{X^{'}}{X^*}\right)+Y^*\left(\frac{Y^{'}}{Y^*}-\frac{Y^*}{Y}\frac{Y^{'}}{Y^*}\right)$$

$$L_2^{'}=\left(1-\frac{X^*}{X}\right)X^{'}+\left(1-\frac{Y^*}{Y}\right)Y^{'}=C\left(2-\frac{X^*}{X}-\frac{X}{X^*}\right).$$

By the relationship of arithmetic mean and geometric mean.

We know that

$$2-\frac{X^*}{X}-\frac{X}{X^*}\le 0.$$

i.e. $L_2^{'}\le 0$, if and only if $(X,Y)=(X^*,Y^*), L_2^{'}=0$. Thus by LaSalle invariance principle $E_+(X^*,Y^*,S^*,I^*)$ is globally asymptotically stable.

IV. Simulation and discussion

In this paper we have discussed the global stability of highly pathogenic avian influenza epidemic model with vertical transmission function in poultry. Vertical transmission function is taken to represent the interaction between susceptible and infected poultry. To illustrate the results numerically, choose $c=2.5,\beta=0.02,d=0.05,b=1.5,$ $m=2,\varepsilon=1,\omega=0.037,$ $\gamma=0.40,$

$\delta = 0.06, \alpha = 0.069, p = 0.015,$ $\left(X(0), Y(0), S(0), I(0)\right) = (60, 25, 30, 15).$ Then $R_0 = 0.91 < 1.$ Figure 1(a) shows that $X(t)$ approaches to its steady-state value 50 while $Y(t)$ tends to zero as time tends to infinity.

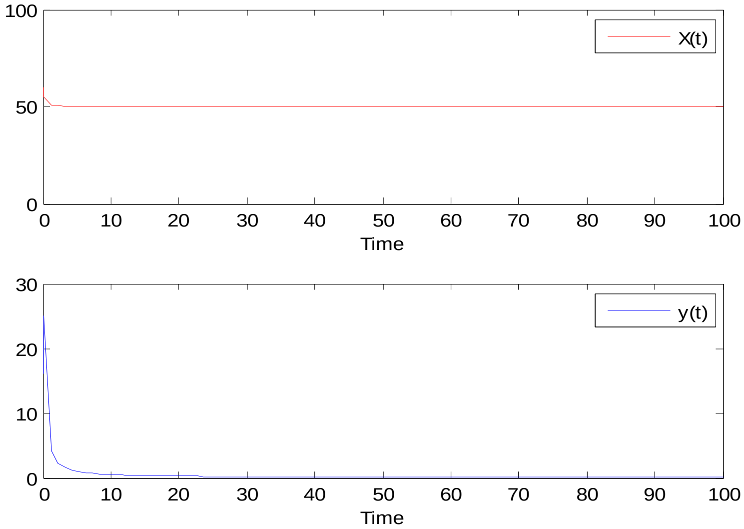

Figure1(a). Here $X(0) = 60, Y(0) = 25, c = 2.5, \beta = .02, d = .05, b = 1.5, m = 2, \varepsilon = 1,$ $\omega = .037, \gamma = .40, \delta = .06, \alpha = .069, p = .015, R_0 = .91 < 1.$

Figure 1(b) shows that $S(t)$ approaches to its steady-state value 21.73 while $I(t)$ tends to zero as time tends to infinity. Thus the equilibrium $\left(X(0), Y(0), S(0), I(0)\right) = (60, 25, 30, 15)$ approaches to disease-free equilibrium $E_0(50, 0, 21.73, 0)$ that is disease dies out.

Again we take the parameters $c = 2.5, \beta = 0.02, d = 0.05, b = 1.5,$ $m = 2, \varepsilon = 1, \omega = 1.22,$ $\gamma = 0.40,$ $\delta = 0.06, \alpha = 0.069, p = 0.015,$ Then $R_0 = 30.31 > 1.$ Therefore by theorem (4.4), E_+ is a globally asymptotically stable.

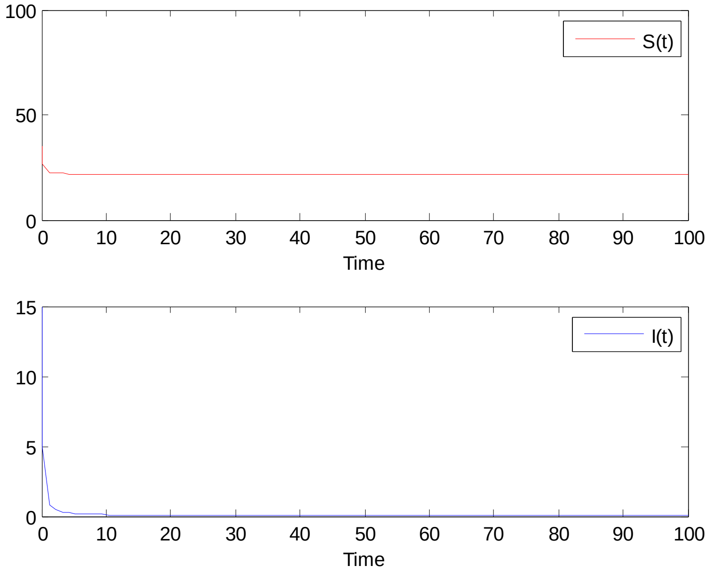

Figure2(b). Here $S(0) = 30, I(0) = 15, c = 2.5, \beta = .02, d = .05, b = 1.5, m = 2, \varepsilon = 1,$ $\omega = .037, \gamma = .40, \delta = .06, \alpha = .069, p = .015, R_0 = .91 < 1.$

Figure 2 (a) and (b) show that the $\left(X(t), Y(t), S(t), I(t)\right)$ approaches to endemic equilibrium

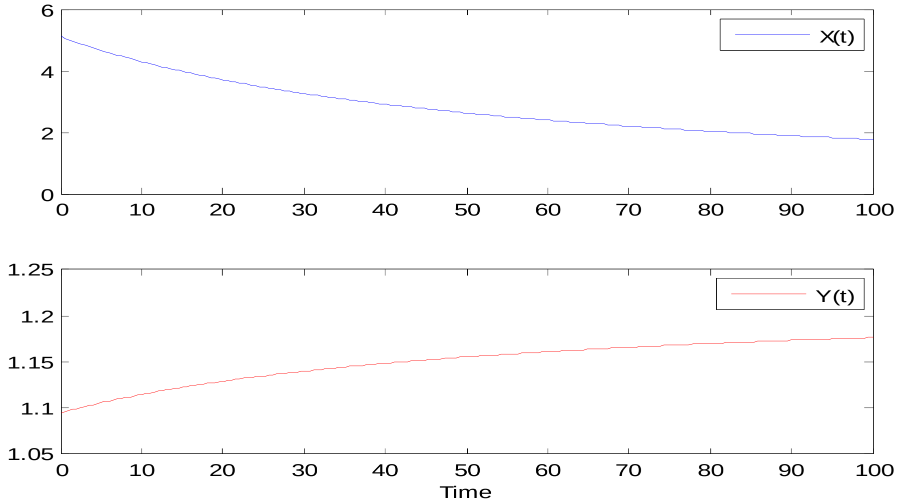

Figure 2(a). Here $c = 2.5, \beta = .02, d = .05, b = 1.5, m = 2, \varepsilon = 1, \omega = 1.22, \gamma = .40, \delta = .06, \alpha = .069, p = .015, R_0 = 30.31 > 1.$

$E_+(1.76, 1.17, 16.31, 0.246)$ as time tends to infinity. Keeping other parameters fixes, if we change the value of p, it is seen that $I(t)$ increases as p increases. It follows from Figure 3.

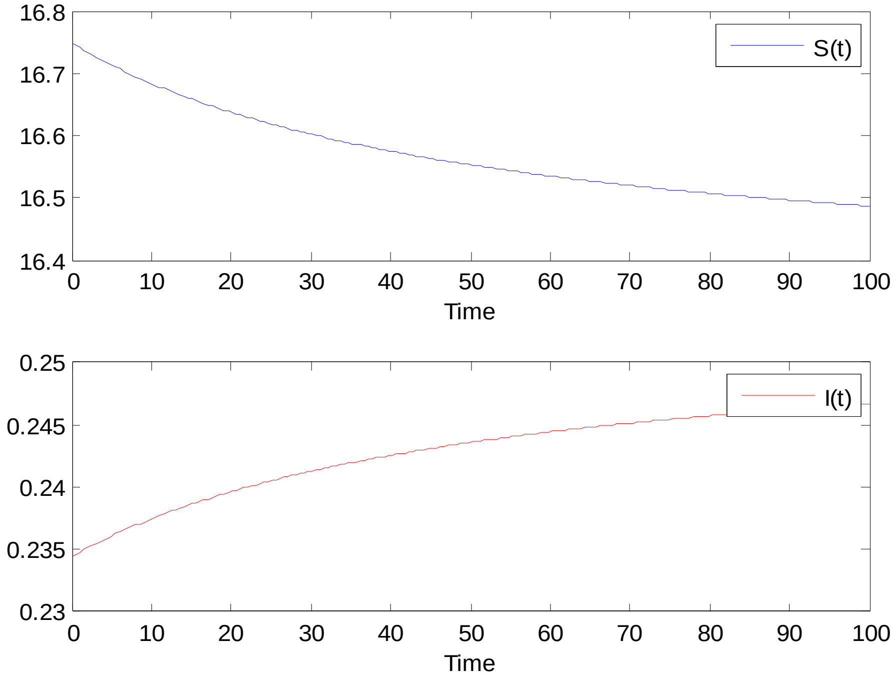

Figure2(b). Here $S(0) = 16.75, I(0) = .234, c = 2.5, \beta = .02, d = .05, b = 1.5, m = 2, \varepsilon = 1,$

$\omega = 1.22, \gamma = .40, \delta = .06, \alpha = .069, p = .015, R_0 = 30.31 > 1.$

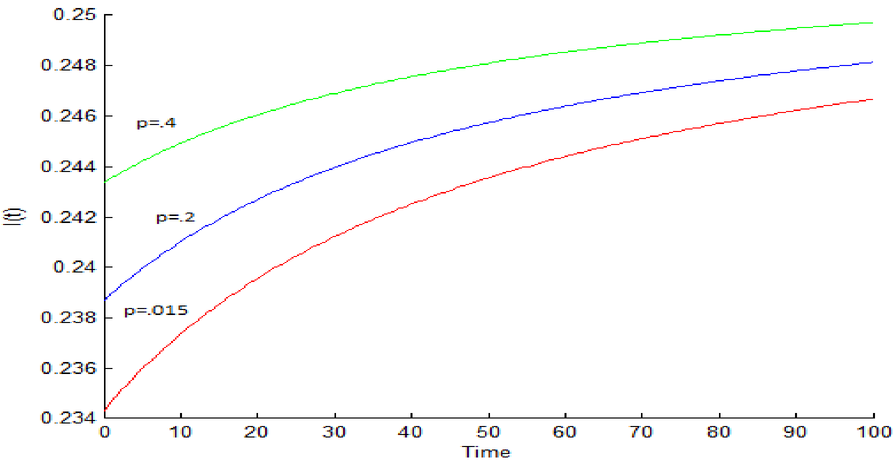

Figure 3 . The dependence of $I(t)$ on the parameters p keeping other parameters fixed.

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