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$x^n + ay^2 = z^2$ Diophantine Equation's Integer Solutions

Ahmet CİHANGİR[1], Hasan ŞENAY[2]
[1]Necmettin Erbakan University, Ahmet Keleşoğlu Faculty Of Education, Primary Education Mathematics Teaching, Meram / KONYA – TURKEY
[2]Mevlana University, Faculty Of Education, Primary Education, Mathematics Teaching, Selçuklu / KONYA– TURKEY
acihangir@konya.edu.tr (A. Cihangir), hsenay@mevlana.edu.tr (H. Şenay)

Abstract: *In this study; where the x, y, z numbers are unknowns, a and n positive integers, existence of x, y, z integer solutions' for each positive n integers of* $x^n + ay^2 = z^2$ *Diophantine equation was shown by induction. Also change of x was analyzed according to the values which a will take and each situation was exemplified.*

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I. Introduction

The conjecture which is stated as like, where the Number theory's famous "*p, q, r*'s are natural numbers and $(x, y) = 1$

$$x^p + y^q = z^r \tag{1}$$

Diophantine equation have *x, y, z* integer solutions"; W. Sierpinki showed that solutions are constituted from (x, y, z) Pythagorean triples in the situation of $(p, q, r) = (2, 2, 2)$ [6]. N. Terai claimed the existence of solutions for $(p, q, r) = (2, 2, 3)$ and $(p, q, r) = (2, 2, 5)$ [1, 2].

So we had shown the existence of integer solutions of equation (1) by induction where *n* is even and $(p, q, r) = (2, 2, n)$ [3]. Also we had shown the existence of equation (1)'s integer solutions by induction for each positive n integer in the situation of $(p, q, r) = (2, 2, n)$ and $(p, q, r) = (n, 2, 2)$ [4].

In this study of us; where the *x, y, z*'s are integers, *a* and *n* positive integers, the existence of integer solutions of

$$x^n + ay^2 = z^2 \tag{2}$$

diophantine equation was shown by induction for each positive *n* integer. Also the change of *x* was analyzed according to the values which *a* will take.

First let's give the lemma that guarantees the existence of all integer solutions of equation (2) in the special situation $n = 2$.

Lemma 1.1 Where the *a, u, v* positive integers, *a* doesn't contain square factor (multiplier) and $u+av \equiv 1 (mod\ 2)$ ve $(u, av) = 1$, all integer solutions of

$$x^2 + ay^2 = z^2 \tag{3}$$

diophantine equations are given as like [6].

$$x = u^2 - av^2,\ y = 2uv,\ z = u^2 + av^2 \tag{4}$$

Let's give aforementioned situations above respectively.

2. $x^n + ay^2 = z^2$ Diophantine Equation's Integer Solutions

Now let's give our theorem regarding integer solutions of equation (2).

Theorem 2.1. Let the integer *z* be given as $x = u^2 - av^2$ where *a, u, v* positive integers, *a* doesn't contain square factor (multiplier), $u+av \equiv 1 (mod\ 2)$ and $(u, av) = 1$. Also the let the operation [5] of

$$(u_1, v_1, x_1) \otimes (u_2, v_2, x_2) = (u_1 u_2 + av_1 v_2, u_1 v_2 + v_1 u_2, x_1 x_2) \tag{5}$$

be defined for the solution triples of (u_1, v_1, x_1), (u_2, v_2, x_2) in order to check $z = u^2 + av^2$ equation by u_1, u_2, v_1, v_2 z_1, z_2 integers. Then, under the operation (5), all integer solutions of $x^n + ay^2 = z^2$ Diophantine equation shall be given as;

$$z = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k} v^{2k}, \quad y = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^k u^{n-2k-1} v^{2k+1}, \quad x = u^2 - av^2 \qquad (6)$$

Proof. Lets make the proof with induction on *n*. According to induction hypothesis for $n = 1$, we see;

$$z = u, \; y = v, \; x = u^2 - av^2$$

In $n = 2$ special condition we could be able to draw attention to

$$z = u^2 + av^2, \quad y = 2uv, \quad x = u^2 - av^2$$

because of Lemma 1.1.

Now, consider claim is correct for *n*. So let's accept it is

$$z = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k} v^{2k}, \quad y = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^k u^{n-2k-1} v^{2k+1}, \; x = u^2 - av^2,$$

and show the truth of claim for $n + 1$. According to operation (5) it is;

$(u, v, x)^{n+1} = (u, v, x)^n \otimes (u, v, x)$

$$= \left(\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k} v^{2k}, \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^k u^{n-2k-1} v^{2k+1}, x^n \right) \otimes (u, v, x)$$

$$= \left(\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k+1} v^{2k} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^{k+1} u^{n-2k-1} v^{2k+2}, \right.$$

$$\left. \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k} v^{2k+1} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^k u^{n-2k} v^{2k+1}, \; x^{n+1} \right)$$

$$= \left(\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k+1} v^{2k} + \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k+1} a^k u^{n-2k+1} v^{2k}, \right.$$

$$\left. \binom{n}{2\left[\frac{n}{2}\right]} a^{\left[\frac{n}{2}\right]} u^{n-2\left[\frac{n}{2}\right]} v^{2\left[\frac{n}{2}\right]+1} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k} a^k u^{n-2k} v^{2k+1} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^k u^{n-2k} v^{2k+1}, x^{n+1} \right)$$

$$= \left(\left(\binom{n+1}{0} u^{n+1} + \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k u^{n-2k+1} v^{2k} + \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k-1} a^k u^{n-2k+1} v^{2k}, \right. \right.$$

$$\left. \binom{n}{2\left[\frac{n}{2}\right]} a^{\left[\frac{n}{2}\right]} u^{n-2\left[\frac{n}{2}\right]} v^{2\left[\frac{n}{2}\right]+1} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \left[\binom{n}{2k} + \binom{n}{2k+1} \right] a^k u^{n-2k} v^{2k+1},\ x^{n+1} \right)$$

$$= \left(\binom{n+1}{0} u^{n+1} + \sum_{k=1}^{\left[\frac{n}{2}\right]} \left[\binom{n}{2k} + \binom{n}{2k-1} \right] a^k u^{n-2k+1} v^{2k}, \right.$$

$$\left. \binom{n}{2\left[\frac{n}{2}\right]} a^{\left[\frac{n}{2}\right]} u^{n-2\left[\frac{n}{2}\right]} v^{2\left[\frac{n}{2}\right]+1} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n+1}{2k+1} a^k u^{n-2k} v^{2k+1},\ x^{n+1} \right)$$

$$= \left(\binom{n+1}{0} u^{n+1} + \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k} a^k u^{n-2k+1} v^{2k},\ \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1} a^k u^{n-2k} v^{2k+1},\ x^{n+1} \right)$$

$$= \left(\sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n+1}{2k} a^k u^{n-2k+1} v^{2k},\ \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1} a^k u^{n-2k} v^{2k+1},\ x^{n+1} \right)$$

so it means that this claim is valid for $n + 1$.

Corollary 2.1.1 ***i***) If $a = 1$ is taken at equation (2), the solution triples which was given by (6) transforms as following:

$$z = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} u^{n-2k} v^{2k}, \quad y = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} u^{n-2k-1} v^{2k+1}, \qquad x = u^2 - v^2$$

ii) If it is accepted that $a = t^2$, $n = 2n_1$ and $(z, ty) = 1$ at equation (2) for a t integer which is different than zero, then the triple of (x^{n_1}, ty, z) becomes a primitive Pythagorean triple [6].

iii) If $a = t^2$, $x = x_1^{\,2}$ and $(z, ty) = 1$ where t and x_1 are integers which are different that zero at equation (2), then the $(x_1^{\,n}, ty, z)$ triple is a primitive Pythagorean triple [6].

Proof. ***i)*** It is from the formulas of (6) [4].

ii) If it is accepted that $a = t^2$, $n = 2n_1$ and $(z, ty) = 1$ at equation (2) for a t integer which is different than zero, then

$$z^2 - ay^2 = x^n \Rightarrow z^2 - t^2 y^2 = x^{2n_1} \Rightarrow z^2 - (ty)^2 = (x^{n_1})^2 \Rightarrow (x^{n_1})^2 + (ty)^2 = z^2$$

the triple of (x^{n_1}, ty, z) becomes a primitive Pythagorean triple [6].

iii) If $a = t^2$, $z = z_1^2$ and $(x,ty) = 1$ where t and z_1 are integers which are different that zero at equation (2), then

$$z^2 - ay^2 = x^n \Rightarrow z^2 - t^2y^2 = (x_1^2)^n \Rightarrow z^2 - (ty)^2 = (x_1^n)^2 \Rightarrow (x_1^n)^2 + (ty)^2 = z^2$$

the (x_1^n, ty, z) triple is a primitive Pythagorean triple [6].

Now lets give the examples below which are the application of theorem and results above.

Example 2.1.1.1 If we accept $a = 1$ at (2) equation then this equation transforms into $z^2 - y^2 = x^n$. Solution formulas corresponding to n = 1, 2, 3, 4, 5, 6, 7 values at solution formulas of (6) are given by the table below.

n	*z*	*y*	*x*
1	u	v	$u^2 - v^2$
2	$u^2 + v^2$	$2uv$	$u^2 - v^2$
3	$u^3 + 3uv^2$	$3u^2v + v^3$	$u^2 - v^2$
4	$u^4 + 6u^2v^2 + v^4$	$4u^3v + 4uv^3$	$u^2 - v^2$
5	$u^5 + 10u^3v^2 + 5uv^4$	$5u^4v + 10u^2v^3 + v^5$	$u^2 - v^2$
6	$u^6 + 15u^4v^2 + 15u^2v^4 + v^6$	$6u^5v + 20u^3v^3 + 6uv^5$	$u^2 - v^2$
7	$u^7 + 21u^5v^2 + 35u^3v^4 + 7uv^6$	$7u^6v + 35u^4v^3 + 21u^2v^5 + v^7$	$u^2 - v^2$

In this situation, the table values below are obtained if *x, y, z* integers are that are shown by $x = x$, $y = y_n$ and $z = z_n$ and that are corresponding to *n* values at formulas of (6) by giving some values in accordant with the hypothesis of Theorem 2.1 in place of *u* and *v*.

u	v	z_2	y_2	z_3	y_3	z_4	y_4	z_5	y_5	z_6	y_6	z_7	y_7	x
2	1	5	4	14	13	41	40	122	121	365	364	1094	1093	3
3	2	13	12	63	62	313	312	1563	1562	**7813**	**7812**	39063	39062	5
4	1	17	8	76	49	353	272	1684	1441	8177	7448	**40156**	**37969**	15
4	3	25	24	172	171	1201	1200	**8404**	**8403**	58825	58824	411772	411771	7
5	2	29	20	185	158	**1241**	**1160**	8525	8282	59189	58460	412865	410678	21
5	3	34	30	260	252	2056	2040	16400	16368	131104	131040	1048640	1048512	16
5	4	41	40	**365**	**364**	3281	3280	**29525**	**29524**	265721	265720	2391485	2391484	9
6	1	**37**	**12**	234	109	1513	888	9966	6841	**66637**	**51012**	450834	372709	35

From above it is calculated as below:

$$(37)^2 - (12)^2 = (35)^2 \Rightarrow (12)^2 + (35)^2 = (37)^2,$$

$$(365)^2 - (364)^2 = (9)^3 \Rightarrow (9)^3 + (364)^2 = (365)^2 \Rightarrow (3)^6 + (364)^2 = (365)^2,$$

$$(1241)^2 - (1160)^2 = (21)^4 \Rightarrow (21)^4 + (1160)^2 = (1241)^2,$$

$$(8404)^2 - (8403)^2 = (7)^5 \Rightarrow (7)^5 + (8403)^2 = (8404)^2,$$

$$(7813)^2 - (7812)^2 = (5)^6 \Rightarrow (5)^6 + (7812)^2 = (7813)^2,$$

$$(40156)^2 - (37969)^2 = (15)^7 \Rightarrow (15)^7 + (37969)^2 = (40156)^2.$$

Again from the table above it is;

$$(29525)^2 - (29524)^2 = (9)^5 \Rightarrow (3^2)^5 + (29524)^2 = (29525)^2$$

$$\Rightarrow (3^5)^2 + (29524)^2 = (29525)^2,$$

$$(66637)^2 - (51012)^2 = (35)^6 \Rightarrow \left[(35)^3\right]^2 + (51012)^2 = (66637)^2$$

and from the expressions at the right side of the equations, the ordered triples of

$$(3^5, 29524, 29525),\ ((35)^3, 51012, 66637)$$

are obtained, and these are Pythagorean triples.

Example 2.1.1.2 If we take $a = 2$ at equation (2) then this equation transforms into $z^2 - 2y^2 = x^n$. If values of n = 1, 2, 3, 4, 5, 6, 7 are taken at solution formulas of (6), the solution formulas that are corresponding to these shall be given by the table below.

n	*z*	*v*	*x*
1	u	v	$u^2 - 2v^2$
2	$u^2 + 2v^2$	2uv	$u^2 - 2v^2$
3	$u^3 + 6uv^2$	$3u^2v + 2v^3$	$u^2 - 2v^2$
4	$u^4 + 12u^2v^2 + 4v^4$	$4u^3v + 8uv^3$	$u^2 - 2v^2$
5	$u^5 + 20u^3v^2 + 20uv^4$	$5u^4v + 20u^2v^3 + 4v^5$	$u^2 - 2v^2$
6	$u^6 + 30u^4v^2 + 60u^2v^4 + 8v^6$	$6u^5v + 40u^3v^3 + 24uv^5$	$u^2 - 2v^2$
7	$u^7 + 42u^5v^2 + 140u^3v^4 + 56uv^6$	$7u^6v + 70u^4v^3 + 84u^2v^5 + 8v^7$	$u^2 - 2v^2$

Here, since the *a* is even, *u* can't be even. In this situation, by giving some values in accordant with the hypothesis of Theorem 2. 1 in place of *u* and *v*, if the *x, y, z* integers that are corresponding to *n* values at formulas of (6) are shown by $x = x$, $y = y_n$ and $z = z_n$, then the table below is generated.

u	v	z_2	y_2	z_3	y_3	z_4	y_4	z_5	y_5	z_6	y_6	z_7	y_7	x
3	1	11	6	45	29	193	132	843	589	3707	2610	16341	7245	7
3	2	17	12	99	70	577	408	3363	2378	19601	13860	114243	80782	1
5	1	27	10	155	77	929	540	5725	3629	35883	23870	227155	155233	23

Example 2.1.1.3 If we take *a* = 3 at equation (2), then the equation (2) transforms into $z^2 - 3y^2 = x^n$. When *n* = 1, 2, 3, 4, 5, 6, 7 is considered at solution triples of (6), the corresponding solution formulas to these are could be seen at the table below.

n	*z*	*y*	*x*
1	u	v	$u^2 - 3v^2$
2	$u^2 + 3v^2$	2uv	$u^2 - 3v^2$
3	$u^3 + 9uv^2$	$3u^2v + 3v^3$	$u^2 - 3v^2$
4	$u^4 + 18u^2v^2 + 9v^4$	$4u^3v + 12uv^3$	$u^2 - 3v^2$
5	$u^5 + 30u^3v^2 + 45uv^4$	$5u^4v + 30u^2v^3 + 9v^5$	$u^2 - 3v^2$
6	$u^6 + 45u^4v^2 + 135u^2v^4 + 27v^6$	$6u^5v + 60u^3v^3 + 54uv^5$	$u^2 - 3v^2$
7	$u^7 + 63u^5v^2 + 315u^3v^4 + 56uv^6$	$7u^6v + 105u^4v^3 + 189u^2v^5 + 27v^7$	$u^2 - 3v^2$

By giving some values in accordant with the hypothesis of Theorem 2.1 in place of *u* and *v* at formula table, if the *x, y, z* integers that are corresponding to *n* values at formulas of (6) are shown by $x = x$, $y = y_n$ and $z = z_n$, then the table below is generated.

u	v	z_2	y_2	z_3	y_3	z_4	y_4	z_5	v_5	z_6	y_6	x
2	1	7	4	26	**15**	**97**	**56**	**362**	**209**	1351	780	1
4	1	19	8	100	51	**553**	**304**	3124	1769	17803	10200	13
4	3	43	24	388	225	3577	2064	32884	18987	302419	174600	-11
5	2	37	20	305	174	**2569**	**1480**	21725	12538	183853	106140	13
5	4	73	40	**845**	**492**	10129	5840	**120725**	**69716**	1440217	831480	-23

From the above table;

$$(26)^2 - 3(15)^2 = (1)^3 \Rightarrow (26)^2 - 3(15)^2 = 1,$$
$$(97)^2 - 3(56)^2 = (1)^4 \Rightarrow (97)^2 - 3(56)^2 = 1,$$
$$(362)^2 - 3(209)^2 = (1)^5 \Rightarrow (362)^2 - 3(209)^2 = 1$$

are obtained and these are solutions of $x^2 - 3y^2 = 1$ Pell equation [6]. Again from the table above

$$(553)^2 - 3(304)^2 = (13)^4 \Rightarrow (13)^4 + 3(304)^2 = (553)^2,$$
$$(2569)^2 - 3(1480)^2 = (13)^4 \Rightarrow (13)^4 + 3(1480)^2 = (2569)^2$$
$$(845)^2 - 3(492)^2 = (-23)^3 \Rightarrow (-23)^3 + 3(492)^2 = (845)^2$$
$$(120725)^2 - 3(69716)^2 = (-23)^5 \Rightarrow (-23)^5 + 3(69716)^2 = (120725)^2$$

is obtained.

Example 2.1.1.4 If we accept *a* = 5 at equation (2) then the equation of (2) shall transforms into form of $z^2 - 5y^2 = x^n$. Solution formulas that are corresponding to *n* = 1, 2, 3, 4, 5, 6, 7 values at solution formulas of (6) could be seen at the table below.

n	z	y	x
1	u	v	$u^2 - 5v^2$
2	$u^2 + 5v^2$	2uv	$u^2 - 5v^2$
3	$u^3 + 15uv^2$	$3u^2v + 5v^3$	$u^2 - 5v^2$
4	$u^4 + 30u^2v^2 + 25v^4$	$4u^3v + 20uv^3$	$u^2 - 5v^2$
5	$u^5 + 50u^3v^2 + 125uv^4$	$5u^4v + 50u^2v^3 + 25v^5$	$u^2 - 5v^2$
6	$u^6 + 75u^4v^2 + 375u^2v^4 + 125v^6$	$6u^5v + 100u^3v^3 + 150uv^5$	$u^2 - 5v^2$
7	$u^7 + 105u^5v^2 + 875u^3v^4 + 875v^6$	$7u^6v + 175u^4v^3 + 525u^2v^5 + 125v^7$	$u^2 - 5v^2$

In this situation if the *x*, *y*, *z* integers that are corresponding to the n values at formulas of (6) by giving some values in accordant with the Theorem 2.1 hypothesis in place of *u* and *v* are shown by $x = x$, $y = y_n$ and $z = z_n$, the table values below are obtained.

u	v	z_2	y_2	z_3	y_3	z_4	y_4	z_5	y_5	z_6	y_6	x
2	1	9	4	38	**17**	**161**	**72**	682	305	2889	1292	-1
3	2	29	12	207	94	1561	696	11643	5210	87029	38916	-11
4	1	21	8	124	53	761	336	4724	2105	29421	13144	11
4	3	61	24	**604**	**279**	6601	2928	70324	31515	754021	337032	-29
6	1	41	12	306	113	2401	984	**19326**	**8305**	157481	69156	31
7	2	69	28	763	334	**8681**	**3864**	99407	44410	1139949	509684	29

From table we obtain;

$$(38)^2 - 5(17)^2 = (-1)^3 \Rightarrow (38)^2 - 5(17)^2 = -1,$$
$$(161)^2 - 5(72)^2 = (-1)^4 \Rightarrow (161)^2 - 5(72)^2 = 1$$

are obtained and these are solutions of $x^2 - 5y^2 = \mp 1$ Pell equations [6]. From table we obtain;

$$(604)^2 - 5(279)^2 = (-29)^3 \Rightarrow (-29)^3 + 5(279)^2 = (604)^2,$$
$$(8681)^2 - 5(3864)^2 = (29)^4 \Rightarrow (29)^4 + 5(3864)^2 = (8681)^2,$$
$$(19326)^2 - 5(8305)^2 = (31)^5 \Rightarrow (31)^5 + 5(8305)^2 = (19326)^2.$$

Theorem 2.2 For $u + av \equiv 1 (mod\ 2)$ and $(u, av) = 1$ let the integer of x be given in the form of $x = u^2 - av^2$ where a, u, v are positive integers, a doesn't contain square factor (multiplier). In this situation;

a) When u is even and v is odd, for a_1, $k \in$ **Z**;

i) If it's $a = 4a_1$, it is $x = 4k$,

ii) If it's $a = 4a_1 + 1$, it is $x = 4k + 3$,

iii) If it's $a = 4a_1 + 2$, then it is $x = 4k + 2$,

iv) If it's $a = 4a_1 + 3$, then it is $x = 4k + 1$.

b) When u is odd and v is even, for a, $k \in$ **Z,** It is in the form of $x = 4k + 1$.

□ **spat.*a*)** If u is even and v is even so if it is $u = 2u_1$ and $v = 2v_1 + 1$, then;

i) If $a = 4a_1$,

$x = u^2 - av^2 = (2u_1)^2 - (4a_1)(2v_1 + 1)^2 = 4u_1^2 - (4a_1)(4v_1^2 + 4v_1 + 1)$
$= 4u_1^2 - (4a_1)[4(v_1^2 + v_1) + 1] = 4[u_1^2 - a_1(2v_1 + 1)^2]$

is found. Here if we say it is $u_1^2 - a_1(2v_1 + 1)^2 = k$ then this means it is $x = 4k$.

ii) If it is $a = 4a_1 + 1$,

$x = u^2 - av^2 = (2u_1)^2 - (4a_1 + 1)(2v_1 + 1)^2 = 4u_1^2 - (4a_1 + 1)(4v_1^2 + 4v_1 + 1)$
$= 4u_1^2 - (4a_1 + 1)[4(v_1^2 + v_1) + 1] = 4[u_1^2 - 4a_1(v_1^2 + v_1) - (v_1^2 + v_1) - a_1] - 1$

is found. Here if we say it is $[u_1^2 - 4a_1(v_1^2 + v_1) - (v_1^2 + v_1) - a_1] = k$ then this means is $x = 4k - 1$ and that is $-1 \equiv 3(mod\ 4)$ and then $x = 4k + 1$.

iii) If it is $a = 4a_1 + 2$,

$x = u^2 - av^2 = (2u_1)^2 - (4a_1 + 2)(2v_1 + 1)^2 = 4u_1^2 - (4a_1 + 2)(4v_1^2 + 4v_1 + 1)$
$= 4u_1^2 - (4a_1 + 2)[4(v_1^2 + v_1) + 1] = 4[u_1^2 - 4a_1v_1^2 - 4a_1v_1 - 4 - 2v_1^2 - 2v_1] - 2$

is found. Here if we say it is $[u_1^2 - 4a_1v_1^2 - 4a_1v_1 - 4 - 2v_1^2 - 2v_1] = k$ then this means is $x = 4k - 2$ and that is

$-2 \equiv 2(mod\ 4)$ and then $x = 4k + 2$.

iv) If it is $a = 4a_1 + 3$,

$$x = u^2 - av^2 = (2u_1)^2 - (4a_1 + 3)(2v_1 + 1)^2 = 4u_1^2 - (4a_1 + 3)(4v_1^2 + 4v_1 + 1)$$

$$= 4u_1^2 - (4a_1 + 3)[4(v_1^2 + v_1) + 1] = 4[u_1^2 - 4a_1v_1^2 - 4a_1v_1 - 4 - 3v_1^2 - 3v_1] - 3$$

is found. Here if we say it is $[u_1^2 - 4a_1v_1^2 - 4a_1v_1 - 4 - 3v_1^2 - 3v_1] = k$ then this means is $x = 4k - 3$ and that is $-3 \equiv 1(mod\ 4)$ and then $x = 4k + 1$.

b) If u is odd and v is even so for $u_1, v_1 \in \mathbf{Z}$ it is $u = 2u_1+1$ and $v = 2v_1$ then

$$x = u^2 - av^2 = (2u_1+1)^2 - a(2v_1)^2 = 4u_1^2 + 4u_1 + 1 - a(4v_1)^2 = 4[u_1^2 + u_1 - av_1^2] + 1$$

is found. Here if we say it is $[u_1^2 + u_1 - av_1^2] = k$ then this means is $x = 4k + 1$.

Lets give examples as the application of this theorem below.

Example 2.2.1 Let *u*, *v* and a_1 are integers and $a = 4a_1 + 1$ and *a* are different than square. Then it is $x = u^2 - av^2 = u^2 - (4a_1+1)v^2$ and *x* takes the values at the table according to values which *u*, *v* and *a* will have.

u	v	$u^2 - v^2$	$u^2 - 5v^2$	$u^2 - 13v^2$	$u^2 - 17v^2$	$u^2 - 21v^2$	$u^2 - 29v^2$	$u^2 - 33v^2$
2	1	3	-1	-9	-13	-17	-25	-29
4	1	15	11	3	-1	-5	-13	-17
4	3	7	-29	-101	-137	-173	-245	-281
6	1	35	31	23	19	15	11	7
6	5	11	-89	-289	-389	-489	-689	-789
8	1	63	59	51	47	43	35	31
8	3	55	19	-53	-89	-125	-197	-233
8	5	39	-61	-261	-361	-461	-661	-761
8	7	15	-181	-573	-769	-965	-1357	-1553

Example 2.2.2 Let it is given in the form of $a = 4a_1 + 3$ where *u* is even, *v* is odd and a_1 are integers. Then $x = u^2 - (4a_1 + 3)v^2$ takes the values at table according to some values which *u*, *v* and *a* will have.

u	v	$u^2 - 3v^2$	$u^2 - 7v^2$	$u^2 - 11v^2$	$u^2 - 15v^2$	$u^2 - 19v^2$	$u^2 - 23v^2$	$u^2 - 27v^2$	$u^2 - 31v^2$	$u^2 - 35v^2$
2	1	1	-3	-7	-11	-15	-19	-23	-27	-31
3	2	-3	-19	-35	-51	-67	-83	-99	-115	-131
4	1	13	9	5	1	-3	-7	-11	-15	-17
4	3	-11	-47	-83	-119	-135	-191	-227	-263	-299
5	2	13	-3	-19	-35	-51	-67	-83	-99	-115
5	4	-23	-87	-151	-215	-279	-343	-407	-471	-535
6	1	33	29	25	21	17	13	9	5	1
6	5	-39	-139	-239	-339	-439	-539	-639	-739	-839
7	2	37	21	5	-11	-27	-43	-59	-75	-91
7	4	1	-63	-35	-71	-107	-143	-179	-243	-307
7	6	-59	-203	-347	-491	-635	-779	-923	-1067	-1131

Example 2.2.3 When it is given in the form of $a = 4a_1+1$ where the *u* is odd, *v* is even and a_1 are integers, $x = u^2 - (4a_1 + 1)v^2$ will take the values at table according to some values which *u*, *v* and *a* will have.

u	v	$u^2 - v^2$	$u^2 - 5v^2$	$u^2 - 13v^2$	$u^2 - 17v^2$	$u^2 - 21v^2$	$u^2 - 29v^2$	$u^2 - 33v^2$	$u^2 - 37v^2$	$u^2 - 41v^2$
3	2	5	-11	-43	-59	-75	-107	-123	-139	-155
5	2	21	5	-27	-43	-59	-91	-107	-123	-139
5	4	9	-55	-183	-247	-311	-439	-503	-567	-631
7	2	45	29	-3	-19	-35	-67	-83	-99	-115
7	4	33	-31	-71	-159	-223	-351	-415	-479	543
7	6	-131	-275	-563	-707	-851	-1139	-1283	-1427	-1571
9	8	17	-239	-751	-1007	-1263	-1519	-1775	-2031	-2287

Example 2.2.4 Let it is given in the form of $a = 2a_1$ where *u* is odd, *v* is even, *a* is square free and a_1 are integers. Then $x = u^2 - (2a_1)v^2$ takes the values at table according to some values which *u*, *v* and *a* will have.

u	v	$u^2 - 2v^2$	$u^2 - 6v^2$	$u^2 - 8v^2$	$u^2 - 10v^2$	$u^2 - 12v^2$	$u^2 - 14v^2$	$u^2 - 18v^2$	$u^2 - 20v^2$	$u^2 - 22v^2$	$u^2 - 24v^2$	$u^2 - 26v^2$
3	2	1	-15	-23	-31	-39	-47	-63	-71	-79	-87	-95
5	2	17	1	-7	-15	-23	-31	-47	-55	-63	-71	-79
5	4	-7	-71	-103	-135	-167	-199	-263	-295	-327	-359	-391
7	2	41	25	17	9	1	-7	-23	-31	-39	-47	-55
13	4	137	73	41	9	-23	-55	-119	-151	-183	-215	-247

Example 2.2.5 Let it is given in the form of $a = 2a_1$ where u is odd, v is odd, a is square free and a_1 are even integers. Then $x = u^2 - (2a_1)v^2$ takes the values at table according to some values which u, v and a will have.

u	v	$u^2 - 8v^2$	$u^2 - 12v^2$	$u^2 - 20v^2$	$u^2 - 24v^2$	$u^2 - 28v^2$	$u^2 - 32v^2$	$u^2 - 40v^2$	$u^2 - 44v^2$	$u^2 - 48v^2$
3	1	1	-3	-11	-15	-19	-23	-31	-35	-39
5	3	-47	-83	-155	-191	-227	-263	-335	-371	-407
7	1	41	37	29	25	21	17	9	5	1
11	3	49	13	-59	-95	-131	-167	-239	-275	-311
29	5	641	541	341	241	141	41	-159	-259	-359

Example 2.2.5 Let it is given in the form of $a = 2a_1$ where u, v, a_1 are odd positive integers *and* a is square free. Then $x = u^2 - (2a_1)v^2$ takes the values at table according to some values which u, v and a will have.

u	v	$u^2 - 2v^2$	$u^2 - 6v^2$	$u^2 - 10v^2$	$u^2 - 14v^2$	$u^2 - 18v^2$	$u^2 - 22v^2$	$u^2 - 26v^2$	$u^2 - 30v^2$	$u^2 - 34v^2$	$u^2 - 38v^2$	$u^2 - 42v^2$
1	1	-1	-5	-9	-13	-17	-21	-25	-29	-33	-37	-41
3	1	7	3	-1	-5	-9	-13	-17	-21	-25	-29	-33
5	3	7	-29	-65	-101	-137	-173	-209	-245	-281	-317	-353
13	3	151	115	79	43	7	-29	-65	-101	-137	-173	-209

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