IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 3 Ver. III (May. - Jun. 2016), PP 111-125
www.iosrjournals.org

On Ecological Model with Host of Prey

Rami Raad Saadi
Networks, Engineering / Iraqi University , Iraq

Abstract: *The objective of this paper is to study the dynamical behavior of three species syn-ecological models. Type of three species mathematical model involving different types of ecological interactions is proposed and analyzed. In the model Holling type –II functional response is a doubted to describe the behavior of predation, for the model, the existence, uniqueness and boundedness of the solution are discussed. The existences and the stability analysis of all possible equilibrium points are studied. Suitable Lyapunov functions are used to study the global dynamics of the proposed model. Numerical simulations are also carried out to investigate the influence of parameters on the dynamical behavior of the model and to support the obtained analytical results of the model.*
Keywords: *Equilibrium Point; Lyapunov Function.*

I. Introduction

Food chains and food webs depict the network of feeding relationships within ecological communities. During the last few decades, a large number of food-chain and food-web systems have been proposed to describe the food transition patterns and processes [1-8]. The basic sequence of energy movement from producer to consumer to decomposer is a food chain. this concept is important in understanding the food and energy relationships of organisms but is much too simplistic to describe the cycling of food in an ecosystem. On the other hand the term food web is used to describe patterns of interlocking food chains. All the intricacies of food cycles cannot be shown on paper even by the most adept artist, even so, food web models offer a picture of the strands supporting the world around us.Living organisms enter into a variety of relationships, such as prey-predator, competition, mutualism, commensalism and so on, among themselves according to the needs of individuals as well as those of species groups. Food webs are one example of interactions between organisms but there are other interactions that go beyond feeding relationships. Recently, number of researchers have been proposed and studied the dynamics of food webs involving some types of these relationships, see for example [9-12] and the references their in.In this paper however , an investigation is devoted to an analytical study of a food web consisting of three species Syn-Ecological system involving a general predator with Holling type-II functional response, in which N_2 is a general predator that preys upon the prey species N_1 and the host N_3, the host N_3 . The prey is a commensal to the host N_3. Further Fig.(1) shows the schematic sketch of the system under investigation. The model equations of the system constitute a set of three first order non-linear ordinary differential equations.

II. The Mathematical Model:

Consider the three species Syn-Ecosymbiosis system consisting of the following interactions prey-predator including commensalisms. It is assumed that the model consists of a prey (for example, Great Egret Bird) whose population density at time T denoted by N_1, the predator (for example, Crocodile) whose population density at time T denoted by N_2, the host (for example, Water Buffalo) whose population density at time T denoted by N_3. Now in order to formulates the mathematical model of the above Syn-Ecosymbiosis system, the following assumptions are adopted:

1. The predator species preys upon the prey, host according to Holling type-II functional response with maximum attack rate $a_i > 0$ *for* $i = 1,2.$ and half saturation constant $b_i > 0$ *for* $i = 1,2.$. While, in the absence of the predator the prey species grows logistically with carrying capacity $k_1 > 0$ and intrinsic growth rate $r_1 > 0$. Moreover in the absence of the prey the predator decay exponential with natural death rate $d_1 > 0$, however in the existence of prey the predator individuals competes each other with intraspecific competition constant rate $d_2 > 0$.
2. The existence of the host N_3 enhance the existence of the prey species N_1 with the commensal constant rate $c > 0$, while the existence of N_1 do not affect (positively or negatively) the existence of N_3.

3. Finally the species N_3 growth logistically with intrinsic growth rates $r_i > 0$ for $i = 1,2$ and carrying capacities $k_i > 0$ for $i = 1,2$ respectively.

Therefore the dynamics of the above proposed model can be represented by the following set of differential equations while the block diagram of this model system can be illustrated in Fig.(1):

$$\begin{aligned}
\frac{dN_1}{dT} &= r_1 N_1\left(1-\frac{N_1}{k_1}\right)-\frac{a_1 N_1}{b_1+N_1}N_2 + c\,N_1\,N_3 \\
\frac{dN_2}{dT} &= e_1\frac{a_1 N_1}{b_1+N_1}N_2 + e_2\frac{a_2 N_3}{b_2+N_3}N_2 - d_1 N_2 - d_2 N_2^2 \\
\frac{dN_3}{dT} &= r_2 N_3\left(1-\frac{N_3}{k_2}\right)-\frac{a_2 N_3}{b_2+N_3}N_2
\end{aligned} \tag{1}$$

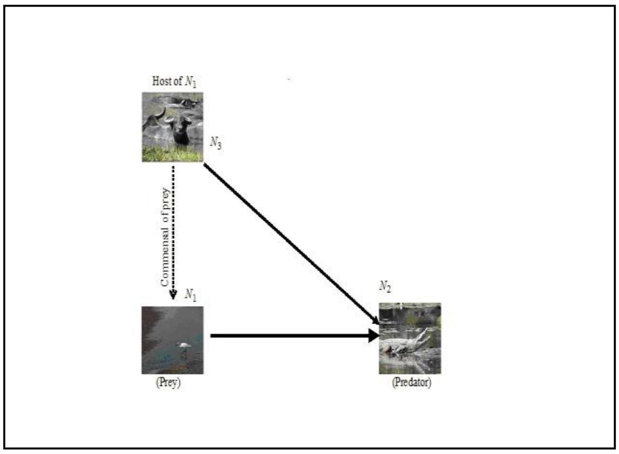

Fig.(1): The block diagram of system (1).

Note that the above proposed model has thirteen parameters in all, which make the analysis difficult. So, in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

$$t = r_1 T \ , \ x = \frac{N_1}{k_1} \ , \ y = \frac{N_2}{k_1} \ , \ z = \frac{c\,N_3}{r_1} \ ,$$

$$u_1 = \frac{a_1}{r_1}\ , u_2 = \frac{b_1}{k_1}\ , u_3 = \frac{a_2}{r_1}\ , \ u_4 = \frac{c\,b_2}{r_1}\ ,$$

$$u_5 = \frac{d_1}{r_1}\ , \ u_6 = \frac{d_2\,k_1}{r_1}, u_7 = \frac{r_2}{r_1}\ , u_8 = \frac{r_1}{c\,k_2}\ , u_9 = \frac{c\,a_2\,k_1}{r_1^2}$$

Then the form of non-dimensional system of system (1) can be written as:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[(1-x)-\frac{u_1 y}{u_2+x}+z\right] = x f_1(x,y,z,w) \\
\frac{dy}{dt} &= y\left[\frac{e_1 u_1 x}{u_2+x}+\frac{e_2 u_3 z}{u_4+z}-u_5-u_6 y\right] = y f_2(x,y,z,w) \\
\frac{dz}{dt} &= z\left[u_7(1-u_8 z)-\frac{u_9 y}{u_4+z}\right] = z f_3(x,y,z,w)
\end{aligned} \tag{2}$$

with $x(0) \ge 0, y(0) \ge 0$ and $z(0) \ge 0$. It is observed that the number of parameters have been reduced from thirteen in the system (1) to eleven in the system (2). Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the positive region $R_+^3 = \{(x,y,z) \in R^3 : x(0) \ge 0, y(0) \ge 0, z(0) \ge 0\}$.

Therefore these functions are Lipschitzian on R_+^3 , and hence the solution of the system (2) exists and is unique. Further, in the following theorem, the boundedness of the solution of the system (2) in R_+^3 is established.

Theorem (1): All the solutions of system (2) which initiate in R_+^3 are uniformly bounded.

Proof:

Let $(x(t), y(t), z(t),)$ be any solution of system (2) with non-negative initial condition $(x_0, y_{0,} z_0)$. Now according to the third equation of system (2) we have

$$\frac{dz}{dt} \le u_7 z(1-u_8 z)$$

So, by using the comparison theorem [11] with the initial point $z(0)=z_0$ we get:

$$z(t) \le \frac{z_0}{z_0 u_8 + (1-z_0 u_8)e^{-u_7 t}}$$

Thus, $\lim_{t\to\infty} z(t) \le \frac{1}{u_8}$. Therefore, $Sup.z(t) \le \frac{1}{u_8}$, $\forall t > 0$.

Finally, according to the first equation of system (2) we have

$$\frac{dx}{dt} \le x(1-x) + xz$$

So, by using the comparison theorem [11] with the initial point we obtain:

$$\lim_{t\to\infty} x(t) \le L \text{ , where } L = 1 + \frac{1}{u_8}$$

Therefore, $Sup.\, x(t) \le L$, $\forall t > 0$.

Now define the function: $M(t) = x(t) + y(t) + z(t)$, and then take the time derivative of $M(t)$ along the solution of the system (2) we get

$$\frac{dM}{dt} \le H - sM$$

where $H = 2\left(L + \frac{L}{2u_8} + \frac{u_7}{u_8}\right)$ and $s = \min\{1, u_5, u_7\}$.

$$\frac{dM}{dt} + sM \le H \text{ , where}$$

Again by solving this differential inequality for the initial value $M(0) = M_0$, we get:

$$M(t) \le \frac{H}{s} + \left(M_0 - \frac{H}{s}\right)e^{-st}$$

then, $\lim_{t\to\infty} M(t) \le \frac{H}{s}$. So, $0 \le M(t) \le \frac{H}{s}$, $\forall t > 0$. Hence all the solutions of system (2) are uniformly bounded and therefore we have finished the proof.

III. The Existence Of Equilibrium Points

In this paper, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2) has five equilibrium points, which are mentioned in the following:

The equilibrium points, $E_0 = (0,0,0)$ known as the washout point always exist. The first two species equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ exists uniquely in $Int.R_+^2$ of $xy-plane$ if there is a positive solution to the following set of equations:

$$(1-x) - \frac{u_1 y}{u_2 + x} = 0 \quad \text{(3a)}$$

$$\frac{e_1 u_1 x}{u_2 + x} - u_5 - u_6 y = 0 \quad \text{(3b)}$$

From equation (3a) we have,

$$y = \frac{(1-x)(u_2+x)}{u_1} \tag{4}$$

By substituting (4) in (3b) and then simplifying the resulting term we obtain that

$$f(x) = \sigma_1 x^3 + \sigma_2 x^2 + \sigma_3 x + \sigma_4 = 0 \tag{5}$$

Where

$$\sigma_1 = u_6 > 0$$
$$\sigma_2 = u_6(2u_2+1)$$
$$\sigma_3 = e_1 u_1^2 - u_1 u_5 + u_2^2 u_6$$
$$\sigma_4 = -u_2(u_1 u_5 + u_2 u_6) < 0$$

Not that Eq.(5) has a unique positive root, namely $\bar{x}$, provided that the following condition :

$$e u_1^2 + u_2^2 u_4 > u_1 u_5 \tag{6}$$

The second two species equilibrium point $E_2 = (\breve{x}, 0, \breve{z}) = \left(\frac{1+u_8}{u_8}, 0, \frac{1}{u_8}\right)$ always exists in the $Int.R_+^2$ of $xz-plane$ where $\breve{x}$ and $\breve{z}$ represent the positive solution of the following system:

$$\left.\begin{aligned} &1 - x + z = 0 \\ &u_7(1-u_8 z) = 0 \end{aligned}\right\} \tag{7}$$

The third two species equilibrium point $E_3 = (0, \hat{y}, \hat{z})$ exists uniquely in the $Int.R_+^2$ of $yz-plane$ if there is a positive solution to the following set of equations:

$$\frac{e_2 u_3 z}{u_4+z} - u_5 - u_6 y = 0 \tag{8a}$$

$$u_7(1-u_8 z) - \frac{u_9 y}{u_4+z} = 0 \tag{8b}$$

From equation (8b) we have,

$$y = \frac{u_7(1-u_8 u_4)z - u_7 u_8 z^2 + u_7 u_4}{u_9} \tag{9}$$

By substituting (9) in (8a) and then simplifying the resulting term we obtain that

$$f(x) = \beta_1 z^2 + \beta_2 z + \beta_3 = 0 \tag{10}$$

Where

$$\beta_1 = u_6 u_7 u_8 > 0$$
$$\beta_2 = e_2 u_3 u_9 - u_5 u_9 - u_7 u_6 + u_7 u_6 u_8 u_4$$
$$\beta_3 = -(u_4 u_5 u_9 + u_4 u_6 u_7) < 0$$

Not that Eq.(10) has a unique positive root, namely $\hat{z}$, provided that the following condition :

$$e_2 u_3 u_9 + u_4 u_6 u_7 u_8 > u_5 u_9 + u_6 u_7 \tag{11}$$

Finally the positive (coexistence) equilibrium point $E_4 = (x^*, y^*, z^*)$ exists in the $Int.R_+^3$ if and only if there is a positive solution of the following set of algebraic equations:

$$1 - x - \frac{u_1 y}{u_2+x} + z = 0 \tag{12}$$

$$\frac{e_1 u_1 x}{u_2+x} + \frac{e_2 u_3 z}{u_4+z} - u_5 - u_6 y = 0 \tag{13}$$

$$u_7(1-u_8) - \frac{u_9 y}{u_4+z} = 0 \tag{14}$$

From (14) we obtain that

$$z = \frac{1}{2}\left[u_7(1-u_4 u_8) + \sqrt{u_7^2(u_4 u_8 - 1)^2 - 4(u_4 u_7 + u_9 y)}\right] \tag{15}$$

$z > 0$ provided that

$$u_4 u_8 < 1 \text{ and } u_7^2(u_4 u_8 - 1)^2 > 4(u_7 u_4 + u_9 y) \tag{16}$$

Then by using (15) in (12) and (13) yield the following two isoclines.

$$f_1(x,y)=u_2+x-u_2x-x^2-u_1y+\frac{1}{2}\left[u_7(1-u_4u_8)+\sqrt{u_7^2(u_4u_8-1)^2-4(u_4u_7+u_9y)}\right](u_2+x)=0 \tag{17}$$

$$f_2(x,y)=x(e_1u_1(u_4+1))+A(e_2u_3-(u_4+u_5))-u_2u_5(u_4+A)=0 \tag{18}$$

where

$$A=\frac{1}{2}\left[u_7(1-u_4u_8)+\sqrt{u_7^2(u_4u_8-1)^2-4u_4u_7}\right]$$

Now from (17) we notice that , when $y\to 0$ then x represent a positive root of the following second order polynomial equation :

$$x^2+\left(u_2-\left(A^*+1\right)\right)x-u_2\left(1+A^*\right)=0 \tag{19}$$

where

$$A^*=\frac{1}{2}\left[u_7(1-u_4u_8)+\sqrt{u_7^2(u_4u_8-1)^2-4u_4u_7}\right]$$

Straight forward computation shows that eq.(19) has a unique positive root namely x_1 if and only if the following condition hold:

$$u_2<A^*+1 \tag{20}$$

Moreover , from eq.(17) we have $\frac{dx}{dy}=\frac{\left(\frac{\partial f_1}{\partial y}\right)}{\left(\frac{\partial f_1}{\partial x}\right)}$, so $\frac{dx}{dy}<0$ if one set of the following sets of conditions holds :

$$\frac{\partial f_1}{\partial y}>0\ ,\ \frac{\partial f_1}{\partial x}<0\quad OR\ \frac{\partial f_1}{\partial y}<0\ ,\ \frac{\partial f_1}{\partial x}>0 \tag{21}$$

Further, from eq.(18) we notice that , when $y\to 0$, then

$x=\dfrac{u_2u_5\left(u_4+A^*\right)}{e_1u_1(u_4+1)+A^*(e_2u_3-(u_4+u_5))}$, note that $x>0$ if and only if

$$u_4+u_5<e_2u_3 \tag{22}$$

In addition since we have $\frac{dx}{dy}=\frac{\left(\frac{\partial f_2}{\partial y}\right)}{\left(\frac{\partial f_2}{\partial x}\right)}$, so $\frac{dx}{dy}>0$ if one set of the following sets of condition holds:-

$$\frac{\partial f_2}{\partial y}>0\ ,\ \frac{\partial f_2}{\partial x}>0\quad OR\ \frac{\partial f_2}{\partial y}<0\ ,\ \frac{\partial f_2}{\partial x}<0 \tag{23}$$

The conditions occur provided that:-

$$\frac{u_2u_5\left(u_4+A^*\right)}{e_1u_1(u_4+1)+A^*(e_2u_3-(u_4+u_5))}<x_1 \tag{24}$$

Then the two isoclines (17) & (18) inters at a unique positive point (x^*,y^*) in the $Int.R_+^2$ of $xy-plane$. Substituting the value of y^* in eq.(15) yield that $z(y^*)=z^*$ which is positive order condition (16) accordingly , the positive equilibrium point $E_4=\left(x^*,y^*,z^*\right)$ exists uniquely in the $Int.R_+^3$, if in addition to conditions (16) , (20) , (22) and (24), the isocline $f_1(x,y)=0$ intersect the $x-axis$ at the positive value namely x_1 .

IV. The Stability Analysis

In this section the stability analysis of all feasible equilibrium points of system (2) is studied analytical with the help of linearization method.

Notation $\lambda_{ix},\lambda_{iy}$ *and* λ_{iz} represent the eigenvalues of $J(E_i)$ that describe the dynamics in the $x-$ direction, $y-$direction and $z-$ direction respectively $(where\ i=0,1,2,3,4)$.

Note that it is easy to verify that the Jacobian matrix of system (2) at the trivial equilibrium point $E_0 = (0,0,0)$ can be written in the form:

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -u_5 & 0 \\ 0 & 0 & u_7 \end{bmatrix}$$

Thus the eigenvalues of $J(E_0)$ are $\lambda_{0x} = 1 > 0, \lambda_{0y} = -u_5 < 0$ *and* $\lambda_{0z} = u_7 > 0$. Then E_0 is a saddle point (unstable).

The Jacobian matrix of system (2) at the first two species equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ can be written as:

$$J(E_1) = \begin{bmatrix} \bar{x}\left(-1 + \dfrac{u_1 \bar{y}}{(u_2 + \bar{x})^2}\right) & \dfrac{-u_1 \bar{x}}{u_2 + \bar{x}} & \bar{x} \\ \dfrac{e_1 u_2 \bar{y}}{(u_2 + \bar{x})^2} & -u_6 \bar{y} & \dfrac{e_2 u_4 u_3 \bar{y}}{u_4^2} \\ 0 & 0 & u_7 - \dfrac{u_9 \bar{y}}{u_4} \end{bmatrix}$$

Hence the characteristic equation of $J(E_1)$ is given by:-

$$\left[\lambda^2 + A_1 \lambda + A_2\right]\left(u_7 - \frac{u_9 \bar{y}}{u_4} - \lambda\right) = 0$$

Where

$$A_1 = -\bar{x}\left(-1 + \frac{u_1 \bar{y}}{(u_2 + \bar{x})}\right) + u_6 \bar{y}$$

$$A_2 = \bar{x}\left(-1 + \frac{u_1 \bar{y}}{(u_2 + \bar{x})^2}\right)(-u_6 \bar{y}) + \frac{e_1 u_1 u_2 \bar{x} \bar{y}}{(u_2 + \bar{x})^3}$$

So , either

$$u_7 - \frac{u_9 \bar{y}}{u_4} - \lambda = 0$$

Which give the eigenvalue of $J(E_1)$ by:

$$\lambda_{1z} = u_7 - \frac{u_9 \bar{y}}{u_4}$$

Or

$$\lambda^2 + A_1 \lambda + A_2 = 0$$

Which gives the other two eigenvalues of $J(E_1)$ with respect to λ_{1x} *and* λ_{1y} provided that the following conditions are satisfied:-

$$\frac{u_1 \bar{y}}{(u_2 + \bar{x})^2} < 1 \tag{25}$$

$$u_7 < \frac{u_9 \bar{y}}{u_4} \tag{26}$$

So, E_1 is locally asymptotically stable. However, it is a saddle point otherwise.

The Jacobian matrix of system (2) at the second two species equilibrium point $E_2 = (\breve{x}, 0, \breve{z}) = \left(\dfrac{1 + u_8}{u_8}, 0, \dfrac{1}{u_8}\right)$ can be written as:

$$J(E_2)=\begin{bmatrix} -\breve{x} & \dfrac{-u_1\breve{x}}{u_2+\breve{x}} & \breve{x} \\ 0 & \dfrac{e_1u_1\breve{x}}{u_2+\breve{x}}+\dfrac{e_2u_3\breve{z}}{u_4+\breve{z}}-u_5 & 0 \\ 0 & \dfrac{-u_9\breve{z}}{u_4+\breve{z}} & -u_7u_8\breve{z} \end{bmatrix}$$

Hence the characteristic equation of $J(E_2)$ is given by:-

$$(-\breve{x}-\lambda)\left[\lambda^2+B_1\lambda+B_2\right]=0$$

Where

$$B_1=-\left[\frac{e_1u_1\breve{x}}{u_2+\breve{x}}+\frac{e_2u_3\breve{z}}{u_4+\breve{z}}-u_5\right]+u_7u_8\breve{z}$$

$$B_2=\left[\frac{e_1u_1\breve{x}}{u_2+\breve{x}}+\frac{e_2u_3\breve{z}}{u_4+\breve{z}}-u_5\right](-u_7u_8\breve{z})$$

So, either

$$-\breve{x}-\lambda=0$$

Which give the eigenvalue of $J(E_2)$ by:

$$\lambda_{2x}=-\breve{x}<0$$

Or

$$\lambda^2+B_1\lambda+B_2=0$$

Which gives the other two eigenvalues of $J(E_2)$ with respect to λ_{2y} *and* λ_{2z} provided that the following conditions are satisfied:-

$$\frac{e_1u_1\breve{x}}{u_2+\breve{x}}+\frac{e_2u_3\breve{z}}{u_4+\breve{z}}<u_5 \tag{27}$$

So, E_2 is locally asymptotically stable. However, it is a saddle point otherwise.

The Jacobian matrix of system (2) at the third two species equilibrium point $E_3=(0,\hat{y},\hat{z})$ can be written as:

$$J(E_3)=\begin{bmatrix} 1-\dfrac{u_1\hat{y}}{u_4}+\hat{z} & 0 & 0 \\ \dfrac{e_1u_1u_2\hat{y}}{u_4^2} & -u_6\hat{y} & \dfrac{e_2u_3u_4\hat{y}}{(u_4+\hat{z})^2} \\ 0 & \dfrac{-u_9\hat{z}}{u_4+\hat{z}} & -u_7u_8\hat{z}+\dfrac{u_9\hat{y}\hat{z}}{(u_4+\hat{z})^2} \end{bmatrix}$$

Hence the characteristic equation of $J(E_3)$ is given by:-

$$\left(1-\frac{u_1\hat{y}}{u_2}+\hat{z}-\lambda\right)\left[\lambda^2+C_1\lambda+C_2\right]=0$$

Where

$$C_1=-\left[-(u_6\hat{y}+u_7u_8\hat{z})+\frac{u_9\hat{y}\hat{z}}{(u_4+\hat{z})^2}\right]$$

$$C_2=-u_6\hat{y}\left[-u_7u_8\hat{z}+\frac{u_9\hat{y}\hat{z}}{(u_4+\hat{z})^2}\right]+\frac{e_2u_3u_4u_9\hat{y}\hat{z}}{(u_4+\hat{z})^3}$$

So, either

$$1-\frac{u_1\hat{y}}{u_2}+\hat{z}-\lambda=0$$

Which give the eigenvalue of $J(E_3)$ by:

$$\lambda_{3x}=1+\hat{z}-\frac{u_1\hat{y}}{u_2}$$

Or

$$\lambda^2 + C_1\lambda + C_2 = 0$$

Which gives the other two eigenvalues of $J(E_3)$ with respect to λ_{3y} *and* λ_{3z} provided that the following conditions are satisfied:-

$$u_6\hat{y} + u_7u_8\hat{z} > \frac{u_9\hat{y}\hat{z}}{(u_4+\hat{z})} \tag{28}$$

$$u_7u_8\hat{z} > \frac{u_9\hat{y}\hat{z}}{(u_4+\hat{z})^2} \tag{29}$$

$$1+\hat{z} < \frac{u_1\hat{y}}{u_2} \tag{30}$$

So, E_3 is locally asymptotically stable. However, it is a saddle point otherwise.

Finely the Jacobian matrix of system (2) at the positive equilibrium point E_4 can be written as:

$$J(E_4) = \begin{bmatrix} -x^*\left[-1+\dfrac{u_1y^*}{(u_2+x^*)^2}\right] & \dfrac{-u_1x^*}{u_2+x^*} & x^* \\ \dfrac{e_1u_1u_2y^*}{(u_2+x^*)^2} & -u_6y^* & \dfrac{e_2u_3u_4z^*}{(u_4+z^*)^2} \\ 0 & \dfrac{-u_9z^*}{u_4+z^*} & z^*\left[-u_7u_8+\dfrac{u_9y^*}{(u_4+z^*)^2}\right] \end{bmatrix}$$

Thus the characteristic equation of $J(E_4)$ is given by:

$$\lambda^3 + K_1\lambda^2 + K_2\lambda + K_3 = 0 \tag{31}$$

Where

$$K_1 = -\left[x^* + \frac{u_9y^*z^*}{(u_4+z^*)^2} - \left(\frac{u_1x^*y^*}{(u_2+x^*)^2} + u_7u_8z^* + u_6y^*\right)\right]$$

$$K_2 = u_6x^*y^*\left[-1+\frac{u_1y^*}{(u_2+x^*)^2}\right] + \frac{e_1u_1^2u_2x^*y^*}{(u_2+x^*)^3} - x^*z^*\left[-1+\frac{u_1y^*}{(u_2+x^*)^2}\right]\left[-u_7u_8+\frac{u_9y^*}{(u_4+z^*)^2}\right] - u_6y^*z^*\left[-u_7u_8+\frac{u_9y^*}{(u_4+z^*)^2}\right] + \frac{e_2u_3u_4u_9(z^*)^2}{(u_4+z^*)^3}$$

$$K_3 = -u_6x^*y^*z^*\left(\left[-1+\frac{u_1y^*}{(u_2+x^*)^2}\right]\left[-u_7u_8+\frac{u_9y^*}{(u_4+z^*)^2}\right] + \frac{e_2u_3u_4u_9(z^*)^2}{(u_4+z^*)^3}\right) - \frac{e_1u_1^2u_2x^*y^*z^*}{(u_2+x^*)^3}\left[-u_7u_8+\frac{u_9y^*}{(u_4+z^*)^2}\right] + \frac{e_1u_1u_2u_9x^*y^*z^*}{(u_2+x^*)^3}$$

So using Routh-Hawirtiz criterion equation (31) has roots (eigenvalues) with negative real parts if and only if $K_1 > 0$ $, K_3 > 0$ *and* $\Delta = K_1K_2 - K_3 > 0$

$K_1 > 0$ If the following conditions satisfy:

$$\frac{u_1y^*}{(u_2+x^*)^2} > 1 \tag{32}$$

$$u_7u_8 > \frac{u_9y^*}{(u_4+z^*)^2} \tag{33}$$

$K_3 > 0$ Always satisfy.
Now direct computation gives that:
$\Delta = S_1 + S_2 + S_3 - S_4 - S_5$

Where

$$S_1 = \left(-x^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]-u_6 y^*\right)\left[-u_6 y^* x^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]-\frac{e_1 u_1^2 u_2 x^*}{(u_2+x^*)^3}\right]$$

$$S_2 = \left(-u_6 y^* + z^*\left[-u_7 u_8 + \frac{u_9 y^*}{(u_4+z^*)^2}\right]\right)\left(\frac{-e_2 u_3 u_4 z^*}{(u_4+z^*)^3} + u_6 y^* z^*\left[-u_7 u_8 + \frac{u_9 y^*}{(u_4+z^*)^2}\right]\right)$$

$$S_3 = x^* z^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]\left[-u_7 u_8 + \frac{u_9 y^*}{(u_4+z^*)^2}\right]\left(-x^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]+z^*\left[-u_7 u_8 + \frac{u_9 y^*}{(u_4+z^*)}\right]\right)$$

$$S_4 = 2u_6 x^* y^* z^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]\left[-u_7 u_8 + \frac{u_9 y_5}{(u_4+z^*)^2}\right]$$

$$S_5 = \frac{e_1 u_1 u_2 u_9 x^* y^* z^*}{(u_2+x^*)^2 (u_2+z^*)}$$

So, $\Delta > 0$ under the following condition:

$$S_1 + S_2 + S_3 - S_4 > S_5 \qquad (34)$$

So, E_4 is locally asymptotically stable if and only if conditions (32), (33) and (34) are hold. However, it is a saddle point otherwise.

V. Global Stability Analysis

In this section the global stability analysis of the equilibrium points, which are locally asymptotically stable of system (2) is studied analytically with the help of Lyapunov method as shown in the following theorems

Theorem (2): Assume that, the equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ of system (2) is locally asymptotically stable and the following conditions hold

$$\frac{u_1 \bar{y}}{(u_2+x)(u_2+\bar{x})} < 1 \qquad (35a)$$

$$\frac{((e_1-1)u_2-\bar{x})u_1}{(u_2+x)(u_2+\bar{x})} < 4u_6\left(1-\frac{u_1 \bar{y}}{(u_2+x)(u_2+\bar{x})}\right) \qquad (35b)$$

$$x + \frac{e_1 u_3 \bar{y}}{u_4+z} + \frac{e_2 u_3 u_7}{u_9} < \bar{x} \qquad (35c)$$

Then the equilibrium point E_1 of system (2) is globally asymptotically stable in the R_+^3.
Proof: Consider the following function

$$V_1(x,y,z) = \left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + \left(y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}}\right) + z$$

Clearly $V_1 : R_+^3 \to R$ is a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_1}{dt} = -\left[1-\frac{u_1 \bar{y}}{(u_2+x)(u_2+\bar{x})}\right](x-\bar{x})^2 + [(e_1-1)u_2-\bar{x}]\frac{u_1(x-\bar{x})(y-\bar{y})}{(u_2+x)(u_2+\bar{x})} - u_6(y-\bar{y})^2$$
$$+ (x-\bar{x})z + \frac{e_2 u_3 (y-\bar{y})z}{u_4+z} + u_7 z - \frac{u_9 yz}{u_4+z} - u_7 u_8 z^2$$

So by using the conditions (35a),(35b) and(35c) we obtain that

$$\frac{dV_1}{dt} \le -\left[\sqrt{1-\frac{u_1\bar{y}}{(u_2+x)(u_2+\bar{x})}}(x-\bar{x})-\sqrt{u_6}(y-\bar{y})\right]^2 + \left(x-\bar{x}+\frac{e_2u_3\,\bar{y}}{u_4+z}+\frac{e_2u_3u_7}{u_9}\right)z$$

Then $\frac{dV_1}{dt}$ is negative definite and hence V_1 is a Lyapunov function. Thus E_1 is a globally asymptotically stable and the proof is complete.

Theorem (3): Assume that, the equilibrium point $E_2 = (\breve{x},0,\breve{z})$ of system (2) is locally asymptotically stable and the following conditions hold

$$1 < 4\sqrt{u_7\,u_8} \tag{36a}$$

$$\frac{u_1\,\breve{x}}{u_2+x}+\frac{e_1u_1\,x}{u_2+x}+\frac{e_2\,u_3z}{u_4+z}+\frac{u_9\breve{z}}{u_4+z} < u_5 \tag{36b}$$

Then the equilibrium point E_2 of system (2) is globally asymptotically stable in the R_+^3.

Proof: Consider the following function

$$V_2(x,y,z) = \left(x-\breve{x}-\breve{x}\ln\frac{x}{\breve{x}}\right)+y+\left(z-\breve{z}-\breve{z}\ln\frac{z}{\breve{z}}\right)$$

Clearly $V_2 : R_+^3 \to R$ is a C^1 positive definite function. Now by differentiating V_2 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_2}{dt} = -(x-\breve{x})^2 - (x-\breve{x})(z-\breve{z}) - u_7u_8\,(z-\breve{z})^2$$
$$-\frac{u_1xy}{u_2+x}+\frac{u_1\breve{x}y}{u_2+x}+\frac{e_1u_1xy}{u_2+x}+\frac{e_2u_3yz}{u_4+z}-u_5y-u_6y^2-\frac{u_9yz}{u_4+z}+\frac{u_9y\breve{z}}{u_4+z}$$

So by using the conditions (36a) and (36b) we obtain that

$$\frac{dV_2}{dt} \le -\left[(x-\breve{x})-\sqrt{u_7u_8}(z-\breve{z})\right]^2 - \left[u_5-\left(\frac{u_1\breve{x}}{u_2+x}+\frac{e_1u_1x}{u_2+x}+\frac{e_2u_3z}{u_4+z}+\frac{u_9\breve{z}}{u_4+z}\right)\right]y$$

Then $\frac{dV_2}{dt}$ is negative definite and hence V_2 is a Lyapunov function. Thus E_2 is a globally asymptotically stable and the proof is complete.

Theorem (4): Assume that, the equilibrium point $E_3 = (0,\hat{y},\hat{z})$ of system (2) is locally asymptotically stable and the following conditions hold

$$\left(\frac{e_2u_3}{(u_4+z)(u_4+\hat{z})}\right)^2 < 4u_6u_7u_8 \tag{37a}$$

$$\frac{e_1u_1y}{u_2+x}+z+1 < \frac{e_1u_1\hat{y}}{u_2+x} \tag{37b}$$

$$\frac{u_9\hat{z}}{(u_4+z)(u_4+\hat{z})} < \frac{u_1x}{u_2+x}+\frac{u_9z}{(u_4+z)(u_4+\hat{z})} \tag{37c}$$

$$z < \hat{z} \tag{37d}$$

Then the equilibrium point E_3 of system (2) is globally asymptotically stable in the R_+^3.

Proof: Consider the following function

$$V_3(x,y,z) = x+\left(y-\hat{y}-\hat{y}\ln\frac{y}{\hat{y}}\right)+\left(z-\hat{z}-\breve{z}\ln\frac{z}{\hat{z}}\right)$$

Clearly $V_3 : R_+^3 \to R$ is a C^1 positive definite function. Now by differentiating V_3 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_3}{dt} = -u_6(y-\hat{y})^2 + (y-\hat{y})(z-\hat{z})\frac{e_2u_3}{(u_4+z)(u_4+\hat{z})} - u_7u_8(z-\hat{z})^2 + x - x^2$$
$$- \frac{u_1xy}{u_2+x} + xz + \frac{e_1u_1xy}{u_2+x} - \frac{e_1u_1x\hat{y}}{u_2+x} - \frac{u_9yz}{(u_4+z)(u_4+\hat{z})}$$
$$- \frac{u_9\hat{y}\hat{z}}{(u_4+z)(u_4+\hat{z})} + \frac{u_9\hat{y}z}{(u_4+z)(u_4+\hat{z})} + \frac{u_9y\hat{z}}{(u_4+z)(u_4+\hat{z})}$$

So by using the conditions (37a), (37b), (37c) and (37d) we obtain that

$$\frac{dV_3}{dt} \le -\left[\sqrt{6}(y-\hat{y}) - \sqrt{u_7u_8}(z-\hat{z})\right]^2 - x\left[\frac{e_1u_1\hat{y}}{u_2+x} - \left(\frac{e_1u_1y}{u_2+x} + z + 1\right)\right]$$
$$- y\left[\frac{u_1x}{u_2+x} + \frac{u_9z}{(u_4+z)(u_4+\hat{z})} - \frac{u_9\hat{z}}{(u_4+z)(u_4+\hat{z})}\right]$$
$$- \frac{u_9\hat{y}}{(u_4+z)(u_4+\hat{z})}(\hat{z}-z)$$

Then $\frac{dV_3}{dt}$ is negative definite and hence V_3 is a Lyapunov function. Thus E_3 is a globally asymptotically stable and the proof is complete.

Theorem (5): Assume that, the equilibrium point $E_4 = (x^*, y^*, z^*)$ of system (2) is locally asymptotically stable and the following conditions hold

$$\left(\frac{e_1u_1u_2}{(u_2+x)(u_2+x^*)}\right)^2 < u_6 \qquad (38a)$$

$$1 < u_7u_8 \qquad (38b)$$

$$\left(\frac{e_2u_3}{(u_4+z)(u_4+z^*)}\right)^2 < u_6u_7u_8 \qquad (38c)$$

$$y < y^* \quad , \quad z < z^* \qquad (38d)$$

Then the equilibrium point E_4 of system (2) is globally asymptotically stable in the R_+^3.

Proof: Consider the following function

$$V_4(x,y,z) = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \left(y - y^* - y^*\ln\frac{y}{y^*}\right) + \left(z - z^* - z^*\ln\frac{z}{z^*}\right)$$

Clearly $V_4 : R_+^3 \to R$ is a C^1 positive definite function. Now by differentiating V_4 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_4}{dt} = -\frac{1}{2}(x-x^*) + \frac{e_1u_1u_2}{(u_2+x)(u_2+x^*)}(x-x^*)(y-y^*) - \frac{u_6}{2}(y-y^*)^2 - \frac{1}{2}(x-x^*)^2$$
$$+ (x-x^*)(z-z^*) - \frac{u_7u_8}{2}(z-z^*)^2 - \frac{u_6}{2}(y-y^*) + \frac{e_2u_3}{(u_4+z)(u_4+z^*)}(y-y^*)(z-z^*)$$
$$- \frac{u_7u_8}{2}(z-z^*)^2 + (x-x^*)^2(y-y^*)\frac{u_1}{(u_2+x)(u_2+x^*)} - (y-y^*)(z-z^*)\frac{u_9}{(u_4+z)(u_4+z^*)}$$

So, by using the conditions (38a), (38b),(38c) and (38d) we obtain

$$\frac{dV_4}{dt} \le -\left[\frac{1}{\sqrt{2}}(x-x^*) - \sqrt{\frac{u_6}{2}}(y-y^*)\right]^2 - \left[\frac{1}{\sqrt{2}}(x-x^*) - \sqrt{\frac{u_7u_8}{2}}(z-z^*)\right]^2$$
$$- \left[\sqrt{\frac{u_6}{2}}(y-y^*)^2 - \sqrt{\frac{u_7u_8}{2}}(z-z^*)\right]^2 + (x-x^*)^2(y-y^*)\frac{u_1}{(u_2+x)(u_2+x^*)}$$

Then $\frac{dV_4}{dt}$ is negative definite and hence V_4 is a Lyapunov function. Thus E_4 is a globally asymptotically stable in the R_+^3 and the proof is complete.

VI. Numerical Simulation

In this paper the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Fig. (2).

Note that, from now onward the **blue**, **green** and **red** colors are used to describing the trajectories of the prey x, the predator y and the Host z.

$$\begin{gathered} u_1 = 0.6,\, u_2 = 0.25,\, u_3 = 3.9, u_4 = 0.05, u_5 = 2, u_6 = 0.5, u_7 = 2 \\ u_8 = 0.75, u_9 = 0.8\,, e_1 = 0.5\,, e_2 = 0.5. \end{gathered} \tag{39}$$

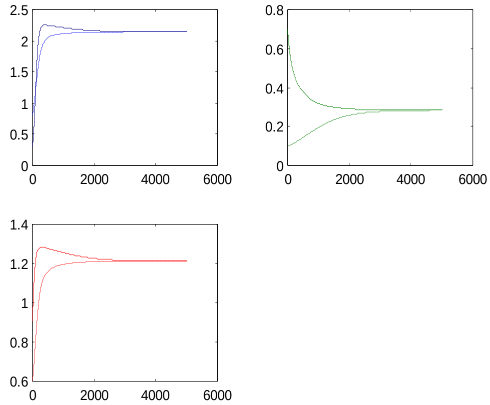

Fig. (2): Time series of the solution of system (2) that started from two different initial points (0.8, 0.7, 0.6) and (0.9, 0.3, 0.1) for the data given by Eq. (39). (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time.

Clearly, Fig.(2) shows that system (2) has a globally asymptotically stable point as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_5 = (x^*, y^*, z^*)$ starting from two different initial points and this is confirming our obtained analytical results.

Now in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (39) with varying one or more parameter each time. It is observed that for the data as given in Eq. (39) with $u_1 \le 0.1$, the solution of system (2) approaches asymptotically to $E_2 = (\breve{x}, 0, \breve{z})$ in the $xz-$ plane as shown in Fig. (3)

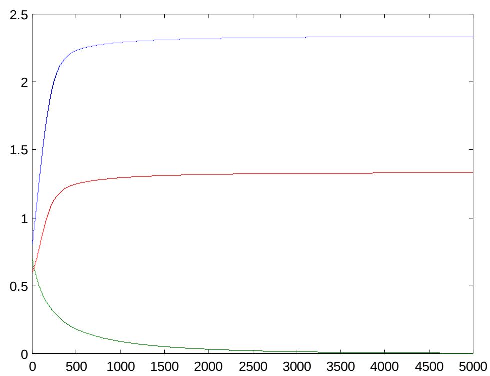

Fig.(3): Time series of the solution of system (2) for the data given by Eq.(39) with $u_1 \le 0.1$, which approaches to $(2.33, 0, 1.33)$ in $xz-$ plane

For the parameters values given in Eq. (39) with $0.2 \le u_1 \le 1.5$, it is observed that the solution of system (2) approaches asymptotically to $E_4 = (x^*, y^*, z^*)$ in the xyz – space as shown in Fig. (4)

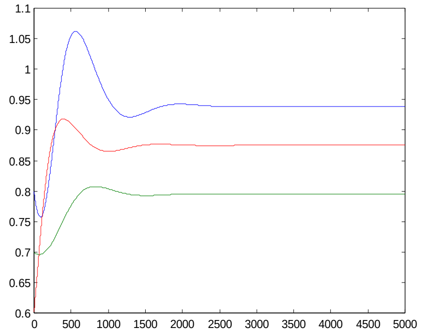

Fig.(4): Time series of the solution of system (2) for the data given by Eq.(39) with $0.2 \le u_1 \le 1.5$, which approaches to $(0.93, 0.79, 0.87)$ in xyz – space.

For the parameters values given in Eq. (39) with $5 \le u_1 \le 8.8$, it is observed that, the solution of system (2) approaches asymptotically to $E_3 = (0, \hat{y}, \hat{z})$ in the yz – plane as shown in Fig. (5).

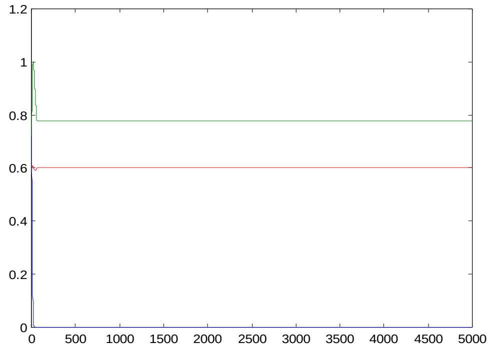

Fig.(5): Time series of the solution of system (2) for the data given by Eq.(39) with $5 \le u_1 \le 8.8$, which approaches to $(0, 0.77, 0.6)$ in yz – plane.

\

For the parameters values given in Eq. (39) with $u_5 = 0.01$, it is observed that the solution of system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0)$ in the xy – plane as shown in Fig. (6).

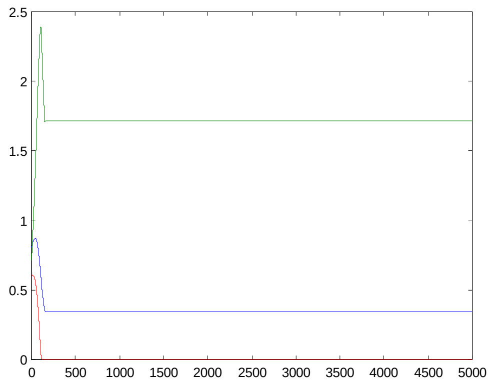

Fig.(6): Time series of the solution of system (2) for the data given by Eq.(39) with $u_5 = 0.01$, which approaches to $(0.34, 1.71, 0)$ in xy – plane.

Keeping the above in view we will summarize our obtained numerical results in the form of table as shown below.

Table (1): numerical behavior of system (2) as changing in a specific parameter keeping other parameters fixed as in Eq.(39)

Parameters varied in system (2)	Numerical behavior of system (2)
$1.6 \le u_1 \le 4.9$ $u_1 \ge 8.9$	Approaches to periodic. Approaches to stable point in $Int.R_+^3$.
$0.01 \le u_2 \le 6$ $u_2 \ge 7$	Approaches to stable point in $Int.R_+^3$. Approaches to stable point in $xz-plane$.
$0.01 \le u_3 \le 3.5$ $3.6 \le u_3 \le 4.8$ $4.9 \le u_3 \le 12$ $u_3 \ge 12.1$	Approaches to stable point in $xz-plane$. Approaches to stable point in $Int.R_+^3$. Approaches to periodic. Approaches to stable point in $xy-plane$.
$u_4 = 0.01$ $0.02 \le u_4 \le 0.2$ $u_4 \ge 0.3$	Approaches to stable point in $xz-plane$. Approaches to stable point in $Int.R_+^3$. Approaches to stable point in $xz-plane$.
$0.5 \le u_5 \le 1.5$ $1.6 \le u_5 \le 2.1$ $u_5 \ge 2.2$	Approaches to periodic. Approaches to stable point in $Int.R_+^3$. Approaches to stable point in $xz-plane$.
$0.01 \le u_6 \le 13.6$ $u_6 \ge 13.7$	Approaches to stable point in $Int.R_+^3$. Approaches to stable point in $xz-plane$.
$0.03 \le u_7 \le 0.4$ $u_7 \ge 0.41$	Approaches to stable point in $xz-plane$. Approaches to stable point in $Int.R_+^3$.
$u_8 = 0.01$ $0.02 \le u_8 \le 2.8$ $u_8 \ge 2.9$	Approaches to stable point in $xz-plane$. Approaches to stable point in $Int.R_+^3$. Approaches to stable point in $xz-plane$.
$0.01 \le u_9 \le 2.2$ $2.3 \le u_9 \le 4$ $u_9 \ge 4.1$	Approaches to stable point in $Int.R_+^3$. Approaches to stable point in $xz-plane$. Approaches to stable point in $xy-plane$
$0.01 \le e_1 \le 0.1$ $0.2 \le e_1 \le 1$	Approaches to stable point in $xz-plane$. Approaches to stable point in $Int.R_+^3$.
$0.01 \le e_2 \le 0.4$ $0.5 \le e_2 \le 0.6$ $0.7 \le e_2 \le 1$	Approaches to stable point in $xz-plane$. Approaches to stable point in $Int.R_+^3$. Approaches to periodic.

VII. Conclusion And Discussion

In this paper however, an investigation is devoted to an analytical study of a food web consisting of three species Syn-Ecological system involving a general predator with Holling type-II functional response.

The existence, uniqueness and boundedness of the solution of the system are discussed. The existence of all possible equilibrium points is studied. The local and global dynamical behaviors of the system are studied analytically as well as numerically. Finally to understand the effect of varying each parameter on the global dynamics of system (2) and to confirm our obtained analytical results, system (2) has been solved numerically for a biological feasible set of hypothetical parameters values and the following results are obtained:

1. For the set of hypothetical parameters values given Eq. (39), the system (2) approaches asymptotically to globally stable positive equilibrium point.
2. For the set of data by Eq.(39), system(2) has a globally asymptotically stable positive point in the $Int.R_+^3$. However as the attack rate u_1 decreases then the predator species will faces extinction and the solution of system (2) approaches to $E_2 = (\breve{x}, 0, \breve{z})$ in the first quadrant $xz-plane$ .while increasing u_1 will causes

destabilizing of system (2) and the point in the $Int.R_{+}^{3}$ .it is observed that the conversion rate parameter e_1 and the intrinsic growth rate.

3. As the half saturation rate u_2 decreases keeping the rest of parameters as in Eq. (39) then again the solution of system(2) approaches asymptotically stable positive point in the $Int.R_{+}^{3}$. Otherwise the systems still have approaches to $E_2 = (\breve{x}, 0, \breve{z})$ in the first quadrant $xz - plane$. It is observed that the half saturation rate parameter u_4 and the carrying capacity rate u_8 .
4. As the attack rate u_3 decreases keeping the rest of parameters as in Eq. (39) then again the solution of system (2) approaches to $E_2 = (\breve{x}, 0, \breve{z})$ the fist quadrant $xz - plane$. Otherwise the systems still have approaches to $E_1 = (\bar{x}, \bar{y}, 0)$ in the first quadrant $xy - plane$.
5. As the natural death u_5 decreases keeping the rest of parameters as in Eq. (39) then again the solution of system (2) approaches to $E_1 = (\bar{x}, \bar{y}, 0)$ the fist quadrant $xy - plane$. Otherwise the systems still have approaches to $E_2 = (\breve{x}, 0, \breve{z})$ in the first quadrant $xz - plane$.
6. As the attack rate u_9 decreases keeping the rest of parameters as in Eq. (39) then again the solution of system(2) approaches asymptotically stable positive point in the $Int.R_{+}^{3}$. Otherwise the systems still have approaches to $E_1 = (\bar{x}, \bar{y}, 0)$ in the first quadrant $xy - plane$.
7. As the conversion rate e_2 decreases keeping the rest of parameters as in Eq. (39) then again the solution of system (2) approaches to $E_2 = (\breve{x}, 0, \breve{z})$ the fist quadrant $xz - plane$. While increasing e_2 will causes destabilizing of system (2) and the solution approaches to a limit cycle in $Int.R_{+}^{3}$.

References

Books:

[1]. Arditi R., and L.R. Ginzberg, Coupling in predator-prey dynamics: ratio dependence, J. of Theoretical Biology, 139**,** p. 311-326, 1989.

[2]. Archana Reddy R. 2009. On the stability of some mathematical models in biosciences- interacting species. PhD Thesis. JNTU.

[3]. Beddington J.R. , Mutual interference between parasites or predators and its effect on searching efficiency, J. of Animal Ecology, 44**,** p. 331-340, 1975.

[4]. Blanchard P., et. al., Differential equations, Boston university , 2002.

[5]. Cushing J. M. 1977. Integro - differential equations and Delay Models in Population Dynamics. Lecture Notes in Biomathematics. Springer- Verlag Heidelberg. Vol. 20.

[6]. DeAngelis D.L., R.A. Goldstein, and R.V. O'Neill, A model for trpphic interactions, Ecology, 56**,** p. 881-892, 1975.

[7]. Freedman H.I., Deterministic mathematical models in population, ecology, Marcel Dekker, Inc., New York, USA, 1980.

[8]. Guckenkerimer J. and P. Holmes, Non linear Oscillation Dynamical Systems and Bifurcations of Vecter fields, Springer-Verlag, New York, Inc., 1983.

[9]. George B. Thomas, JR. and Ross L. Finney , Calculus and Analytic Geometry, Addison-Wesley Publishing Company, 1981.

[10]. Hirsch M.W. and S. Smale, Differential Equation, Dynamical System, and Linear Algebra, New York, Academic Press, 1974.

[11]. Hale J.H., Ordinary Differential Equation, New York, Wiley-Interscience, 1969.

[12]. Horn R. A., Johanson C. R., matrix analysis, Cambridge University press, 1985.