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Common Fixed Point Theorems Of Weak Generalized $(\alpha,\psi)-$ Contractive Maps In Partially Ordered Partial b - Metric Spaces

Vedula Perraju[1]
Principal, Mrs.A.V.N.College, Visakhapatnam, Andhra Pradesh, India

Abstract: *In this paper, we availed the opportunity of extending the concepts of Babu.G.V.R, Sarma.K.K.M, and Kumari.V.A [9] by introducing a pair* (f,g) *of weak generalized* $(\alpha,\psi)-$ *contractive maps with rational expressions and prove the existence of common fixed points when* (f,g) *is a pair of weakly compatible maps and the range of* g *is complete in partially ordered patrial b - metric spaces where* f *is a triangular* $(\alpha,g)-$ *admissible map. Further, we also extend the same conclusions by relaxing the condition 'range of* g *is complete ', but by imposing reciprocally continuity of* (f,g) *and compatibility of* (f,g) *in complete partially ordered patrial b - metric spaces. Our results are the extensions of the results of Babu.et.al [9] for partially ordered patrial b - metric spaces.*
Keywords*: Partially ordered patrial b - metric spaces,* $\alpha-$ *admissible,* $(\alpha,g)-$ *admissible, triangular* $\alpha-$ *admissible, triangular* $(\alpha,g)-$ *admissible,* $(\alpha,\psi)-$ *contractive mapping, a pair* (f,g) *of weak generalized* $(\alpha,\psi)-$ *contractive maps with rational expressions.*
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I. Introduction and preliminaries

The existence and uniqueness of common fixed points by using weak commutativity assumptions under more general contraction conditions having rational expressions in partially ordered patrial b - metric space is our present interest. In 2012, Samet.et.al [17] introduced a new concept namely $(\alpha,\psi)-$ contractive mappings which generalize contractive mappings and proved the existence of fixed points of such mappings in metric space setting. In Babu.et.al [9], Ψ denotes the family of non-decreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that ψ is continuous on $[0,\infty)$ and $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t>0$, where ψ^n is the nth iterate of ψ.

Remark(Babu.et.al [9]): Any function $\psi\in\Psi$ satisfies $\lim_{n\to\infty}\psi^n(t)=0$ and $\psi(t)<t$ for any t > 0

Definition 1.1: (Samet.et.al [17]) Let (X,d) be a metric space $f:X\to X$, and $\alpha:X\times X\to[0,\infty)$. We say that f is $\alpha-$ admissible, if $x,y\in X, \alpha(x,y)\geq 1 \Rightarrow \alpha(fx,fy)\geq 1.$ (1.1.1)

Definition 1.2: (Babu.et.al[10]) Let f,g be two self maps on X . Let $\alpha:X\times X\to[0,\infty)$ be a function. We say that f is $(\alpha,g)-$ admissible map, if for $x,y\in X, \alpha(gx,gy)\geq 1\Rightarrow \alpha(fx,fy)\geq 1.$ (1.2.1)

Definition 1.3: (Samet.et.al[17]) Let (X,d) be a metric space, and let $f:X\to X$ and $\alpha:X\times X\to[0,\infty)$. We say that f is triangular $\alpha-$ admissible, if

(i) f is $\alpha-$ admissible; and

(ii) $\alpha(x,y)\geq 1,\ \alpha(y,z)\geq 1 \Rightarrow \alpha(x,z)\geq 1$ for all $x,y,z\in X$. (1.3.1)

Definition 1.4: (Samet.et.al[17]) Let (X,d) be a metric space and $f:X\to X$ be a self map . If there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that $\alpha(x,y)d(fx,fy)\leq\psi(d(x,y))$ for all $x,y\in X$, then we say that f is a $(\alpha,\psi)-$contractive mapping.

Definition 1.5: (Arshad.et.al[1]) Let $(X,d,\prec)$ be a partially ordered metric space. A self mapping f on X is called an almost Jaggi contraction if it satisfies the following condition:

Email: perrajuvedula2004@gmail.com

there exist $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$ and $L\geq 0$ such that

$$d(fx,fy)\leq\alpha\frac{d(x,fx)d(y,fy)}{d(x,y)}+\beta d(x,y)+L\min\{d(x,y),d(x,fy),d(fx,y)\} \quad (1.5.1)$$

for any $x,y\in X$ with $x\prec y$

Definition 1.6: Let f,g be two self mappings on X . Let $\alpha:X\times X\to[0,\infty)$ be a function. We say that the map f is triangular $(\alpha,g)-$ admissible map if

(i) f is $(\alpha,g)-$ admissible; and

(ii) $\alpha(gx,gy)\geq 1$, $\alpha(gy,gz)\geq 1\Rightarrow\alpha(gx,gz)\geq 1$ for all $x,y,z\in X$ (1.6.1)

Remark: Let f be a triangular $(\alpha,g)-$ admissible mapping and suppose $f(X)\subseteq g(X)$. Assume that there exists $x_0\in X$ such that $\alpha(gx_0,fx_0)\geq 1$. Define a sequence $\{x_n\}$ by $gx_{n+1}=fx_n$. Then $\alpha(gx_m,gx_n)\geq 1$ for all $m,n\in$ □ with m < n.

Definition 1.7: Let $(X,\prec)$ be a partially ordered metric space and suppose that $f:X\to X$ be a mapping. If there exist two functions $\alpha:X\times X\to[0,\infty)$, $\psi\in\Psi$ and $L\geq 0$ such that

$$\alpha(x,y)d(fx,fy)\leq\psi(M(x,y))+L.N(x,y) \quad (1.7.1)$$

where

$$M(x,y)=\begin{cases} max\{d(x,y),\dfrac{d(x,fx)d(fy,y)}{d(x,y)},\dfrac{d(x,fy)d(fx,y)}{d(x,y)},\dfrac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{2d(x,y)}\} \\ \text{if } x\neq y, x\prec y \\ =0 \text{ if } x=y \end{cases}$$

and $N(x,y)=min\{d(x,fx),d(x,fy),d(y,fx)\},x,y\in X$ with $x\prec y$, then we say that f is a weak generalized $(\alpha,\psi)-$ contractive map with rational expressions.

We extend the above definition for two maps f and g .

Definition 1.8: Let $(X,\prec)$ be a partially ordered metric space and let f,g be two self maps on X . If there exist two functions $\alpha:X\times X\to[0,\infty)$, $\psi\in\Psi$ and $L\geq 0$ such that

$$\alpha(gx,gy)d(fx,fy)\leq\psi(M(x,y))+L.N(x,y) \quad (1.8.1)$$

where

$$M(x,y)=\begin{cases} max\{d(gx,gy),\dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)},\dfrac{d(gx,fy)d(fx,gy)}{d(gx,gy)},\dfrac{d(gx,fx)d(gx,fy)+d(gy,fy)d(gy,fx)}{2d(gx,gy)}\} \\ \text{if } x\neq y, x\prec y \\ =0 \text{ if } x=y \end{cases}$$

and $N(x,y)=min\{d(x,fx),d(x,fy),d(y,fx)\},x,y\in X$ with $x\prec y$, then we say that (f,g) is a pair of weak generalized $(\alpha,\psi)-$ contractive map with rational expressions.

Babu.et.al.[9] concluded that the class of (f,g) weak generalized (α,ψ) contractive maps with rational expressions is more general than the class of almost Jaggi contraction maps which in turn, it is more general than the class of all Jaggi contraction maps.

Theorem 1.9: (Babu.et.al.[9]Theorem 3.1.) Let $(X,\prec)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a metric space. Let $f,g:X\to X$ be two self maps on X . Suppose that f is a triangular $(\alpha,g)-$ admissible and $g-$ non-decreasing mapping. Suppose that there exist two functions $\alpha:X\times X\to[0,\infty)$, $\psi\in\Psi$ and $L\geq 0$ such that such that (f,g) is a pair of weak generalized $(\alpha,\psi)-$ contractive maps with rational expressions. Also, assume that:

(i) $fX \subseteq gX$;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ with $gx_0 \prec fx_0$

(iii) $g(X)$ is a complete subset of X

(iv) if $\{gx_n\}$ is a non-decreasing sequence in X such that $gx_n \to gx$ as $n \to \infty$ then $gx = Sup\{gx_n\}$ and $gx_n \prec ggx$

Then f and g have a coincidence point.

Corollary 1.10: (Babu.et.al.[9]Corollary 3.2.) Let $(X,\prec)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $f: X \to X$ be a non-decreasing mapping. Suppose that there exists a function $\alpha: X \times X \to [0,\infty)$ and a constant $k \in (0,1)$ such that

$$\alpha(x,y)d(fx,fy) \leq k\ max\{d(x,y), \frac{d(x,fx)+d(y,fy)}{d(x,y)}\} \qquad (1.10.1)$$

for all $x, y \in X\ with\ x \prec y, x \neq y$. Also, assume that

(i) f is $\alpha-$ admissible

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ with $x_0 \prec f(x_0)$; either

(iii) f is continuous; (or)

(iv) $\{x_n\}$ is non-decreasing in X such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$ and also $\alpha(x_n, x) \geq 1 and\ \alpha(x, fx) \geq 1$. Then f has a fixed point.

Theorem 1.11: (Babu.et.al.[9]Theorem 3.3)In addition to hypotheses of Theorem 3.1 of Babu.et.al.[9] if f and g are weakly compatible then f and g have unique common fixed point in X .

Theorem 1.12: (Babu.et.al.[9]Theorem 3.4.)Let $(X,\prec)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X;d)$ is a complete metric space. Let $f, g: X \to X$ be two self maps on X . Suppose that f is a triangular $(\alpha, g)-$ admissible and $g-$ non-decreasing mapping. Suppose that there exist two functions $\alpha: X \times X \to [0,\infty), \psi \in \Psi$ and $L \geq 0$ such that (f, g) is a pair of weak generalized $(\alpha,\psi)-$ contractive maps with rational expressions. Also, assume that

(i) $fX \subseteq gX$;

(ii) f and g are compatible;

(iii) there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ with $gx_0 \prec fx_0$;

(iv) f and g are reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point in X .

Corollory 1.13: (Babu.et.al.[9]Corollary 4.1.) Let $(X,\prec)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space and let $f: X \to X$ be a weak generalized $(\alpha,\psi)-$ contractive map with rational expressions. If there exists x_0 in X such that $x_0 \prec f(x_0)$ with $\alpha(x_0, f(x_0)) \geq 1$ and f is non-decreasing. Further assume that for any non-decreasing sequence $\{x_n\}$, where $x_n = f(x_{n-1})$, $n = 1,2,3,...$ in X converges to u then $x_n \prec u$ for all $n \geq 0$. Then f has a fixed point in X .

Corollory 1.14: (Babu.et.al.[9]Corollary 4.2.) Let $(X,\prec)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space and let $f: X \to X$ be a weak generalized $(\alpha,\psi)-$ contractive map with rational expressions. If there exists x_0 in X such that $x_0 \prec f(x_0)$, if f is non-decreasing and continuous. Then f has a fixed point in X .

II. Main result

In this section, we introduce a pair (f,g) of weak generalized $(\alpha,\psi)-$ contractive maps with rational expressions and prove the existence of common fixed points when (f,g) is a pair of weakly compatible maps and the range of g is complete in partially ordered partial b - metric spaces where f is a triangular $(\alpha,g)-$ admissible map. Further, we prove the same conclusion by relaxing the condition 'range of g is complete', but by imposing reciprocal continuity of (f,g) and compatibility of (f,g) in complete partially ordered partial b - metric metric spaces. Our results generalize the results of Babu.et.al [9]. In the following, Ψ_s denotes the family of non-decreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that ψ is continuous on $[0,\infty)$ and $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t>0$, ψ^n is the n^{th} iterate of ψ

Remark: Any function $\psi\in\Psi_s$ satisfies $\lim_{n\to\infty}\psi^n(t)=0$ and $\psi(t)<\frac{t}{s}$ for any $t>0$ and $s\geq 1$ is the coefficient of the partially ordered partial b - metric space $(X,\leq,p)$. We begin this section with the following definition Shukla [19] introduced the notation of a partial b - metric space as follows.

Definition 2.1: (Shukla.S [19]) Let X be a non empty set and let $s\geq 1$ be a given real number. A function $p:X\times X\to[0,\infty)$ is called a partial b - metric if for all $x,y,z\in X$ the following conditions are satisfied.

(i) $x=y$ if and only if $p(x,x) = p(x,y) = p(y,y)$

(ii) $p(x,x) \leq p(x,y)$

(iii) $p(x,y) = p(y,x)$

(iv) $p(x,y) \leq s\{p(x,z)+p(z,y)\}$ - $p(z,z)$. The pair (X,p) is called a partial b - metric space. The number $s\geq 1$ is called a coefficient of (X,p).

Definition 2.2: (Z.Mustafa [15]) Suppose $(X,\leq)$ is a partially ordered set and p is a partial b – metric with $s\geq 1$ as the coefficient of (X,p). Then we say that the triplet $(X,\leq,p)$ is a partially ordered partial b - metric space. We observe that every ordered partial b - metric space is a partially ordered partial b - metric space.

Definition 2.3: (Z.Mustafa [15]) A sequence $\{x_n\}$ in a partial b - metric space (X,p) is said to be:

(i) convergent to a point $x\in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$

(ii) a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite

(iii)a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x\in X$ such that $\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Now we introduce the notions of compatibility, weak compatibility and reciprocal continuity of two self maps on a partially ordered partial b - metric space.

Definition 2.4: Two self maps f and g of a partially ordered partial b - metric space $(X,\leq,p)$ are said to be compatible if $\lim_{n\to\infty} p(fgx_n,gfx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that

$\lim_{m,n\to\infty} p(fx_m,fx_n) = \lim_{n\to\infty} p(fx_n,u) = p(u,u)=0$, $\lim_{m,n\to\infty} p(gx_m,gx_n) = \lim_{n\to\infty} p(gx_n,u) = p(u,u)=0$ for some $u\in X$

Definition 2.5: Two self f and g of a partially ordered partial b - metric space $(X,\leq,p)$ are said to be weakly compatible if they commute at their coincidence points. That is $fu=gu$ for some $u\in X$, then $fgu=gfu$.

Definition 2.6: Two self maps f and g are said to be reciprocally continuous in a partially ordered partial b - metric space $(X,\leq,p)$ if $\lim_{m,n\to\infty} p(fgx_m, fgx_n) = \lim_{n\to\infty} p(fgx_n, fz) = p(fz, fz) = 0$ and $\lim_{m,n\to\infty} p(gfx_m, gfx_n) = \lim_{n\to\infty} p(gfx_n, fz) = p(gz, gz) = 0$ whenever $\{x_n\}$ is a sequence in X with $\lim_{m,n\to\infty} p(fx_m, fx_n) = \lim_{n\to\infty} p(fx_n, z) = p(z,z) = 0$ and $\lim_{m,n\to\infty} p(gx_m, gx_n) = \lim_{n\to\infty} p(gx_n, z) = p(z,z) = 0$ for some $z \in X$

Definition 2.7: Let $(X,\leq,p)$ be a partially ordered partial b - metric space and let f,g be two self mappings on X. If there exist two functions $\alpha: X\times X \to [0,\infty)$, $\psi \in \Psi_s$ and $L \geq 0$ such that

$$\alpha(gx,gy)p(fx,fy) \leq \psi(M(x,y)) + L.N(x,y) \tag{2.7.1}$$

Where $M(x,y) =$

$$\begin{cases} max\{p(gx,gy), \dfrac{p(gx,fx)p(gy,fy)}{2sp(gx,gy)}, \dfrac{p(gx,fy)p(fx,gy)}{2sp(gx,gy)}, \dfrac{p(gx,fx)p(gx,fy)+p(gy,fy)p(gy,fx)}{4sp(gx,gy)}\} \\ \text{if } x\neq y, x \prec y \\ = 0 \text{ if } x = y \end{cases}$$

and $N(x,y) = min\{p(x,fx), p(x,fy), p(y,fx)\}, x,y \in X$ with $x \prec y$, then we say that (f,g) is a pair of weak generalized $(\alpha,\psi)-$ contractive map with rational expressions.

Now we state the following useful lemmas, whose proofs can be found in Sastry. et. al [18].

Lemma 2.8: Let $(X,\leq,p)$ be a complete partially ordered partial b - metric space with $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that

(i) $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} p(x_n, y_n) = 0 \Rightarrow x = y$

(ii) $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$ Then $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = p(x,y)$ and hence $x = y$

Lemma 2.9: (i) $p(x,y) = 0 \Rightarrow x = y$

(ii) $\lim_{n\to\infty} p(x_n, x) = 0 \Rightarrow p(x,x) = 0$ and hence $x_n \to x$ as $n \to \infty$

Lemma 2.10: Let $(X,\leq,p)$ be a partially ordered partial b - metric space with coefficient $s \geq 1$.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Then (i) $\{x_n\}$ is a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow$ $\exists \epsilon > 0$ and sequences $\{m_i\},\{n_i\} \ni m_k > n_k > k \in \mathbb{N}$; $p(x_m, x_n) > \epsilon$ and $p(x_{n_k}, x_{m_k - 1}) \leq \epsilon$

Proof (i) Suppose $\{x_n\}$ is a Cauchy sequence then $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite. Therefore $0 = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{m,n\to\infty} p(x_m, x_n)$

Therefore $\lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) \neq 0$ if it exists

$\Rightarrow \exists \epsilon > 0$ and for every N and $m,n > N \ni p(x_m, x_n) > \epsilon$

$\because \lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \Rightarrow \exists M \ni p(x_n, x_{n+1}) < \epsilon \forall n > M$

Let $N_1 > M$ and n_1 be the smallest such that $m > n_1$ and $p(x_{n_1}, x_m) > \epsilon$ for at least one m.
Let m_1 be the smallest such that $m_1 > n_1 > N_1 > 1$ and $p(x_{n_1}, x_{m_1}) > \epsilon$
so that $p(x_{n_1}, x_{m_1-1}) \le \epsilon$. Let $N_2 > N_1$ and choose $m_2 > n_2 > N_2 > 2$ å $p(x_{n_2}, x_{m_2}) > \epsilon$ and $p(x_{n_2}, x_{m_2-1}) \le \epsilon$.
Continuing this process we can get sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k > k$ and $p(x_m, x_n) > \epsilon$; $p(x_{n_k}, x_{m_k-1}) \le \epsilon$

Now we state and prove the first main result:

Theorem 2.11: Let $(X, \le, p)$ be a partially ordered partial b - metric space with coefficient $s \ge 1$. Let $f, g: X \to X$ be two self maps on X . Suppose that f is a triangular $(\alpha, g)-$ admissible and $g-$ non-decreasing mapping. Suppose that there exist two functions $\alpha: X \times X \to [0,\infty)$, $\psi \in \Psi_s$ and $L \ge 0$ such that (f, g) is a pair of weak generalized $(\alpha,\psi)-$ contractive maps with rational expressions. Also, assume that

(i) $fX \subseteq gX$;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \ge 1$ with $gx_0 \prec fx_0$

(iii) $g(X)$ is a complete subset of X

Then f and g have a coincidence point.

Proof: Let $x_0 \in X$ be as in (ii) , $\alpha(gx_0, fx_0) \ge 1$ with $gx_0 \prec fx_0$. Since $fX \subseteq gX$, we choose $x_1 \in X$ such that $gx_1 = fx_0$. Since $gx_0 \prec fx_0 = gx_1$, and f is g-non-decreasing, we have $fx_0 \prec fx_1$ so that $gx_1 \prec gx_2$. By using the similar argument we choose a sequence $\{x_n\}$ in X with $fx_n = gx_{n+1}$ for $n = 1, 2, \ldots$ (2.11.1)

Further, since $gx_1 \prec gx_2$ and f is $g-$ non-decreasing, we have $fx_1 \prec fx_2$ so that $gx_2 \prec gx_3$. Inductively, it follows that $gx_n \prec gx_{n+1}$ for all $n = 0, 1, 2, \ldots$ (2.11.2)

Now, $\alpha(gx_0, gx_1) = \alpha(gx_0, fx_0) \ge 1$, and by using the property that f is an $(\alpha, g)-$ admissible map, we have $\alpha(fx_0, fx_1) \ge 1$, *i.e*., $\alpha(gx_1, gx_2) \ge 1$. By a repeated application of this property $\alpha(fx_1, fx_2) \ge 1$ *i.e*. $\alpha(gx_2, gx_3) \ge 1$ and inductively, it follows that $\alpha(fx_n, fx_{n+1}) \ge 1$ *i.e* . $\alpha(gx_{n+1}, gx_{n+2}) \ge 1$ for all $n = 0, 1, 2, 3, \ldots$ (2.11.3)

If $gx_{n+1} = gx_{n+2}$, for some n, then $gx_{n+1} = fx_{n+1}$ so that x_{n+1} is a coincidence point of f and g .

If $gx_{n+1} \ne gx_{n+2}$ for all n, then we have $p(gx_{n+1}, gx_{n+2}) \ne 0$. Now, from (2.11.1); (2.11.2) and (2.11.3),
We have

$$p(gx_{n+2}, gx_{n+1}) = p(fx_{n+1}, fx_n) \le \alpha(gx_{n+1}, gx_n) p(fx_{n+1}, fx_n) \le \psi(M(x_{n+1}, x_n)) + L.N(x_{n+1}, x_n)$$

where

$$M(x_{n+1}, x_n) = max\{p(gx_{n+1}, gx_n), \frac{p(gx_{n+1}, fx_{n+1}) p(gx_n, fx_n)}{2sp(gx_{n+1}, gx_n)}, \frac{p(gx_{n+1}, fx_n) p(fx_{n+1}, gx_n)}{2sp(gx_{n+1}, gx_n)},$$

$$\frac{p(gx_{n+1}, fx_{n+1}) p(gx_{n+1}, fx_n) + p(gx_n, fx_{n+1}) p(gx_n, fx_n)}{4sp(gx_{n+1}, gx_n)}\}$$

$$= max\{p(gx_{n+1}, gx_n), \frac{p(gx_{n+1}, gx_{n+2}) p(gx_n, gx_{n+1})}{2sp(gx_{n+1}, gx_n)}, \frac{p(gx_{n+1}, gx_{n+1}) p(gx_{n+2}, gx_n)}{2sp(gx_{n+1}, gx_n)},$$

$$\frac{p(gx_{n+1}, gx_{n+2}) p(gx_{n+1}, gx_{n+1}) + p(gx_n, gx_{n+2}) p(gx_n, gx_{n+1})}{4sp(gx_{n+1}, gx_n)}\}$$

$\leq max\{p(gx_{n+1},gx_n),\frac{p(gx_{n+1},gx_{n+2})}{2s},\frac{p(gx_{n+2},gx_n)}{2s},\frac{p(gx_{n+1},gx_{n+2})+p(gx_n,gx_{n+2})}{4s}\}$

(since $p(gx_{n+1},gx_{n+1})\leq p(gx_n,gx_{n+1})$)

$\leq max\{p(gx_{n+1},gx_n),\frac{p(gx_{n+1},gx_{n+2})}{2s},\frac{p(gx_{n+2},gx_n)}{2s}\}$

$\leq max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2}),\frac{1}{2s}[sp(gx_n,gx_{n+1})+sp(gx_{n+1},gx_{n+2})-p(gx_{n+1},gx_{n+1})]\}$

$\leq max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2}),\frac{1}{2}[p(gx_n,gx_{n+1})+p(gx_{n+1},gx_{n+2})]\}$

$\leq max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2})\}$

and

$N(x_{n+1},x_n)=min\{p(gx_{n+1},fx_{n+1}),p(gx_{n+1},fx_n),p(gx_n,fx_{n+1})\}$

$=min\{p(gx_{n+1},gx_{n+2}),p(gx_{n+1},gx_{n+1}),p(gx_n,gx_{n+2})\}$

$=p(gx_{n+1},gx_{n+1})$

$\therefore$ On taking $L=0$

$\Rightarrow\ p(gx_{n+2},gx_{n+1})\leq\psi(max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2}))$

$<\frac{1}{s}max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2})\}$ (2.11.4)

Suppose $max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2}))=p(gx_{n+1},gx_{n+2})$

then $sp(gx_{n+2},gx_{n+1})<p(gx_{n+2},gx_{n+1})$, a contradiction.

$\therefore\ max\{p(gx_{n+1},gx_n),p(gx_{n+1},gx_{n+2}))=p(gx_n,gx_{n+1})$

$\therefore\ p(gx_{n+1},gx_{n+2})<p(gx_{n+1},gx_n)$

$\therefore$ Sequence $\{p(gx_n,gx_{n+1})\}$ is strictly decreasing and converges to r say,

$\therefore\ \lim_{n\to\infty}p(gx_n,gx_{n+1})=r\geq 0$. Suppose $r>0$

$\therefore\ p(gx_{n+1},gx_{n+2})\leq\psi(p(gx_{n+1},gx_n))<\frac{1}{s}p(gx_{n+1},gx_n)$

Allowing as $n\to\infty\Rightarrow r\leq\psi(r)<\frac{r}{s}$, a contradiction.

$\therefore\ r=0$

Now we claim sequence $\{gx_n\}$ is a Cauchy sequence. Assume that $\{gx_n\}$ is not a Cauchy sequence. Then by lemma 2.10 $\exists\epsilon>0$ and sequences $\{m_k\}$, $\{n_k\}$; $m_k>n_k>k$ such that $p(gx_{m_k},gx_{n_k})\ \geq\epsilon$ and $p(gx_{m_k-1},gx_{n_k})\ <\epsilon$.

$\therefore\ \epsilon\leq p(gx_{m_k},gx_{n_k})\}$

$=p(fx_{m_k-1},fx_{n_k-1})\leq\psi(M(x_{m_k-1},x_{n_k-1}))+L.N(x_{m_k-1},x_{n_k-1})$ (2.11.5)

where

$M(x_{m_k-1},x_{n_k-1})$

$=max\ \{p(gx_{m_k-1},gx_{n_k-1})\ ,\ \frac{p(gx_{n_k-1},fx_{n_k-1})p(gx_{m_k-1},fx_{m_k-1})}{2sp(gx_{m_k-1},gx_{n_k-1})},\ \frac{p(gx_{m_k-1},fx_{n_k-1})p(gx_{n_k-1},fx_{m_k-1})}{2sp(gx_{m_k-1},gx_{n_k-1})},$

$\frac{p(gx_{m_k-1},fx_{m_k-1})p(gx_{m_k-1},fx_{n_k-1})+p(gx_{n_k-1},fx_{m_k-1})p(gx_{n_k-1},fx_{n_k-1})}{4sp(gx_{m_k-1},gx_{n_k-1})}\}$

$$= max \{p(gx_{m_k-1}, gx_{n_k-1}) , \frac{p(gx_{n_k-1}, gx_{n_k})p(gx_{m_k-1}, gx_{m_k})}{2sp(gx_{m_k-1}, gx_{n_k-1})} , \frac{p(gx_{m_k-1}, gx_{n_k})p(gx_{n_k-1}, gx_{m_k})}{2sp(gx_{m_k-1}, gx_{n_k-1})} ,$$

$$\frac{p(gx_{m_k-1}, gx_{m_k})p(gx_{m_k-1}, gx_{n_k}) + p(gx_{n_k-1}, gx_{m_k})p(gx_{n_k-1}, gx_{n_k})}{4sp(gx_{m_k-1}, gx_{n_k-1})}\}$$

$$= max \{p(gx_{m_k-1}, gx_{n_k-1}) ,0, \frac{p(gx_{m_k-1}, gx_{n_k})p(gx_{n_k-1}, gx_{m_k})}{2sp(gx_{m_k-1}, gx_{n_k-1})} , 0 \}$$, for large k

$$\leq max \{sp(gx_{m_k-1}, gx_{n_k}) + sp(gx_{n_k}, gx_{n_k-1}) - p(gx_{n_k}, gx_{n_k}),$$

$$\frac{\epsilon(sp(gx_{n_k-1}, gx_{m_k-1}) + sp(gx_{m_k-1}, gx_{m_k}) - p(gx_{m_k-1}, gx_{m_k-1}))}{2sp(gx_{m_k-1}, gx_{n_k-1})}\}$$

$\leq max\{s\epsilon, \frac{\epsilon}{2}\}$, for large k

$= s\epsilon$ for large k

$\therefore \epsilon \leq \psi(M(x_{m_k-1}, x_{n_k-1})) \leq \psi(s\epsilon) < \epsilon$, a contradiction. (2.11.6)

Consequently $\{gx_n\}$ is a Cauchy sequence.

$\therefore \lim_{k\to\infty} p(gx_{m_k}, gx_{n_k})\} = 0$

Therefore $\{gx_n\}$ is a Cauchy sequence in $(X, \leq, p)$. Since $g(X)$ is complete, there exists $z \in g(X)$ such that $\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} fx_n = gx = z$ for some $x \in X \Rightarrow gx_{n+1} \prec gx$. (2.11.7)

$\therefore \alpha(gx_{n+1}, gx) \geq 1$

To show that $gx = fx$.

Suppose $gx \neq fx$.

we have $p(gx_{n+1}, fx) = p(fx_n, fx) \leq \alpha(gx_{n+1}, gx)p(fx_n, fx) \leq \psi(M(x_n, x)) + L.N(x_n, x)$

where

$$M(x_n, x) = max\{p(gx_n, gx), \frac{p(gx_n, fx_n)p(gx, fx)}{2sp(gx_n, gx)},$$

$$\frac{p(gx_n, fx)p(fx_n, gx)}{2sp(gx_n, gx)}, \frac{p(gx_n, fx_n)p(gx_n, fx) + p(gx, fx)p(gx, fx_n)}{4sp(gx_n, gx)}\}$$

$$= max\{p(gx_n, gx), \frac{p(gx_n, gx_{n+1})p(gx, fx)}{2sp(gx_n, gx)}, \frac{p(gx_n, fx)p(gx_{n+1}, gx)}{2sp(gx_n, gx)},$$

$$\frac{p(gx_n, gx_{n+1})p(gx_n, fx) + p(gx, fx)p(gx, gx_{n+1})}{4sp(gx_n, gx)}\}$$

$$= max\{p(gx_n, gx), 0, \frac{p(gx_n, fx)p(gx_{n+1}, gx)}{2sp(gx_n, gx)}, \frac{p(gx, fx)p(gx_{n+1}, gx)}{4sp(gx_n, gx)}\}$$

$$\leq max\{p(gx_n, gx), \frac{p(gx_n, fx)\{sp(gx_{n+1}, gx_n) + sp(gx_n, gx) - p(gx_n, gx_n)}{2sp(gx_n, gx)},$$
$$\frac{p(gx, fx)\{sp(gx_{n+1}, gx_n) + sp(gx_n, gx) - p(gx_n, gx_n)}{4sp(gx_n, gx)}\}$$
$$\leq max\{p(gx_n, gx), \frac{p(gx_n, fx)\{sp(gx_{n+1}, gx_n) + sp(gx_n, gx) - p(gx_n, gx_n)}{2sp(gx_n, gx)},$$
$$\frac{p(gx, fx)\{sp(gx_{n+1}, gx_n) + sp(gx_n, gx) - p(gx_n, gx_n)}{4sp(gx_n, gx)}\}$$
$$\leq max\{p(gx_n, gx), \frac{p(gx_n, fx)[sp(gx_{n+1}, gx_n) + sp(gx_n, gx)]}{2sp(gx_n, gx)}, \frac{p(gx, fx)[sp(gx_{n+1}, gx_n) + sp(gx_n, gx)]}{4sp(gx_n, gx)}\}$$

$$\leq max\{p(gx_n, gx), \frac{p(gx_n, fx)}{2}, \frac{p(gx, fx)}{4}\}$$
$$N(x_n, x) = min\{p(gx_n, fx_n), p(gx_n, fx), p(gx, fx_n)\}$$
$$= min\{p(gx_n, gx_{n+1}), p(gx_n, fx), p(gx, fx_n)\} = 0$$
$$\therefore\ p(gx_{n+1}, fx) \leq \psi(M(x_n, x) = \psi[max\{p(gx_n, gx), \frac{p(gx_n, fx)}{2}, \frac{p(gx, fx)}{4}\}]$$

Allowing as $n \to \infty$

$$p(gx, fx) < \frac{1}{s} max\{p(gx, gx), \frac{p(gx, fx)}{2}\}$$

$p(gx, fx) < \frac{1}{s} p(gx, fx)$, (since $p(gx, gx) \leq p(gx, fx)$), a contradiction.

$\therefore\ p(gx, fx) = 0 \Rightarrow gx = fx$

$\therefore\ x$ is a coincidence point of f and g.

Corollary 2.12: Let $(X, \prec, p)$ be a partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be a non - decreasing mapping on X. Suppose that f is a triangular $(\alpha, g)-$ admissible and $g-$ non-decreasing mapping. Suppose that there exists a function $\alpha : X \times X \to [0, \infty)$, and a constant $k \in (0,1)$ such that $\alpha(x, y)sp(fx, fy) \leq k\ max\{p(x, y), \frac{p(x, fx)p(y, fy)}{2sp(x, y)}\}$ for all $x, y \in X$ with $x \prec y$ and $x \neq y$. Also, assume that

(i) f is $\alpha-$ admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ with $gx_0 \prec fx_0$ either

(iii) f is a continuous (or)

(iv) x_n is non-decreasing in X such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$; and also $\alpha(x_0, x) \geq 1$ and $\alpha(x, fx) \geq 1$.

f has a fixed point. Then f and g have a coincidence point.

Proof: The conclusion of this corollary follows by taking $g = I_x$; $\psi(t) = kt, t \geq 0$ and L = 0, in Theorem 2.11.

Theorem 2.13: In addition to the hypotheses of Theorem 2.11, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof: From the proof of Theorem 2.11 we have gx_n is non-decreasing sequence that converges to gx and $fx = gx$. Further we can show that $p(ggz, gz) = 0$. Since f and g are weakly compatible, we have $fgx = gfx$ *i.e.,* $gx = fgx = ggx$. Hence $fz = gz = z$, so that f and g have a common fixed point z.

Uniqueness: Let z and z' be two common fixed points of f and g *i.e.,* $fz = gz = z$ and $fz' = gz' = z'$
To show that $z = z'$
we have $p(z,z') = p(fz, fz') \le \alpha(gz, gz')p(fz, fz') \le \psi(M(z,z')) + L.N(z,z')$
where

$$M(z,z') = max\{p(gz,gz'), \frac{p(gz, fz)p(gz', fz')}{2sp(gz,gz')}, \frac{p(gz, fz')p(fz,gz')}{2sp(gz,gz')},$$
$$\frac{p(gz, fz)p(gz, fz') + p(gz', fz')p(gz', fz)}{4sp(gz,gz')}\}$$
$$= max\{p(gz,gz'), \frac{p(gz,gz)p(gz',gz')}{2sp(gz,gz')}, \frac{p(gz,gz')p(gz,gz')}{2sp(gz,gz')},$$
$$\frac{p(gz,gz)p(gz,gz') + p(gz',gz')p(gz',gz)}{4sp(gz,gz')}\}$$
$$\le max\{p(gz,gz'), \frac{p(gz,gz)}{2s}, \frac{p(gz,gz')}{2s}, \frac{p(gz,gz) + p(gz',gz')}{4s}\}$$
$$\le max\{p(gz,gz'), \frac{p(gz,gz)}{2s}, \frac{p(gz,gz')}{2s}\}$$
$$= p(gz,gz')$$
$$= p(z,z')$$
$$N(z,z') = min\{p(gz, fz), p(gz, fz'), p(gz', fz)\}$$
$= p(z,z)$ for $L = 0$
$$\Rightarrow \; p(z,z') \le \psi(M(z,z'))$$
$\Rightarrow \; p(z,z') < \frac{1}{s}p(z,z')$, a contradiction.
$$\therefore \; p(z,z') = 0$$
$$\therefore z = z'$$
Hence uniqueness

Theorem 2.14: Let $(X,\prec,p)$ be a partially ordered partial b - metric space with coefficient $s \ge 1$. Let $f,g : X \to X$ be two self maps on X. Suppose that f is a triangular $(\alpha,g)-$ admissible and $g-$ non-decreasing mapping. Suppose that there exist two functions $\alpha : X \times X \to [0,\infty)$, $\psi \in \Psi_s$ and $L \ge 0$ such that (f,g) is a pair of weak generalized $(\alpha,\psi)-$ contractive maps with rational expressions. Also, assume that

(i) $fX \subseteq gX$;

(ii) f and g are compatible

(iii) there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \ge 1$ with $gx_0 \prec fx_0$

(iv) f and g are reciprocally continuous.

Then f and g have a coincidence point.

Moreover, f and g have a unique common fixed point in X.

Proof: As in the proof of Theorem 2.11, for $x_0 \in X$ of (iii), we choose $\{x_n\}$ in X that satisfies $fx_n = gx_{n+1}$ for $n = 1, 2, \ldots$ and that $\{gx_n\}$ is a Cauchy sequence in X. Since $(X, \prec, p)$ is complete, there exists $z \in X$ such that $\lim_{n\to\infty} gx_n = z$. Hence $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z$.

Since f and g are reciprocally continuous, we have $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$, Since f and g are compatible, we have $\lim_{n\to\infty} p(fgx_n, gfx_n) = 0$ so that $p(fz, gz) = 0$.

Hence $fz = gz$ so that z is a coincidence point of f and g.

Now, since every compatible pair is weakly compatible, by applying Theorem 2.13 it follows that f and g have a unique common fixed point in X.

By choosing $g = I_x$ in Theorem 2.11, we have the following corollary.

Corollary 2.15: Let $(X, \prec, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be weak generalized $(\alpha,\psi)-$ contractive map with rational expressions.

If there exists $x_0 \in X$ such that $x_0 \prec fx_0$ with $\alpha(x_0, fx_0) \geq 1$ and f is non - decreasing. Further, assume that for any non-decreasing sequence $\{x_n\}$, where $x_n = fx_{n-1}, n = 1, 2, \ldots$ in X converge to u, then $x_n \prec u$ for all $n \geq 0$. Then f has a fixed point in X.

By choosing $g = I_x$ in Theorem 2.14, we have the following corollary.

Corollary 2.16: Let $(X, \prec, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $f : X \to X$ be weak generalized $(\alpha,\psi)-$ contractive map with rational expressions. If there exists $x_0 \in X$ such that $x_0 \prec fx_0$, if f is non - decreasing and continuous. Then f has a fixed point in X.

Example 2.17: Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}\}$ with usual ordering.

Define $p(x,y) = \begin{cases} 0 \, if \; x = y \\ 1 \, if \; x \neq y \in \{0,1\} \\ |x-y| \, if \; x, y \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\} \\ 4 \, otherwise \end{cases}$

Clearly, $(X, \leq, p)$ is a partially ordered partial b - metric space with coefficient $s = \frac{8}{3}$

(Kumam.P.et.al [15])

Define $f, g : X \to X$ by

$$f1 = f\frac{1}{2} = f\frac{1}{3} = f\frac{1}{4} = f\frac{1}{5} = \frac{1}{2} = f\frac{1}{6} = f\frac{1}{7} = f\frac{1}{8} = f\frac{1}{9} = f\frac{1}{10} = f0$$

$$\Rightarrow f(X) = \{\frac{1}{2}\}$$

and

$$g(\frac{1}{n}) = \begin{cases} \frac{1}{2} \, if \; 1 \leq n \leq 5 \\ \frac{1}{9} \, if \; 6 \leq n \leq 10 \\ g0 = \frac{1}{9} \end{cases} \quad \Rightarrow g(X) = \{\frac{1}{2}, \frac{1}{9}\}$$

$\therefore f(X)\subset g(X)\subset X$ .and $g(x)\le g(y)\Rightarrow f(x)\le f(y)$

$\therefore f$ is g - non decreasing

Define $\psi_s:[0,\infty)\to[0,\infty)$ by $\psi(t)=\frac{t}{3}<\frac{3t}{8}=\frac{t}{s}$

$\alpha(x,y)=2\,\forall\, x,y\in X$

For $x,y\in X$ and $x\ne y \Rightarrow p(fx,fy)=0$ and $M(x,y)\ge 0$ for all $gx\ne gy$,

$\therefore\ p(fx,fy)\le\alpha(x,y)p(fx,fy)\le\psi(M(x,y))+L.N(x,y)$

Clearly $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}$ are coincidence points.

$f\frac{1}{2}=\frac{1}{2}=g\frac{1}{2}\Rightarrow fg\frac{1}{2}=\frac{1}{2}=gf\frac{1}{2}\ \Rightarrow f$ and g are weakly compatible at $\frac{1}{2}\in X$

Clearly $g\frac{1}{10}=\frac{1}{9}<\frac{1}{2}=f\frac{1}{10}$.

Let $x_0=\frac{1}{10}\Rightarrow gx_0<fx_0=\frac{1}{2}=g\frac{1}{2}=gx_1$

$\Rightarrow fx_1=f\frac{1}{2}=\frac{1}{2}=g\frac{1}{2}=gx_1$

Therefore $\frac{1}{2}\in X$ is the unique fixed point. The hypothesis and conclusions of of theorem 2.11 satisfied.

Open Problem : Is theorem 2.11 true if Ψ_s defined independent of s ?

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