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Examples of Shrinkage Estimators of the Mean, Dominating the Maximum Likelihood Estimator in Larges Dimension

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Abstract: *In this paper we are interested to the estimation of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ in $\Re^p$, by a shrinkage estimators deduced from the empirical average estimator. We study bounds and limits of risk ratios of some minimax shrinkage estimators in the both cases σ^2 known and unknown. We show that the limit of risk ratios of polynomial estimator, estimator proposed by T.F. Li and W.H. Kuo [9] and the estimator proposed by D. Benmansour and T. Mourid, [3] to the maximum likelihood estimator X tend to values less than one.*

Keywords: *James-Stein estimator, multivariate normal distribution, non-central chi-square distribution, quadratic risk, shrinkage estimator*.

I. Introduction

Since paper of C. Stein [11], many studies were carried out in the direction of shrinkage estimators of the mean θ of a multivariate Gaussian random variable $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ in $\Re^p$. In these works, one has estimated the mean θ of a multidimensional Gaussian distribution $N_p\left(\theta, \sigma^2 I_p\right)$ in $\Re^p$ by shrinkage estimators deduced from the empirical average which are better in quadratic loss than the empirical average.

More precisely, if X represents an observation or a sample of multidimensional Gaussian law $N_p\left(\theta, \sigma^2 I_p\right)$, so the aim is to estimate θ by an estimator δ relatively at the quadratic loss function :

$$L(\delta, \theta) = \left\|\delta(X) - \theta\right\|_p^2$$

where $\|.\|_p$ is the usual norm in $\Re^p$. We associate his risk function :

$$R(\delta, \theta) = E_\theta\left(L(\delta, \theta)\right).$$

W. James and C. Stein [8] introduced a class of James-Stein estimators improving the maximum likelihood estimator $\delta_0 = X$, when the dimension of the space of the observations $p \geq 3$, noted :

$$\delta_{JS} = \left(1 - \frac{p-2}{\|X\|^2}\right)X \quad \text{in the case where } \sigma^2 \text{ is known}$$

and

$$\delta_{JS}^{\sigma} = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right)X \quad \text{in the case where } \sigma^2 \text{ is unknown,}$$

where $S^2 \sim \sigma^2 \chi_n^2$ is the estimate of σ^2.

A.J. Baranchik [1] proposed the positive-part of James-Stein estimator dominating the James-Stein estimator when $p \geq 3$, noted :

$\delta_{JS}^{+} = \max\left(0, \left(1 - \frac{p-2}{\|X\|^2}\right)\right) X$ in the case where σ^2 is known,

$\delta_{JS}^{\sigma+} = \max\left(0, \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right)\right) X$ in the case where σ^2 is unknown.

G. Casella and J.T. Hwang [5] studied the case where σ^2 is known $(\sigma^2 = 1)$ and showed that if the limit of the ratio $\frac{\|\theta\|^2}{p}$, when p tends to infinity is a constant $c > 0$, then

$$\lim_{p \to +\infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \lim_{p \to +\infty} \frac{R(\delta_{JS}^{+}, \theta)}{R(X, \theta)} = \frac{c}{1+c}.$$

Thus they showed the stability of the dominating of James-Stein estimator and its positive-part, to the maximum likelihood estimator, when the dimension of space parameter p tends to infinity.

Li. Sun [13] has considered the following model : $(y_{ij}/\theta_j, \sigma^2) \sim N(\theta_j, \sigma^2)$ $i = 1, \dots, n$, $j = 1, \dots, m$ where $E(y_{ij}) = \theta_j$ for the group j and $var(y_{ij}) = \sigma^2$ is unknown. The James-Stein estimators are written in this case

$\delta^{JS} = (\delta_1^{JS}, \dots, \delta_m^{JS})^t$,

where $\delta_j^{JS} = \left(1 - \frac{(m-3)S^2}{(N+2)T^2}\right)(\bar{y}_j - \bar{y}) + \bar{y}$, $j = 1, \dots, m$,

where $S^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}(y_{ij} - \bar{y}_j)^2$, $T^2 = n\sum_{j=1}^{m}(\bar{y}_j - \bar{y})^2$, $\bar{y}_j = \frac{\sum_{i=1}^{n} y_{ij}}{n}$, $\bar{y} = \frac{\sum_{j=1}^{m} \bar{y}_j}{m}$, $N = (n-1)m$,

he has given a lower bound for the ratio $\frac{R(\delta^{JS}, \theta)}{R(X, \theta)}$, which allows him to conclude that

$$\lim_{m \to +\infty} \frac{R(\delta^{JS}, \theta)}{R(X, \theta)} = \lim_{m \to +\infty} \frac{R(\delta^{JS+}, \theta)}{R(X, \theta)} = \frac{q}{q + \frac{\sigma^2}{n}},$$

if $\lim_{m \to +\infty} \frac{\sum_{j=1}^{m}(\theta_j - \bar{\theta})^2}{m} = q$ exists.

D. Benmansour and A. Hamdaoui [2] are interested the case where σ^2 is unknown. We showed that if $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c(> 0)$, then the risk ratio of James-Stein estimator δ_{JS}^{σ} to the maximum likelihood estimator X, tends to the value $\frac{\frac{2}{n+2} + c}{1+c}$ when p tends to infinity and n is fixed. Under the same condition namely $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, authors showed that the risk ratio of James-Stein estimator δ_{JS}^{σ} to the maximum likelihood estimator X , tends to the value $\frac{c}{1+c}$ when n and p tend simultaneously to infinity. They also found the same results for the positive-part of James-Stein estimator.

Moreover, A. Hamdaoui and D. Benmansour [7] studied the behaviour of risk ratios of general class of shrinkage estimator proposed by D. Benmansour and T. Mourid [3] given by $\delta_{l,\delta_{JS}^{\sigma},\psi}(X)=\delta_{l,\delta_{JS}}=\delta_{JS}^{\sigma}+l\psi\left(S^2,\|X^2\|\right)X$, in the case where σ^2 is unknown. Then, they showed that if $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, the risk ratio of general class of shrinkage estimator $\delta_{l,\delta_{JS}^{\sigma},\psi}$, tend a value less then 1, when n and p tend simultaneously to infinity, provided the function ψ satisfies certain conditions.
When the dimension p is moderate, A.C. Brandwein and W.E. Strawderman [4] considered the following model $(X,U)\sim f\left(\|X-\theta\|^2+\|U\|^2\right)$, where $\dim X=\dim\theta=p$ and $\dim U=k$. The classical example of this model is, of course, the normal model of density $\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{p+k}e^{-\frac{\|X-\theta\|^2}{2\sigma^2}}$. They showed that the estimator $\delta=X+\left\{\frac{\|U\|^2}{k+2}\right\}g(X)$ dominate X , so that δ is minimax, provided the function g satisfies certain conditions.

Y. Maruyama [10] has also studied the minimaxity of shrinkage estimator when the dimension of parameter's space is moderate. Then he considered the following model : $z\sim N_d(\theta,I_d)$ and the so called $\bar{\pi}$ -norm given by : $\|z\|_p=\left\{\sum_{i=1}^{i=d}|z_i|^p\right\}^{\frac{1}{p}}$, $p>0$. He stydied the minimaxity of shrinkage estimators defined as follows : $\hat{\theta}_\phi=(\hat{\theta}_{1\phi},...,\hat{\theta}_{d\phi})$ with : $\hat{\theta}_{i\phi}=\left(1-\phi(\|z\|_p)\Big/\phi\left(\|z\|_p^{2-\alpha}|z_i|^{\alpha}\right)\right)z_i$ where $0\le\alpha\le(d-2)/(d-1)$, $p>0$.

In this paper, by taking the same model, namely $X\sim N_p\left(\theta,\sigma^2I_p\right)$, our aim is :
Firstly, when σ^2 is known, we show the same results linked of risk ratios of James-Stein estimator, obtained in G. Casella, and J.T. Hwang [5], for two classes of shrinkage estimators dominating the James-Stein estimator, so the first class is polynomial estimators proposed by T.F. Li and W.H. Kuo [9] and the second is the class of estimators proposed by D. Benmansour and T. Mourid [3].
Secondly, we give another proof different to that given in A. Hamdaoui and D. Benmansour [7], which shows the stability of the minimaxity of two classes of estimators dominating the James-Stein estimator, when the dimension p of the parameter space and the size n of the sample, tends simultaneously to infinity.

In Section 2, we recall two essential results obtained in the paper of D. Benmansour and A. Hamdaoui [2]. First, we shown that under the condition $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, the risk ratio of James-Stein estimator δ_{JS}^{σ}, to the maximum likelihood estimator X , tends to the value $\frac{\frac{2}{n+2}+c}{1+c}$ when p tends to infinity and n fixed. The second result indicates that under the same condition $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, the risk ratio of James-Stein estimator δ_{JS}^{σ}, to the maximum likelihood estimator X, tends to the value $\frac{c}{1+c}$, when n and p tend simultaneously to infinity.

In Section 3, we give main results. In the first part of this Section, we show the same results obtained by G. Casella, and J.T. Hwang [5], (respectively by D. Benmansour and A. Hamdaoui [2]), according to the case where σ^2 known (respectively σ^2 unknown), for the class of polynomial estimators proposed by T.F. Li and W.H. Kuo [9]. The same results are proved in the second part of this section, for the class of estimators proposed by D. Benmansour and T. Mourid [3]. Thus, we give another proof different to that given in A. Hamdaoui and D. Benmansour [7], which shown the stability of the minimaxity of both classes of estimators, cited as above, when the dimension p of the parameter space and sample size n, tends simultaneously to infinity.

In section 4, we give a graphic illustration of different risk ratios for various values of n and p. An appendix is given at the end of this paper.

II. Preliminary

Let us recall that if $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, where the parameter σ^2 is unknown, the risk of the maximum likelihood estimator X is $p\sigma^2$, and the form of James-Stein estimator is

$$\delta_{JS}^{\sigma} = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right)X \,, \tag{2.1}$$

where $S^2 \sim \sigma^2 \chi_{\mathrm{n}}^2$ is the estimate of σ^2.

From R. Christian [6], the risk of the James-Stein estimator given in (2.1) is

$$R(\delta_{JS}^{\delta}, \theta) = \sigma^2 \left\{ p - \frac{n}{n+2}(p-2)^2 E\left(\frac{1}{p-2+2K}\right)\right\},$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the Poisson distribution of parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

Theorem 1 *(D. Benmansour and A. Hamdaoui [2]). If* $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c(>0)$, *we have*

$$\lim_{p \to +\infty} \frac{R\left(\delta_{JS}^{\sigma}, \theta\right)}{R(X, \theta)} = \frac{c + \frac{2}{n+2}}{c+1}. \tag{2.2}$$

Corollary 2 (*D. Benmansour and A. Hamdaoui [2]). If* $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c(>0)$, *we have*

$$\lim_{n,p \to +\infty} \frac{R\left(\delta_{JS}^{\sigma}, \theta\right)}{R(X, \theta)} = \frac{c}{c+1}. \tag{2.3}$$

III. Main results

In the next we prove the main results of this paper. At first we show that the limit of risk ratio of estimator proposed by T.F. Li and W.H. Kuo [9], tend to $\frac{c}{c+1}(<1)$, when p tends to infinity in the case σ^2 known and when n and p tend simultaneously to infinity in the case σ^2 unknown. Secondly, we show the same results for the class of estimator proposed by D. Benmansour and T. Mourid [3].

3.1. Family of Tze Fen Li and Wen Hou Kuo

σ **known :** Let $X \sim N_p(\theta, I_p)$, and for all r $\left(2 < r < \frac{p+2}{2}\right)$, we consider the estimator

$$\delta_r = \delta_{JS} + dX\|X\|^{-r}, \tag{3.1}$$

where $d = (r-2)2^{\frac{r-2}{2}} \dfrac{\Gamma\left(\frac{p-r}{2}\right)}{\Gamma\left(\frac{p-2r+2}{2}\right)}$.

We know that the risk of the estimator δ_r is

$$R(\delta_r, \theta) = R(\delta_{JS}, \theta) - 2d(r-2)E\left\{\|X\|^{-r}\right\} + d^2 E\left\{\|X\|^{-2r+2}\right\}. \tag{3.2}$$

For the next we need the following lemma.

Lemma 3 *If* $X \sim N_p(\theta, I_p)$ *then, for all* r $(0 < r < p)$*:* $E\left(\|X\|^{-r}\right)$ *is a function strictly decreasing with respect to* $\|\theta\|^2$.

Proof.

$$\begin{aligned} E\left(\|X\|^{-r}\right) &= E\left(\|X\|^2\right)^{\frac{-r}{2}}, \\ &= 2^{\frac{-r}{2}} E\left\{\frac{\Gamma\left(\frac{p}{2}+K-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+K\right)}\right\}, \\ &= 2^{\frac{-r}{2}} \sum_{k\geq 0} \frac{\Gamma\left(\frac{p}{2}+k-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+k\right)} e^{-\lambda} \frac{\lambda^k}{k!}, \end{aligned} \tag{3.3}$$

where $\lambda = \dfrac{\|\theta\|^2}{2}$.

The equality (3.3) comes from formula (5.2) lemma 10 in the appendix.

$$\begin{aligned} \frac{\partial}{\partial\lambda} E\left(\|X\|^{-r}\right) &= 2^{\frac{-r}{2}} \sum_{k\geq 0} \frac{\Gamma\left(\frac{p}{2}+k-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+k\right)} \left(-e^{-\lambda}\frac{\lambda^k}{k!} + k\frac{\lambda^{k-1}}{k!}e^{-\lambda}\right) \\ &= 2^{\frac{-r}{2}} \left\{\sum_{k\geq 0} \frac{\Gamma\left(\frac{p}{2}+k-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+k\right)} k\frac{\lambda^{k-1}}{k!}e^{-\lambda} - \sum_{k\geq 0} \frac{\Gamma\left(\frac{p}{2}+k-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+k\right)} \frac{\lambda^k}{k!}e^{-\lambda}\right\} \\ &= 2^{\frac{-r}{2}} \left\{\sum_{k_1\geq 0} \frac{\Gamma\left(\frac{p}{2}+k_1+1-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+k_1+1\right)} \frac{\lambda^{k_1}}{k_1!}e^{-\lambda} - E\left\{\frac{\Gamma\left(\frac{p}{2}+K-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+K\right)}\right\}\right\} \end{aligned}$$

$$= 2^{\frac{-r}{2}} \left\{ E\left\{ \frac{\Gamma\left(\frac{p}{2}+K+1-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+K+1\right)} \right\} - E\left\{ \frac{\Gamma\left(\frac{p}{2}+K-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+K\right)} \right\} \right\}$$

$$= 2^{\frac{-r}{2}} E\left\{ \frac{\Gamma\left(\frac{p}{2}+K-\frac{r}{2}\right)\left\{\left(\frac{p}{2}+K-\frac{r}{2}\right)-\left(\frac{p}{2}+K\right)\right\}}{\Gamma\left(\frac{p}{2}+K+1\right)} \right\}$$

$$= \frac{-r}{2} 2^{\frac{-r}{2}} E\left\{ \frac{\Gamma\left(\frac{p}{2}+K+1-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}+K+1\right)} \right\} < 0 .$$

Then $E\left(\|X\|^{-r}\right)$ is strictly decreasing with respect to $\|\theta\|^2$. ■

Theorem 4 *If* $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p} = c(> 0)$ *, then*

$$\lim_{p \to +\infty} \frac{R(\delta_r, \theta)}{R(X, \theta)} = \frac{c}{c+1} .$$

Proof. On the one hand, from T.F. Li and W.H. Kuo [9], we have

$$R(\delta_r, \theta) \le R(\delta_{JS}, \theta) \ \ \forall p \ge 3 \text{ and } \forall \|\theta\|^2 > 0 .$$

Hence

$$\lim_{p \to +\infty} \frac{R(\delta_r, \theta)}{R(X, \theta)} \le \lim_{p \to +\infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)}$$

and from G. Casella and J.T. Hwang [5], we have $\lim_{p \to +\infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \frac{c}{1+c}$, thus

$$\lim_{p \to +\infty} \frac{R(\delta_r, \theta)}{R(X, \theta)} \le \frac{c}{1+c} .$$

On the other hand, following the formula (3.2) we have

$$R(\delta_r, \theta) \ge R(\delta_{JS}, \theta) - 2d(r-2)E\left(\|X\|^{-r}\right).$$

According the formula (3.3), and the fact that $E\left(\|X\|^{-r}\right)$ is strictly decreasing with respect to $\|\theta\|^2$ (see lemma 3) we have

$$E\left(\|X\|^{-r}\right) \le 2^{\frac{-r}{2}} \frac{\Gamma\left(\frac{p}{2}-\frac{r}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} . \tag{3.4}$$

Therefore

$$\frac{R(\delta_r,\theta)}{R(X,\theta)} \geq \frac{R(\delta_{JS},\theta)}{R(X,\theta)} - \frac{(r-2)^2}{p} \frac{\left\{\Gamma\left(\frac{p-r}{2}\right)\right\}^2}{\Gamma\left(\frac{p-2r}{2}+1\right)\Gamma\left(\frac{p}{2}\right)}.$$

Using the Stirling formula ($\lim_{y \to +\infty} \frac{\Gamma(y+1)}{\sqrt{2\pi}\, y^{y+1/2} e^{-y}} = 1$), we have

$$\lim_{p \to +\infty} \frac{R(\delta_r,\theta)}{R(X,\theta)} \geq \lim_{p \to +\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)}$$

$$- \lim_{p \to +\infty} \frac{(r-2)^2}{p} \frac{\left\{\sqrt{2\pi}\left(\frac{p-r}{2}-1\right)^{\left(\frac{p-r}{2}-1\right)+\frac{1}{2}} e^{-\left(\frac{p-r}{2}-1\right)}\right\}^2}{\sqrt{2\pi}\left(\frac{p-2r}{2}\right)^{\left(\frac{p-2r}{2}\right)+\frac{1}{2}} e^{-\left(\frac{p-2r}{2}\right)} \sqrt{2\pi}\left(\frac{p}{2}-1\right)^{\left(\frac{p}{2}-1\right)+\frac{1}{2}} e^{-\left(\frac{p}{2}-1\right)}}.$$

where e^t is the exponential function. Thus

$$\lim_{p \to +\infty} \frac{R(\delta_r,\theta)}{R(X,\theta)} \geq \lim_{p \to +\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} - \lim_{p \to +\infty} \left\{\frac{2e(r-2)^2}{p^2}\right\}$$

$$\geq \lim_{p \to +\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} = \frac{c}{1+c},$$

because $\lim_{p \to +\infty} \left\{\frac{2e(r-2)^2}{p^2}\right\} = 0$. ■

σ **unknown :** Let $X \sim N_p(\theta, \sigma^2 I_p)$, $Y = \frac{X}{\sigma} \sim N_p\left(\frac{\theta}{\sigma}, I_p\right)$ and for all $r\left(2 < r < \frac{p+2}{2}\right)$, we consider

$$\delta_r^\sigma = \delta_{JS}^\sigma + g\left(S^2\right)^{\frac{r}{2}} X \|X\|^{-r}, \tag{3.5}$$

where $g = \frac{(r-2)}{2} \frac{(n+p)\Gamma\left(\frac{n+r}{2}\right)}{(n+2)\Gamma\left(\frac{n+2r}{2}\right)} \frac{\Gamma\left(\frac{p-r}{2}\right)}{\Gamma\left(\frac{p-2r+2}{2}\right)}$.

We know that the risk of the estimator δ_r^σ is

$$R(\delta_r^\sigma,\theta) = R(\delta_{JS}^\sigma,\theta) + \sigma^2 \left\{2g\left[2^{\frac{r}{2}}(p-r)\frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - \frac{p-2}{n+2} 2^{\frac{r+2}{2}} \frac{\Gamma\left(\frac{n+r+2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]\right\} E\left(\|Y\|^{-r}\right)$$

$$+ \sigma^2 g^2 2^r \frac{\Gamma\left(\frac{n+2r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} E\left(\|Y\|^{-2r+2}\right). \tag{3.6}$$

Theorem 5 *If* $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c(>0)$, *then*

$$\lim_{n,p \to +\infty} \frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} = \frac{c}{c+1}.$$

Proof. On the one hand, from T.F. Li and W.H. Kuo [9], it is clear that

$$R(\delta_r^\sigma, \theta) \le R(\delta_{JS}^\sigma, \theta) \quad \forall p \ge 3, \ \forall n \ge 1 \text{ and } \forall \|\theta\|^2 > 0.$$

Then

$$\lim_{p \to +\infty} \frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \le \lim_{p \to +\infty} \frac{R(\delta_{JS}^\sigma, \theta)}{R(X,\theta)}$$

and

$$\lim_{n,p \to +\infty} \frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \le \lim_{n,p \to +\infty} \frac{R(\delta_{JS}^\sigma, \theta)}{R(X,\theta)}.$$

Thus, from formulas (2.2) and (2.3) we have

$$\lim_{p \to +\infty} \frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \le \frac{\frac{2}{n+2} + c}{1+c}$$

and

$$\lim_{n,p \to +\infty} \frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \le \frac{c}{1+c}.$$

On the other hand, from the formula (3.6) we have

$$\frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \ge \frac{R(\delta_{JS}^\sigma, \theta)}{R(X,\theta)} - 2g\frac{(p+2)}{p(n+2)} 2^{\frac{(r+2)}{2}} \frac{\Gamma\left(\frac{n+r+2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} E\left(\|Y\|^{-r}\right)$$

$$\ge \frac{R(\delta_{JS}^\sigma, \theta)}{R(X,\theta)}$$

$$-2\frac{(r-2)}{2}\frac{(n+p)}{p(n+2)}\frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n+2r}{2}\right)}\frac{\Gamma\left(\frac{p-r}{2}\right)}{\Gamma\left(\frac{p-2r+2}{2}\right)}\frac{(p-2)}{(n+2)}2^{\frac{(r+2)}{2}}\frac{\Gamma\left(\frac{n+r+2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}2^{-\frac{r}{2}}\frac{\Gamma\left(\frac{p-2}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}, \quad (3.7)$$

the inequality (3.7) comes from the Lemma 3 and the formula (3.3). Hence

$$\frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \ge \frac{R(\delta_{JS}^\sigma, \theta)}{R(X,\theta)}$$

$$-2(r-2)\frac{(n+p)(p-2)}{p(n+2)^2}\frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n+2r}{2}\right)}\frac{\left[\Gamma\left(\frac{p-r}{2}\right)\right]^2}{\Gamma\left(\frac{p-2r+2}{2}\right)\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(\frac{n+r}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{\Gamma\left(\frac{p}{2}\right)}.$$

Using the Stirling formula, we have

$$\lim_{n,p \to +\infty} \frac{R(\delta_r^\sigma, \theta)}{R(X,\theta)} \ge \lim_{n,p \to +\infty} \frac{R(\delta_{JS}^\sigma, \theta)}{R(X,\theta)}$$

$$-_{n \xrightarrow{} +\infty}\lim 2e(r-2)\frac{1}{(n+2)^2}\frac{\left(\frac{n+r}{2}\right)\left\{\sqrt{2\pi}\left(\frac{n+r-2}{2}\right)^{\left(\frac{n+r-2}{2}+\frac{1}{2}\right)}e^{-\left(\frac{n+r-2}{2}\right)}\right\}^2}{\sqrt{2\pi}\left(\frac{n+2r-2}{2}\right)^{\left(\frac{n+2r-2}{2}+\frac{1}{2}\right)}e^{-\left(\frac{n+2r-2}{2}\right)}\sqrt{2\pi}\left(\frac{n-2}{2}\right)^{\left(\frac{n-2}{2}+\frac{1}{2}\right)}e^{-\left(\frac{n-2}{2}\right)}}$$

$$\geq_{n,p \xrightarrow{} +\infty}\lim \frac{R(\delta_{JS}^{\sigma},\theta)}{R(X,\theta)} -_{n \xrightarrow{} +\infty}\lim \frac{2e(r-2)\left(\frac{n+r}{2}\right)}{(n+2)^2}$$

$$\geq_{n,p \xrightarrow{} +\infty}\lim \frac{R(\delta_{JS}^{\sigma},\theta)}{R(X,\theta)} = \frac{c}{1+c},$$

because $_{n \xrightarrow{} +\infty}\lim \frac{2e(r-2)\left(\frac{n+r}{2}\right)}{(n+2)^2} = 0$. ■

3.2. Family of Djamel Benmansour and Tahar Mourid

σ **known :** Let the family of estimators

$$\delta_{l,\delta_{JS},\psi} = \delta_{l,\delta_{JS}} = \delta_{JS} + l\psi\left(\|X\|^2\right)X, \tag{3.8}$$

where l is a positive parameter and ψ is a function with support $[0,b]$, $b \leq p-2$, strictly positive and $0 \leq \psi(u) \leq \frac{2}{l}\left(\frac{p-2}{u}-1\right)$ for all $u \in [0,b]$.

The risk of the estimator $\delta_{l,\delta_{JS}}$ is

$$R\left(\delta_{l,\delta_{JS}},\theta\right) = R(\delta_{JS},\theta) + \int_0^b lu\psi(u)\left\{l\psi(u) + 2 - \frac{2(p-2)}{u}\right\}\chi_p^2(\lambda;du)$$

$$-2l\lambda\int_0^b \psi(u)\chi_{p+2}^2(\lambda;du) \text{ où } \lambda = \|\theta\|^2. \tag{3.9}$$

Theorem 6 *If* $_{p \xrightarrow{} +\infty}\lim \frac{\|\theta\|^2}{p} = c(>0)$*, then*

$$_{p \xrightarrow{} +\infty}\lim \frac{R\left(\delta_{l,\delta_{JS}},\theta\right)}{R(X,\theta)} = \frac{c}{c+1}.$$

Proof. On the one hand, from D. Benmansour and T. Mourid [3], we have

$$\frac{R\left(\delta_{l,\delta_{JS}},\theta\right)}{R(X,\theta)} \leq \frac{R(\delta_{JS},\theta)}{R(X,\theta)} \quad \forall p \geq 3 \text{ and } \forall\|\theta\|^2 > 0,$$

hence

$$_{p \xrightarrow{} +\infty}\lim \frac{R\left(\delta_{l,\delta_{JS}},\theta\right)}{R(X,\theta)} \leq_{p \xrightarrow{} +\infty}\lim \frac{R(\delta_{JS},\theta)}{R(X,\theta)}.$$

From G. Casella and J.T. Hwang [5], we have $_{p \xrightarrow{} +\infty}\lim \frac{R(\delta_{JS},\theta)}{R(X,\theta)} = \frac{c}{c+1}$, thus

$$\lim_{p\to+\infty}\frac{R\left(\delta_{l,\delta_{JS}},\theta\right)}{R(X,\theta)}\le\frac{c}{c+1}.$$

On the other hand, according to the formula (3.9) we have

$$R\left(\delta_{l,\delta_{JS}},\theta\right)\ge R(\delta_{JS},\theta)-2(p-2)l\int_0^b\psi(u)\chi_p^2(\lambda;du)-2l\lambda\int_0^b\psi(u)\chi_{p+2}^2(\lambda;du)$$

and from the formula (5.1) of lemma 9 in the appendix, we have

$$-2\lambda\int_0^b\psi(u)\chi_{p+2}^2(\lambda;du)\ge-\int_0^b u\psi(u)\chi_{p-2}^2(\lambda;du)$$

and the fact that $0\le\psi(u)\le\frac{2}{l}\left(\frac{p-2}{u}-1\right)$, we obtain

$$R\left(\delta_{l,\delta_{JS}},\theta\right)\ge R(\delta_{JS},\theta)-4(p-2)^2\int_0^b\frac{1}{u}\chi_p^2(\lambda;du)-2(p-2)\int_0^b\chi_{p-2}^2(\lambda;du).$$

According to the formula (5.1) of lemma 9 in the appendix, we have

$$-(p-2)\int_0^b\frac{1}{u}\chi_p^2(\lambda;du)\ge-P\left(\chi_{p-2}^2(\lambda)\le b\right),$$

where $P\left(\chi_{p-2}^2(\lambda)\le b\right)$ indicate the probability of the set $\left\{\chi_{p-2}^2(\lambda)\le b\right\}$.

The fact that $b<p-2$, we have

$$-(p-2)\int_0^b\frac{1}{u}\chi_p^2(\lambda;du)\ge-P\left(\chi_{p-2}^2(\lambda)\le p-2\right),$$

then
$$R\left(\delta_{l,\delta_{JS}},\theta\right)\ge R(\delta_{JS},\theta)-6(p-2)P\left(\chi_{p-2}^2(\lambda)\le p-2\right),$$

hence

$$\lim_{p\to+\infty}\frac{R\left(\delta_{l,\delta_{JS}},\theta\right)}{R(X,\theta)}\ge\lim_{p\to+\infty}\frac{R(\delta_{JS},\theta)}{R(X,\theta)}-6\lim_{p\to+\infty}P\left(\chi_{p-2}^2(\lambda)\le p-2\right).$$

From G. Casella, and J.T. Hwang. [5], we have $\lim_{p\to+\infty}P\left(\chi_{p-2}^2(\lambda)\le p-2\right)=0$.

As $\lim_{p\to+\infty}\frac{R(\delta_{JS},\theta)}{R(X,\theta)}=\frac{c}{c+1}$, thus we find

$$\lim_{p\to+\infty}\frac{R\left(\delta_{l,\delta_{JS}},\theta\right)}{R(X,\theta)}\ge\lim_{p\to+\infty}\frac{R(\delta_{JS},\theta)}{R(X,\theta)}$$
$$\ge\frac{c}{c+1}.$$

■

σ **unknown :** Let the family of estimators

$$\delta_{l,\delta_{JS}^\sigma,\psi}^\sigma=\delta_{l,\delta_{JS}^\sigma}^\sigma=\delta_{\delta_{JS}^\sigma}^\sigma+l\psi\left(\|X\|^2,S^2\right)X. \tag{3.10}$$

where l is a positive parameter, and $\psi\left(.,S^2\right)$ is a function with support $[0,b]$, $b\le p-2$, strictly positive with $0\le\psi\left(u,S^2\right)\le\frac{2}{l}\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\right)I_{\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\ge1\right)}$ for all $u\in[0,b]$ and $S^2>0$.

Where : $I_{\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\geq 1\right)}$ indicates the indicating function of the set $\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\geq 0\right)$.

Proposition 7 *The risk function of the estimator* $\delta^{\sigma}_{l,\delta^{\sigma}_{JS}}$ *is*

$$R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS},\psi},\theta\right)=R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)+\int_{0}^{b}\int_{\frac{u}{\alpha}}^{+\infty}lu\,\psi\left(u,S^2\right)\left\{l\psi\left(u,S^2\right)+2-\frac{2(p-2)}{n+2}\frac{S^2}{u}\right\}\chi^2_n\left(0;dS^2\right)\chi^2_p(\lambda;du)$$
$$-2l\lambda\sigma^2E\left\{\psi\left(\chi^2_{p+2}(\lambda),S^2\right)\right\},\qquad(3.11)$$

where $\alpha=\frac{p-2}{n+2}$ and $\lambda=\frac{\|\theta\|^2}{\sigma^2}$.

Proof. According immediately to the following equality

$E\left(\langle X,\theta\rangle\psi\left(\|X\|^2,S^2\right)\right)=\sigma^2\lambda E\left(\psi\left(\chi^2_{p+2}(\lambda),S^2\right)\right)$

and of the independence of two random variables X and S^2. ■

Theorem 8 *If* $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, *then*

$$\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}=\frac{\frac{2}{n+2}+c}{c+1}$$

and

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}=\frac{c}{c+1}.$$

Proof. On the one hand, it is clear from the condition $0\leq\psi\left(u,S^2\right)\leq\frac{2}{l}\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\right)I_{\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\geq 1\right)}$ that

$R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)\leq R\left(\delta^{\sigma}_{JS},\theta\right)\quad\forall p\geq 3$ and $\forall n\geq 1$,

hence

$$\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\leq\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{JS},\theta\right)}{R(X,\theta)}$$

and

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\leq\lim_{n,p\to+\infty}\frac{R\left(\delta^{\sigma}_{JS},\theta\right)}{R(X,\theta)}.$$

Therefore, according to the formulas (2.2) and (2.3), we find

$$\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\leq\frac{\frac{2}{n+2}+c}{c+1}$$

and

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\leq\frac{c}{c+1}.$$

On the other hand, from to the proposition 7, we have

$$R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS},\psi},\theta\right)\geq R\left(\delta^{\sigma}_{JS},\theta\right)-\frac{2(p-2)}{n+2}l\int_{0}^{b}\int_{\frac{u}{\alpha}}^{+\infty}\psi\left(u,S^{2}\right)S^{2}\chi^{2}_{n}\left(0;dS^{2}\right)\chi^{2}_{p}(\lambda;du)$$
$$-2l\lambda\sigma^{2}E\left\{\psi\left(\chi^{2}_{p+2}(\lambda),S^{2}\right)\right\}.$$

According to the formula (5.1) of lemma 9 in the appendix and the fact that $0\leq\psi\left(u,S^{2}\right)\leq\frac{2}{l}\left(\frac{p-2}{n+2}\frac{S^{2}}{u}-1\right)I_{\left(\frac{p-2}{n+2}\frac{S^{2}}{u}-1\geq 1\right)}$, we obtain

$$R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS},\psi},\theta\right)\geq R\left(\delta^{\sigma}_{JS},\theta\right)-4\frac{(p-2)^{2}}{(n+2)^{2}}\int_{0}^{b}\int_{\frac{u}{\alpha}}^{+\infty}\frac{\left(S^{2}\right)^{2}}{u}\chi^{2}_{n}\left(0;dS^{2}\right)\chi^{2}_{p}(\lambda;du)$$
$$-\frac{2(p-2)}{n+2}\sigma^{2}\int_{0}^{b}\int_{\frac{u}{\alpha}}^{+\infty}S^{2}\chi^{2}_{n}\left(0;dS^{2}\right)\chi^{2}_{p-2}(\lambda;du)$$
$$\geq R\left(\delta^{\sigma}_{JS},\theta\right)-4\frac{(p-2)^{2}}{(n+2)^{2}}\int_{0}^{b}\int_{0}^{+\infty}\frac{\left(S^{2}\right)^{2}}{u}\chi^{2}_{n}\left(0;dS^{2}\right)\chi^{2}_{p}(\lambda;du)$$
$$-\frac{2(p-2)}{n+2}\sigma^{2}\int_{0}^{b}\int_{0}^{+\infty}S^{2}\chi^{2}_{n}\left(0;dS^{2}\right)\chi^{2}_{p-2}(\lambda;du).$$

Using the Lemma 9 in the appendix, the independence of two random variables X and S^{2}, and the fact that $-P\left(\chi^{2}_{p-2}(\lambda)\leq b\right)\geq -P\left(\chi^{2}_{p-2}(\lambda)\leq p-2\right)$, we have

$$R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS},\psi},\theta\right)\geq R\left(\delta^{\sigma}_{JS},\theta\right)-6(p-2)\frac{n}{n+2}\sigma^{4}P\left(\chi^{2}_{p-2}(\lambda)\leq p-2\right),$$

hence

$$\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\geq\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{JS},\theta\right)}{R(X,\theta)}-\lim_{p\to+\infty}\frac{6(p-2)n\sigma^{4}}{p\sigma^{4}(n+2)}P\left(\chi^{2}_{p-2}(\lambda)\leq p-2\right).$$

From G. Casella and J.T. Hwang [5], we have

$\lim_{p\to+\infty}P\left(\chi^{2}_{p-2}(\lambda)\leq p-2\right)=0$ and by using formulas (2.2) and (2.3), we have

$$\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\geq\lim_{p\to+\infty}\frac{R\left(\delta^{\sigma}_{JS},\theta\right)}{R(X,\theta)}$$
$$\geq\frac{\frac{2}{n+2}+c}{c+1}$$

and

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\sigma}_{l,\delta^{\sigma}_{JS}},\theta\right)}{R(X,\theta)}\geq\lim_{n,p\to+\infty}\frac{R\left(\delta^{\sigma}_{JS},\theta\right)}{R(X,\theta)}$$
$$\geq\frac{c}{c+1}.$$

■

IV. Simulation results

We illustrate graphically in what follows the risk ratios: $\frac{R(\delta_{JS}^{\sigma},\theta)}{R(X,\theta)}$, $\frac{R(\delta_{\sigma}^{r},\theta)}{R(X,\theta)}$ and $\frac{R(\delta_{l,\delta_{JS}^{\sigma}}^{\sigma},\theta)}{R(X,\theta)}$ as function of $\lambda=\frac{\|\theta\|^2}{\sigma^2}$, when $\psi(u,S^2)=\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\right)I_{\left(\frac{p-2}{n+2}\frac{S^2}{u}-1\geq1\right)}$ and different values of n and p.

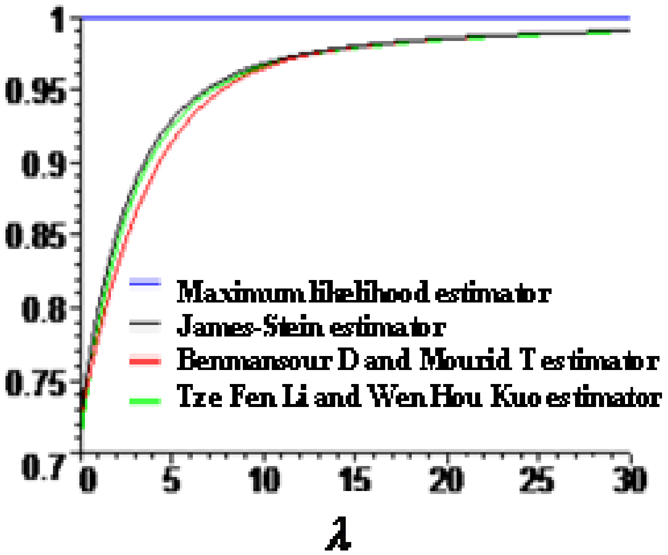

Fig.1 Graph of risk ratios $\frac{R(\delta_{JS}^{\sigma},\theta)}{R(X,\theta)}$, $\frac{R(\delta_{r}^{\sigma},\theta)}{R(X,\theta)}$ and $\frac{R(\delta_{l,\delta_{JS}^{\sigma}}^{\sigma},\theta)}{R(X,\theta)}$, for $n=10$, $p=3$ and $r=2.25$ as function of $\lambda=\frac{\|\theta\|^2}{\sigma^2}$.

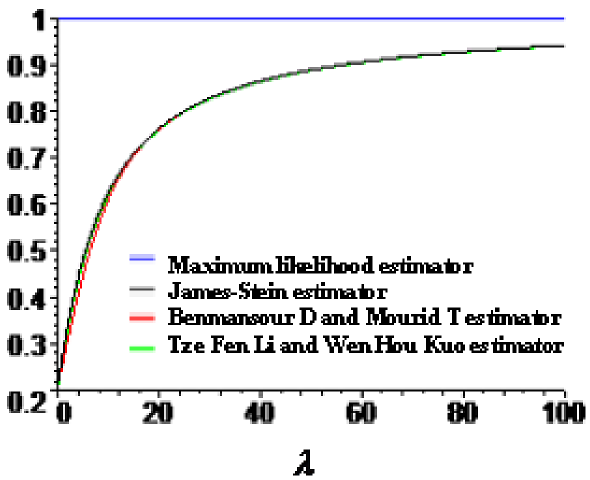

Fig.2 Graph of risk ratios $\frac{R(\delta_{JS}^{\sigma},\theta)}{R(X,\theta)}$, $\frac{R(\delta_{r}^{\sigma},\theta)}{R(X,\theta)}$ and $\frac{R(\delta_{l,\delta_{JS}^{\sigma}}^{\sigma},\theta)}{R(X,\theta)}$, for $n=100$, $p=10$ and $r=3$ as function of $\lambda=\frac{\|\theta\|^2}{\sigma^2}$.

V. Appendix

Lemma 9 *(G. Casella and J.T. Hwang [5]). For any real function* h *such as* $E\{h(\chi_q^2(\lambda))\chi_q^2(\lambda)\}$ exist, we have

$$E\{h(\chi_q^2(\lambda))\chi_q^2(\lambda)\} = qE\{h(\chi_{q+2}^2(\lambda))\} + 2\lambda E\{h(\chi_{q+4}^2(\lambda))\}. \quad (5.1)$$

Lemma 10 *(D. Benmansour and T. Mourid [3]). Let a random variable* $U \sim \chi_p^2(\lambda)$, *for* $s \geq 0$, $r > -\frac{p}{2}$, $t = \frac{1}{1+2s}$ *and* K *a random variable of Poisson's law* $P\left(\frac{\lambda t}{2}\right)$, *we have*

$$E\left(U^r e^{-sU}\right) = 2^r t^{\frac{p}{2}+r} e^{\frac{\lambda(t-1)}{2}} E\left\{\frac{\Gamma\left(\frac{p}{2}+K+r\right)}{\Gamma\left(\frac{p}{2}+K\right)}\right\}. \quad (5.2)$$

VI. Conclusion

In context of study of asymptotic behaviour of the risk ratios of shrinkage estimator of the mean θ of a multivariate Gaussian random $N_p(\theta, \sigma^2 I_p)$ in $\Re^p$. G. Casella and J.T. Hwang [5], studied the case where σ^2 is known ($\sigma^2 = 1$), they showed that if $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p} = c(>0)$ then the risk ratios $\frac{R(\delta_{JS}, \theta)}{R(X, \theta)}$ and $\frac{R(\delta_{JS}^+, \theta)}{R(X, \theta)}$ tend to $\frac{c}{c+1}$. D. Benmansour and A. Hamdaoui [2] have taking the same model, namely $X \sim \mathrm{N_p}(\theta, \sigma^2 I_p)$ with σ^2 unknown and estimated by the statistic $S^2 \sim \sigma^2 \chi_n^2$ independent of X. They showed that if $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c(>0)$ then the risk ratios $\frac{R(\delta_{JS}^{\sigma}, \theta)}{R(X, \theta)}$ and $\frac{R(\delta_{JS}^{\sigma+}, \theta)}{R(X, \theta)}$ tend to $\frac{\frac{2}{n+2}+c}{c+1}$ when p tends to infinity and n fixed in the one hand, and in the other hand, they showed that in the same condition, namely $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c(>0)$, the risk ratios $\frac{R(\delta_{JS}^{\sigma}, \theta)}{R(X, \theta)}$ and $\frac{R(\delta_{JS}^{\sigma+}, \theta)}{R(X, \theta)}$ tend to $\frac{c}{c+1}$ when p and n tend simultaneously to infinity, without assuming any order relation or functional relation between p and n.

In our work by taking the same model $X \sim N_p(\theta, \sigma^2 I_p)$, we study the asymptotic behaviour of the risk ratios of some minimaxs shrinkage estimators, then we show that the limit of risk ratios of polynomial estimator, estimator proposed by T.F. Li and W.H. Kuo [9] and the estimator proposed by D. Benmansour and T. Mourid [3], to the maximum likelihood estimator, tend to values less than one.

An idea would be to see whether one can obtain similar results of the asymptotic behaviour of risk ratios in the general case of the symmetrical spherical model, for general classes of shrinkage estimators.

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