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Fractional Physical Models via Natural Transform

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Abstract: In this paper, we propose a new technique, namely homotopy perturbation natural transform method (HPNTM) for solving fractional physical models. It consists of coupling the natural transform method and the homotopy perturbation method. This technique yields an analytical solution in terms of a rapidly convergent infinite power series with easily computable terms. The fractional derivatives are described in the Caputo sense. The results obtained reveal the efficiency, simplicity and applicability of the proposed method for solving other fractional physical models.
Keywords: Natural transform, Homotopy perturbation natural transform method (HPNTM),Fractional Jaulent–Miodek system.

I. Introduction

With the progress of science and engineering, nonlinear fractional differential equations had been used as the models to describe real physical phenomena in solid state physics, plasma waves, fluid mechanics, chemical physics and so forth. Thus, for the last few decades, huge attention has been focused for finding the solutions(both analytical and numerical)of these problems, which is associated with energy-dependent Schrödinger potential[1-3].Systems of nonlinear partial differential equations [4,5] come up in lots of scientific physical models. In contemporary years, significant research has been done to study the classical Jaulent–Miodek equations. Various methods such as unified algebraic method [6], a domian decomposition method [7], tanh-sech method [8], and homotopy analysis method [9]had been implemented for solving of coupled Jaulent–Miodek equations, the comprehensive analysis of the nonlinear fractional order coupled Jaulent–Miodek equation is only an initiation. The purpose of this paper is to find an approximate solution for the system of fractional physical-Jaulent- Miodek system in novel form via natural transform. The natural transform, initially was defined by Waqar et al. [10] as the N - transform, which studied their properties and applications. Later, Belgacem et al. [11,12] defined its inverse and studied some additional fundamental properties of this integral transform and named it the natural transform. Applications of natural transform in the solution of differential and integral equations and for the distribution and Bohemians spaces can be found in [12, 13, 14, 15, 16, 17, 18]. Now, we mention the following basic definitions of natural transform and its properties as follows:

Definition1.1[11].
Over the set of functions

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,\left|\ f(t)\ \right|<Me^{|t|/\tau_j},\ if\ t\in(-1)^j\times[0,\infty)\right\}$$

The natural transform of $f(t)$ is $N[f(t)]=R(s;u)=\int_0^\infty f(ut)e^{-st}dt, u>0, s>0$ (1)

where $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables u and s are the natural transform variables.

Theorem1.2. We derives the relationship between natural and Laplace, Sumudu transform in successive theorems as follow[11]:

1- If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A, $G(u)$ is Sumudu transform then,

$$N[f(t)]=R(s;u)=\frac{1}{u}\int_0^\infty f(t)e^{-\frac{st}{u}}dt=\frac{1}{u}F(\frac{s}{u}) \tag{2}$$

2.If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A then, $G(u)$ is Sumudu transform of function $f(t)$ in A, then:

$$N[f(t)] = R(s;u) = \frac{1}{s}\int_0^\infty f(\frac{ut}{s})e^{-t}dt = \frac{1}{s}G(\frac{u}{s}) \quad (3)$$

3- If $f^n(t)$ is the *n*th derivative of function $f(t)$ then, its natural transform is given by:

$$N[f^n(t)] = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^{(k)}(0), n \geq 1 \quad (4)$$

4. If $F(s,u)$, $G(s,u)$ are the natural transform of respective functions $f(t)$, $g(t)$ both defined in set A then,

$$N[f * g] = uF(s,u)G(s,u) \quad (5)$$

where $f * g$ is convolution of two functions f and g .

5. If $N[f(t)]$ is the natural transform of the function $f(t)$, then the natural transform of fractional derivative of order α is defined as:

$$N[f^{(\alpha)}(t)] = \frac{s^\alpha}{u^\alpha}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0) \quad (6)$$

6. Let the function $f(t)$ belongs to set A be multiplied with weight function $e^{\pm t}$ then,

$$N[e^{\pm t}f(t)] = \frac{s}{s \mp u}R[\frac{s}{s \mp u}] \quad (7)$$

7. Let the function $f(at)$ belongs to set A, where a is non-zero constant then,

$$N[f(at)] = \frac{1}{a}R[\frac{s}{a}, u] \quad (8)$$

8. If $w^n(t)$ is given by $w^n(t) = \int_0^t ....\int_0^t f(t)(dt)^n dt$, then, the natural transform of $w^n(t)$ is given by:

$$N[w^n(t)] = \frac{u^n}{s^n}R(s,u) \quad (9)$$

9. The natural transform of T-periodic function $f(t) \in A$ such that $f(t+nT) = f(t), n = 0,1,2,...$ is given by:

$$N[f(t)] = R(s,u) = [1 - e^{\frac{-sT}{u}}]^{-1}\frac{1}{u}\int_0^T e^{-\frac{st}{u}}f(t)dt \quad (10)$$

10. The function $f(t)$ in set A is multiplied with shift function t^n, then,

$$N[t^n f(t)] = \frac{u^n}{s^n}\frac{d}{du^n}^n u^n R(s,u) \quad (11)$$

II. The basic idea of homotopy perturbation natural transform method(HPNTM).

To illustrate the basic idea of HPNTM, we consider the following nonlinear fractional differential equation:

$$D_t^\alpha U(x,t) + L(U(x,t)) + F(U(x,t)) = q(x,t), \; t > 0, 0 < \alpha < 1 \quad (12)$$

subject to initial condition :

$$U(x,0) = f(x)$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional Caputo derivative of the function $U(x,t)$,

L is the linear differential operator, F is the nonlinear differential operator, and $q(x,t)$ is the source term. Now, applying the natural transform on both sides of (12) we have:

$$\frac{s^\alpha}{u^\alpha}N[U] - \sum_{k=0}^{\alpha-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}U^{(k)}(0) + N[LU] + N[FU] = N[q(x,t)] \quad (13)$$

On simplifying

$$N[U] - \frac{u^\alpha}{s^\alpha}\sum_{k=0}^{\alpha-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}U^{(k)}(0) + \frac{u^\alpha}{s^\alpha}[N[LU] + N[FU] - N[q(x,t)]] = 0 \quad (14)$$

Operating with natural inverse on both sides of (14):

$$U(x,t)=Q(x,t)-N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[L(U(x,t))+F(U(x,t))]] \quad .(15)$$

where $Q(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t)=\sum_{n=0}^{\infty} p^n U_n(x,t), \quad (16)$$

where the homotopy parameter p is considered as a small parameter $p\in[0,1]$.
We can decompose the nonlinear term as:

$$FU(x,t)=\sum_{n=0}^{\infty} p^n H_n(U), \quad (17)$$

where H_n are He's polynomials of $U_0(x,t),U_1(x,t),U_2(x,t),....,U_n(x,t)$ and it can be calculated by the following formula:

$$H_n(U_0(x,t),U_1(x,t),U_2(x,t),....,U_n(x,t))=\frac{1}{n!}\frac{\partial^n}{\partial p^n}[F(\sum_{i=0}^{\infty} p^i U_i)]_{p=0} \quad (18)$$

By substituting (16) and (17) and using HPM we get:

$$\sum_{n=0}^{\infty} p^n U_n(x,t)=Q(x,t)-p(N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[L(\sum_{n=0}^{\infty} p^n U_n(x,t))+(\sum_{n=0}^{\infty} p^n H_n(U(x,t)))]]). \quad (19)$$

This is coupling of natural transform and homotopy perturbation method using He's polynomials. By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0: U_0(x,t)=Q(x,t). \quad (20)$$

$$p^1: U_1(x,t)=-(N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[L(U_0(x,t))+(H_0(U(x,t)))]]), \quad (21)$$

$$p^2: U_2(x,t)=-(N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[L(U_1(x,t))+(H_1(U(x,t)))]]), \quad (22)$$

$$p^3: U_3(x,t)=-(N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[L(U_2(x,t))+(H_2(U(x,t)))]]). \quad (23)$$

Proceeding in the same manner, the rest of the components $U_n(x,t)$ can be completely obtained, and the series solution is thus entirely determined. Finally, we approximate the solution $U(x,t)$ by truncated series.

$$U(x,t)=\lim_{N\to\infty}\sum_{n=0}^{N} U_n(x,t) \quad (24)$$

These series solutions generally converge very rapidly.

3. An application:

Consider the time-fractional coupled Jaulent-Mdek (JM) equations:

$${}_0D_t^{\alpha}U+U_{xxx}+\frac{3}{2}VV_{xxx}+\frac{9}{2}V_xV_{xx}-6UU_x-6UVV_x-\frac{3}{2}U_xV^2=0, \quad (25)$$

$${}_0D_t^{\alpha}V+V_{xxx}-6U_xV-6UV_x-\frac{15}{2}V_xV^2=0,\ \ 0<\alpha<1 \quad (26)$$

with initial conditions [9]:

$$U(x,0)=\frac{1}{8}\lambda^2[1-4\sec h^2(\frac{\lambda x}{2})],$$

$$V(x,0)=\lambda\sec h[\frac{\lambda x}{2}].$$

where λ is an arbitrary constant. For α=1,the exact solutions of Eqs.(25) and (26) are given by[9]:

$$U(x,t)=\frac{1}{8}\lambda^2[1-4\sec h^2(\frac{1}{2}\lambda(x+\frac{1}{2}\lambda^2 t)]$$

$$V(x,t)=\lambda\sec h[\frac{1}{2}\lambda(x+\frac{1}{2}\lambda^2 t)]$$

Now, applying the natural transform on both sides of Eqs.(25) and (26) we have:

$$\frac{s^\alpha}{u^\alpha}N[U]-\sum_{k=0}^{\alpha-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}U^k(0)+N[U_{xxx}+\frac{3}{2}VV_{xxx}+\frac{9}{2}V_xV_{xx}-6UU_x-6UVV_x-\frac{3}{2}U_xV^2]=0$$

$$\frac{s^\alpha}{u^\alpha}N[V]-\sum_{k=0}^{\alpha-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}V^k(0)+N[V_{xxx}-6U_xV-6UV_x-\frac{15}{2}V_xV^2]=0$$

On simplifying

$$N[U]=\frac{1}{8s}\lambda^2[1-4\sec h^2(\frac{\lambda x}{2})]-\frac{u^\alpha}{s^\alpha}N[U_{xxx}+\frac{3}{2}VV_{xxx}+\frac{9}{2}V_xV_{xx}-6UU_x-6UVV_x-\frac{3}{2}U_xV^2], \tag{27}$$

$$N[V]=\frac{1}{s}\lambda\sec h(\frac{\lambda x}{2})-\frac{u^\alpha}{s^\alpha}N[V_{xxx}-6U_xV-6UV_x-\frac{15}{2}V_xV^2]. \tag{28}$$

Operating with natural inverse on both sides of Eqs.(27) and (28) we get:

$$U=\frac{1}{8}\lambda^2[1-4\sec h^2(\frac{\lambda x}{2})]-N^{-1}[\frac{u^\alpha}{s^\alpha}N[U_{xxx}+\frac{3}{2}A_n+\frac{9}{2}B_n-6C_n-6D_n-\frac{3}{2}E_n]], \tag{29}$$

$$V=\lambda\sec h(\frac{\lambda x}{2})-N^{-1}[\frac{u^\alpha}{s^\alpha}N[V_{xxx}-6F_n-6G_n-\frac{15}{2}H_n]]. \tag{30}$$

where nonlinear terms are: $A_n,B_n,C_n,D_n,E_n,F_n,G_n,H_n$ and it can be calculated by the following formula:

$$A_n,B_n,C_n,D_n,E_n,F_n,G_n,H_n\ (U_0(x,t),U_1(x,t),....,U_n(x,t))=\frac{1}{n!}\frac{\partial^n}{\partial p^n}[F(\sum_{i=0}^{\infty}p^iU_i)]_{p=0}, \tag{31}$$

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t)=\sum_{n=0}^{\infty}p^nU_n(x,t)$$

where the homotopy parameter p is considered as a small parameter $p\in[0,1]$.

$$\sum_{n=0}^{\infty}p^nU_n=\frac{1}{8}\lambda^2[1-4\sec h^2(\frac{\lambda x}{2})]-N^{-1}[\frac{u^\alpha}{s^\alpha}N[\sum_{n=0}^{\infty}p^nU_{nxxx}+\frac{3}{2}A_n+\frac{9}{2}B_n-6C_n-6D_n-\frac{3}{2}E_n]]$$

$$\sum_{n=0}^{\infty}p^nV_n=\lambda\sec h(\frac{\lambda x}{2})-N^{-1}[\frac{u^\alpha}{s^\alpha}N[\sum_{n=0}^{\infty}p^nV_{nxxx}-6F_n-6G_n-\frac{15}{2}H_n]]$$

This is coupling of natural transform and homotopy perturbation method using He's polynomials. By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0:U(x,0)=\frac{1}{8}\lambda^2[1-4\sec h^2(\frac{\lambda x}{2})], \tag{32}$$

$$p^0:V(x,0)=\lambda\sec h(\frac{\lambda x}{2}), \tag{33}$$

$$p^1:U_1=-N^{-1}[\frac{u^\alpha}{s^\alpha}N[U_{0xxx}+\frac{3}{2}A_0+\frac{9}{2}B_0-6C_0-6D_0-\frac{3}{2}E_0]],$$

$$p^1 : V_1 = -N^{-1}[\frac{u^\alpha}{s^\alpha} N[V_{0xxx} - 6F_0 - 6G_0 - \frac{15}{2} H_0]],$$

$$U_1 = -\frac{t^\alpha}{\Gamma(\alpha+1)}[U_{0xxx} + \frac{3}{2}V_0 V_{0xxx} + \frac{9}{2}V_{0x}V_{0xx} - 6U_0 U_{0x} - 6U_0 V_0 V_{0x} - \frac{3}{2}U_{0x}{V_0}^2],$$

$$V_1 = -\frac{t^\alpha}{\Gamma(\alpha+1)}[V_{0xxx} - 6U_{0x}V_0 - 6U_0 V_{0x} - \frac{15}{2}V_{0x}{V_0}^2],$$

Hence,

$$U_1 = 2\lambda^5 \frac{t^\alpha}{\Gamma(\alpha+1)} \cos ech^3[\lambda x]\sinh^4[\frac{\lambda}{2}x], \tag{34}$$

$$V_1 = -\lambda^4 \frac{t^\alpha}{\Gamma(\alpha+1)} \cos ech^2[\lambda x]\sinh^3[\frac{\lambda}{2}x], \tag{35}$$

$$U_2 = -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}[U_{1xxx} + \frac{3}{2}V_1 V_{0xxx} + \frac{3}{2}V_{1xxx}V_0 + \frac{9}{2}V_{1x}V_{0xx} + \frac{9}{2}V_{1xx}V_{0x} - 6U_0 U_{1x} - 6U_{0x}U_1 - 6U_1 V_0 V_{0x}$$

$$- 6U_0 V_1 V_{0x} - 6U_0 V_0 V_{1x} - \frac{3}{2}U_{1x}V_0^2 - 3U_{0x}V_0 V_1],$$

$$V_2 = -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}[V_{1xxx} - 6U_{0x}V_1 - 6U_{1x}V_0 - 6U_0 V_{1x} - 6U_1 V_{0x} - \frac{15}{2}V_{1x}{V_0}^2 - 15V_{0x}V_0 V_1],$$

Hence,

$$U_2 = -\frac{1}{16}\lambda^8 \frac{t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \sec h^4[\frac{\lambda}{2}x](-2 + \cosh[\lambda x]), \tag{36}$$

$$V_2 = \frac{1}{32}\lambda^7 \frac{t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \sec h^3[\frac{\lambda}{2}x](-3 + \cosh[\lambda x]). \tag{37}$$

The series solutions are:

$$U = U_0 + U_1 + U_2 + ...,$$

$$V = V_0 + V_1 + V_2 + ...$$

Then,

$$U(x,t) = \frac{1}{8}\lambda^2[1 - 4\sec h^2(\frac{\lambda x}{2})] + 2\lambda^5 \frac{t^\alpha}{\Gamma(\alpha+1)} \cos ech^3[\lambda x]\sinh^4[\frac{\lambda}{2}x]$$

$$-\frac{1}{16}\lambda^8 \frac{t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \sec h^4[\frac{\lambda}{2}x](-2 + \cosh[\lambda x]) + \ldots, \tag{38}$$

$$V = \lambda \sec h(\frac{\lambda x}{2}) - \lambda^4 \frac{t^\alpha}{\Gamma(\alpha+1)} \cos ech^2[\lambda x]\sinh^3[\frac{\lambda}{2}x] +$$

$$\frac{1}{32}\lambda^7 \frac{t^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \sec h^3[\frac{\lambda}{2}x](-3 + \cosh[\lambda x]) + ... \tag{39}$$

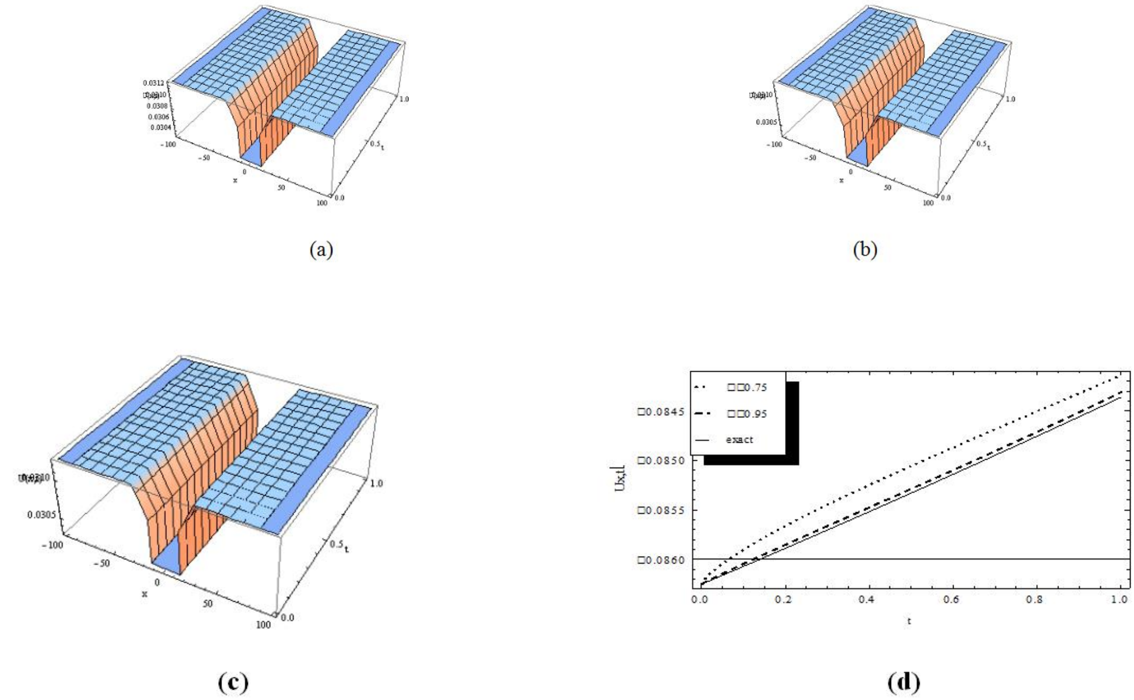

Fig 1: The approximate solution of second order of $U(x,t)$ of application 1 when (a) $\alpha = .75$, (b) $\alpha = .95$, (c) $\alpha = 1$ which is the exact solution, and plot2D of second order of $U(x,t)$ versus t at x =1, $\lambda = 0.5$ for different values of $\alpha$and comparison the results with the exact solution as shown in (d).

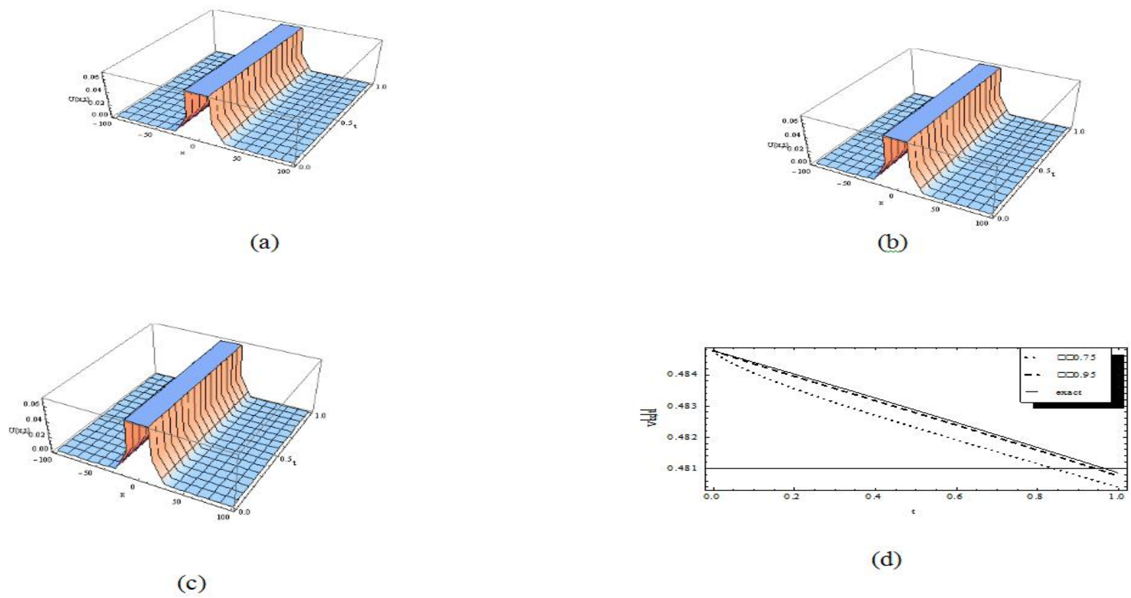

Fig 2: The approximate solution of second order of $V(x,t)$ of application 1 when (a) $\alpha = 0.75$, (b) $\alpha = 0.95$, (c) $\alpha = 1$ which is the exact solution, and plot2D of second order of $V(x,t)$ versus t at x =1, $\lambda = 0.5$ for different values of α and comparison the results with the exact solution as shown in (d).

III. Conclusion

In this paper, fractional Jaulent-Miodek system has been solved by HPNTM method. The obtained results are compared with the exact solutions. The application of the proposed method for solutions of fractional Jaulent-Miodek system are in excellent agreement with exact solution and can be applied easily for other fractional physical models .The obtained solutions are shown graphically by using the Mathematica Packages to calculate the functions obtained from the HPNTM.

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