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Certain Study on Eigen Values of Bicomplex Matrices

Dhruva Dixit
(DD) Department Of Mathematics, Institute Of Basic Science,Khandari, Dr.B.R.Ambedkar University, Agra - U.P, India, Email Address

Abstract: *Theory of matrices is an integral part of algebra as well as Theory of equations. Matrices plays an important role in every branch of Physics, Computer Graphics and are also used in representing the real world's data and there are so many applications of matrices ,for this reason we thought of studying bicomplex matrices. Bicomplex matrix has a number of applications in various fields of science as well as real world but it has captured less attention what it deserves. The monograph by Price [4] contains few exercises pertaining to matrices with bicomplex entries. Futagawa[2] and Riley[5] also contribute to the theory of bicomplex analysis .In this paper we discussed the method to find out eigen values of bicomplex matrices and obtained the result that a bicomplex matrix of order 2 has 2^2 roots and these roots can be factorized into linear factor in $2!$ essentially different ways. This result is further generalized for order n.*

Keywords: *Bicomplex matrices, Bicomplex polynomial, eigen value, Fundamental Theorem of Bicomplex Algebra.*

Symbols: C_0 *: set of real numbers ,* C_1 *: set of complex numbers ,* C_2 *: set of Bicomplex numbers.*

I. Introduction

Definition Of Bicomplex Matrix:

Let $A = [\xi_{mn}]_{m\times n}$ be a bicomplex matrix, that is a matrix having bicomplex number entries.

$$A=\begin{bmatrix}\xi_{11} & \cdots & \xi_{1n}\\ \vdots & \ddots & \vdots \\ \xi_{1m} & \cdots & \xi_{mn}\end{bmatrix} \quad \xi_{pq} \in C_2,\ 1\le p\le m\ \&\ 1\le q \le n.$$

$$A=\begin{bmatrix} z_{11}+i_2w_{11} & z_{12}+i_2w_{12} & \dots & \dots & z_{1n}+i_2w_{1n}\\ z_{21}+i_2w_{21} & z_{22}+i_2w_{22} & \dots & \dots & z_{2n}+i_2w_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ z_{m1}+i_2w_{21} & z_{m2}+i_2w_{m2} & \dots & \dots & z_{mn}+i_2w_{mn}\end{bmatrix}$$

$$A=\begin{bmatrix} z_{11} & z_{12} & \dots & \dots & z_{1n}\\ z_{21} & z_{22} & \dots & \dots & z_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ z_{m1} & z_{m2} & \dots & \dots & z_{mn}\end{bmatrix} + i_2 \begin{bmatrix} w_{11} & w_{12} & \dots & \dots & w_{1n}\\ w_{21} & w_{22} & \dots & \dots & w_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ w_{m1} & w_{m2} & \dots & \dots & w_{mn}\end{bmatrix}$$

Where z_{mn} & w_{mn} ϵ C_1

$$A=\begin{bmatrix} x_{11} & x_{12} & \dots & \dots & x_{1n}\\ x_{21} & x_{22} & \dots & \dots & x_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ x_{m1} & x_{m2} & \dots & \dots & x_{mn}\end{bmatrix} + i_1 \begin{bmatrix} y_{11} & y_{12} & \dots & \dots & y_{1n}\\ y_{21} & y_{22} & \dots & \dots & y_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ y_{m1} & y_{m2} & \dots & \dots & y_{mn}\end{bmatrix} + i_2 \begin{bmatrix} u_{11} & u_{12} & \dots & \dots & u_{1n}\\ u_{21} & u_{22} & \dots & \dots & u_{2n}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ u_{m1} & u_{m2} & \dots & \dots & u_{mn}\end{bmatrix} + i_1 i_2$$

$$\begin{bmatrix} v_{11} & v_{12} & \dots & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & \dots & v_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ v_{m1} & v_{m2} & \dots & \dots & v_{mn} \end{bmatrix}$$

where x_{mn}, y_{mn}, u_{mn} & $v_{mn} \epsilon C_0$ and $Z_{pq} = x_{pq} + i_1 y_{pq}$; $w_{pq} = u_{pq} + i_1 v_{pq}$

Every bicomplex matrix A=$[\xi_{mn}]_{m\times n}$ can be expressed uniquely as : (cf. Srivastava [6])

${}^1Ae_1 + {}^2Ae_2$ s. t.

${}^1A = \left[{}^1\xi_{mn}\right]$

${}^2A = \left[{}^2\xi_{mn}\right]$

2.1.1 Algebraic Structure Of Bicomplex Matrices:

Let S be the set of all square matrices of order n×n define the binary composition of addition, real scalar multiplication & multiplication as follows: If A= $\begin{bmatrix} \xi_{11} & \xi_{12} & .. & .. & \xi_{1n} \\ \xi_{21} & \xi_{22} & .. & .. & \xi_{2n} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ \xi_{n1} & \xi_{n2} & .. & .. & \xi_{nn} \end{bmatrix}$ &

B= $\begin{bmatrix} \eta_{11} & \eta_{12} & .. & .. & \eta_{1n} \\ \eta_{21} & \eta_{22} & .. & .. & \eta_{2n} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ \eta_{n1} & \eta_{n2} & .. & .. & \eta_{nn} \end{bmatrix}$ are arbitrary member of S then

A+B= $\begin{bmatrix} \xi_{11}+\eta_{11} & \xi_{12}+\eta_{12} & \dots & \dots & \xi_{1n}+\eta_{1n} \\ \xi_{21}+\eta_{21} & \xi_{22}+\eta_{22} & \dots & \dots & \xi_{2n}+\eta_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{n1}+\eta_{n1} & \xi_{n2}+\eta_{n2} & \dots & \dots & \xi_{nn}+\eta_{nn} \end{bmatrix}$, αA= $\begin{bmatrix} \alpha\xi_{11} & \alpha\xi_{12} & \dots & \dots & \alpha\xi_{1n} \\ \alpha\xi_{21} & \alpha\xi_{22} & \dots & \dots & \alpha\xi_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha\xi_{n1} & \alpha\xi_{n2} & \dots & \dots & \alpha\xi_{nn} \end{bmatrix}$ &

A.B= $\begin{bmatrix} \xi_{11}\eta_{11}+\dots+\xi_{1n}\eta_{n1} & \xi_{11}\eta_{12}+\dots+\xi_{1n}\eta_{n2} & \dots & \xi_{11}\eta_{1n}+\dots+\xi_{1n}\eta_{nn} \\ \xi_{21}\eta_{11}+\dots+\xi_{2n}\eta_{n1} & \xi_{21}\eta_{12}+\dots+\xi_{2n}\eta_{n2} & \dots & \xi_{21}\eta_{1n}+\dots+\xi_{2n}\eta_{nn} \\ \dots & \dots & \dots & \dots \\ \dots & .. & \dots & \dots \\ \xi_{n1}\eta_{11}+\dots+\xi_{nn}\eta_{n1} & \xi_{n1}\eta_{12}+\dots+\xi_{nn}\eta_{n2} & \dots & \xi_{n1}\eta_{1n}+\dots+\xi_{nn}\eta_{nn} \end{bmatrix}$

With these binary compositions S is an algebra.

Theorem1: Let A ==$[\xi_{ij}]_{n\times n}$ be a bicomplex square matrix . Then

Det A = Det $[{}^1A]$ e_1 + Det $[{}^2A]e_2$

Bicomplex Singular & non singular matrix:

A square matrix is said to be non singular if $|A| \notin O_2$. If $|A| \in O_2$ then A is called singular matrix.

Determinant of A is non singular then $|{}^1A| \neq 0$ & $|{}^2A| \neq 0$ (cf . Anjali [1]).

2.1.2 Let $A = [\xi_{ij}]_{n\times n}$ be bicomplex square matrix . then
$Adj[A] = Adj[{}^1A]\, e_1 + Adj\, [{}^2A]\, e_2$, This is proved by Anjali [1].

Theorem2 (Kumar [3]) : Let A & B be two square bicomplex matrix of same order n , such that $|A| \notin O_2$ and $|B| \notin O_2$, then their product (AB) wil be invertible , and the inverse of AB will be $B^{-1}A^{-1}$.

Theorem3 (Kumar [3]):
Let A & B be two square bicomplex matrices then determinant of their product will be equal to product of their individual determinants i.e. $|AB| = |A|.|B|$

2.1.3 Some Properties of Bicomplex Matrices :
(1) If A is any bicomplex square matrix of order n then det(A) and determinant of transpose A are equal.
(2) A & B are two bicomplex matrices of order n such that B is obtained from A by interchanging any two row\ column of A then $|A| = -|B|$
(3) If any one of row\ column in a square bicomplex matrix has each element in $O2 = I_1 \cup I_2$ then matrix will be singular or non invertible
* Proofs of these results are straight forward (cf. Kumar [3])

2.2.1 Bicomplex Polynomial:
The polynomial of the form $P(\xi) = \sum_{k=0}^{n} \alpha_k \xi^k$ where $\alpha_k , \xi^k \in C_2$ is called bicomplex polynomial in C_2.
Zeros of bicomplex polynomial: If $P(\xi_0) = 0$ for some ξ_0 , then we say that ξ_0 is a zero of this bicomplex polynomial $P(\xi_0)$.

2.2.2 Fundamental Theorem of bicomplex algebra:
Theorem 4:
A bicomplex polynomial of degree n with non singular leading coefficient has exactly n^2 roots counted according to their multiplicities.
Proof : Let $Pn(\xi) = \sum_{k=0}^{n} \alpha_k\, \xi^k$, $\alpha_n \notin O_2$ be a bicomplex polynomial of degree n, If $\alpha_k = {}^1\alpha_k\, e_1 + {}^2\alpha_k\, e_2$,k = 1,2,3……..n. The roots of the polynomial $P_n(\xi)$ will be the solution of the equation $P_n(\xi) = 0$ or equivalently of the equation $\sum_{k=0}^{n} \alpha_k \xi^k = 0$. The idempotent equation is $P_n(\xi) = {}^1P_n({}^1\xi)e_1 + {}^2P_n({}^2\xi)\, e_2$ where
${}^1P_n(\xi) = \sum_{k=0}^{n} {}^1\alpha_k\ {}^1\xi^k$ & $\sum_{k=0}^{n} {}^2\alpha_k\ {}^2\xi^k$

Thus $\sum_{k=0}^{n} {}^1\alpha\ {}^1\xi^k\, e_1 + \sum_{k=0}^{n} {}^2\alpha\ {}^2\xi^k\, e_2 = 0$

Since e_1 and e_2 are linearily independent with respect to complex coefficients.

$$\sum_{k=0}^{n} {}^1\alpha_k\ {}^1\xi^k = {}^1\alpha_n \left({}^1\xi\right)^n + {}^1\alpha_{n-1} \left({}^1\xi\right)^{n-1} + \ldots\ldots\ldots + {}^1\alpha_1 \left({}^1\xi\right) + {}^1\alpha_0 = 0 \qquad (2.1)$$

$$\sum_{k=0}^{n} {}^2\alpha_k\ {}^2\xi^k = {}^2\alpha_n \left({}^2\xi\right)^n + {}^2\alpha_{n-1} \left({}^2\xi\right)^{n-1} + \ldots..+ {}^2\alpha_2 \left({}^2\xi\right) + {}^2\alpha_0 = 0 \qquad (2.2)$$

Note that $\alpha_n \notin O_2 \Rightarrow {}^1\alpha_n \neq 0$ & ${}^2\alpha_n \neq 0$. Hence polynomial in (2.1) & (2.2) are of degree n .

By Fundamental Theorem of complex algebra they have precisely n roots each .
Let roots of equation (2.1) be $a_1 , a_2 , \ldots\ldots\ldots + a_n \in C_1$
& roots of equation (2.2) be $b_1 , b_2 , \ldots\ldots\ldots b_n \in C_1$
It can be verified that bicomplex number η will be a root of the Bicomplex polynomial $P_n(\xi)$ if & only if
$\eta = a_p e_1 + b_q e_2$, $1 \leq p \leq n$, $1 \leq q \leq n$.
Hence the theorem.

2.2.3 Theorem 5 (Price [4]):
If no two roots of $P(\xi) = 0$ are equal then $P(\xi)$ can be factored into linear factors in n ! essentially different ways.
Proof : Suppose that $P(\xi) = 0$ has a root α_1 then $P(\xi) = (\xi - \alpha_1)Q_1(\xi)$ … (by Factor Th.) and equation $Q_1(\xi) = 0$ has a root α_2 and $P(\xi) = [(\xi - \alpha_1)\,(\xi - \alpha_2)]\, Q_2(\xi)$,
Now the equation $Q_2(\xi)$ has a root α_3 and $P(\xi) = [(\xi - \alpha_1)(\xi - \alpha_2)(\xi - \alpha_3)]Q_3(\xi)$, by repeating this process we have $P(\xi) = 0$ has n roots and from theorem , It has n^2 roots .
To factor $P(\xi)$, use any one of $(n-1)^2$ roots of $Q_1(\xi) = 0$ for the second factor.
Similarly,

To factor P(ξ) , use any one of $(n-2)^2$ roots of $Q_2(\xi) = 0$ for third factor . Now by repeating this process , we have $n^2(n-1)^2 \ldots\ldots 2^2.1^2$ or $(n!)^2$ strings of factors. Since each set of n factors can be arranged in 2! different ways (order) there are $(n!)^2/_{n!}$ or n! essentially different ways to factor P(ξ).

3.1.1 Characteristic value problem:

Given a square bicomplex matrix A of order *n*, the problem is how to determine the scalar λ and non zero vector X which simultaneously satisfy the equation.

AX = λX ... (3.1)

This is known as characteristic value problem.

$$\text{Let } A=[\xi_{nn}]_{n\times n} \quad \& \quad X=\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Since λX = λIX, equation (3.1) can be written as

$$\begin{bmatrix} \xi_{11} & \cdots & \cdots & \xi_{1n} \\ \xi_{21} & \cdots & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \cdots & \cdots & \xi_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda & 0 & .. & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

$$\text{or} \begin{bmatrix} \xi_{11}-\lambda & \cdots & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22}-\lambda & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \cdots & \cdots & \xi_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

or (A – λI) X = 0 ...(3.2)

This is homogenous system of linear equations whose coefficient matrix is (A – λI). Since a non zero vector X is required, it is necessary for this coefficient matrix to have determinant equal to zero i,e |A-λI|=0

3.1.2. Definitions:

Characteristic Equations:
The equation |A – λI| = 0 i.e. $|{}^1A - {}^1\lambda|\, e_1 + |{}^2A - {}^2\lambda I|\, e_2 = 0$
is called characteristic equation of A.

Characteristic matrix:
The matrix (A – λI) = $({}^1A - {}^1\lambda I)\, e_1 + ({}^2A - {}^2\lambda I)\, e_2$
is called characteristic matrix of A.

Characteristic Polynomial:
The expansion of determinant |A – λI| yields a polynomial in λ, P(λ) which is called characteristic polynomial of matrix A.
Eigen values of bicomplex matrix A:
By fundamental theorem of bicomplex algebra we know that a bicomplex polynomial of degree *n* has n^2 roots in C_2 so by idempotent combination of roots of $|{}^1A - {}^1\lambda I| = 0$ &
$|{}^2A - {}^2\lambda I| = 0$
we get the roots of bicomplex matrix A and these n^2 roots are called eigen values of bicomplex matrix A.
If no two roots of P (λ) = 0 are equal then P (λ) can be factored into linear factors in $\lfloor n$ essentially different ways. (cf. Price [4])

3.2.1 Now we proceed to find eigen values of a bicomplex matrix A. (For the sake of brevity we consider 2 × 2 matrices)

Theorem1: A 2×2 bicomplex matrix have 4 roots and these roots can be factored into linear factor in 2! essentially different ways.

Proof: Let A be a 2×2 bicomplex matrix in C_2 so

$$A=\begin{bmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{bmatrix} \quad \text{where } \xi_{11}\ \xi_{12},\ \xi_{21},\ \xi_{22} \in C_2$$

So in terms of Idempotent components, A can be written as

$A = {}^1Ae_1 + {}^2Ae_2$

$$\begin{bmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{bmatrix}=\begin{bmatrix}{}^1\xi_{11} & {}^1\xi_{12}\\ {}^1\xi_{21} & {}^1\xi_{22}\end{bmatrix}e_1+\begin{bmatrix}{}^2\xi_{11} & {}^2\xi_{12}\\ {}^2\xi_{21} & {}^2\xi_{22}\end{bmatrix}e_2$$

Characteristic matrix of A

$$\begin{bmatrix}\xi_{11}-\lambda & \xi_{12}\\ \xi_{21} & \xi_{22}-\lambda\end{bmatrix}=\begin{bmatrix}{}^1\xi_{11}-\lambda & {}^1\xi_{12}\\ {}^1\xi_{21} & {}^1\xi_{22}-\lambda\end{bmatrix}e_1+\begin{bmatrix}{}^2\xi_{11}-\lambda & {}^2\xi_{12}\\ {}^2\xi_{21} & {}^2\xi_{22}-\lambda\end{bmatrix}e_2$$

Now characteristic polynomial of ${}^1A$ is ${}^1P({}^1\lambda) = 0$

i.e., $({}^1\xi_{11} - {}^1\lambda)\,({}^1\xi_{22} - {}^1\lambda) - {}^1\xi_{12}{}^1\xi_{21} = 0$

or ${}^1\lambda^2 - {}^1\lambda({}^1\xi_{11} + {}^1\xi_{22}) + ({}^1\xi_{11}\,{}^1\xi_{22} - {}^1\xi_{21}\,{}^1\xi_{12}) = 0$

so $${}^1\lambda = \frac{({}^1\xi_{11} + {}^1\xi_{22}) \pm \sqrt{({}^1\xi_{11} + {}^1\xi_{22})^2 - 4({}^1\xi_{11}\,{}^1\xi_{22} - {}^1\xi_{12}\,{}^1\xi_{21}}}{2},$$

hence the roots are

$${}^1\lambda_1 = \frac{({}^1\xi_{11} + {}^1\xi_{22}) + \sqrt{({}^1\xi_{11} + {}^1\xi_{22})^2 - 4({}^1\xi_{11}\,{}^1\xi_{22} - {}^1\xi_{12}\,{}^1\xi_{21}}}{2} \qquad \ldots 3.3)$$

And

$${}^1\lambda_2 = \frac{({}^1\xi_{11} + {}^1\xi_{22}) - \sqrt{({}^1\xi_{11} + {}^1\xi_{22})^2 - 4({}^1\xi_{11}\,{}^1\xi_{22} - {}^1\xi_{12}\,{}^1\xi_{21}}}{2} \qquad \ldots(3.4)$$

Similarly characteristic polynomial of ${}^2A$ is ${}^2P\,({}^2\lambda) = 0$

$${}^2\lambda^2 - {}^2\lambda\,({}^2\xi_{11} + {}^2\xi_{22}) + ({}^2\xi_{11}{}^2\xi_{22} - {}^2\xi_{21}{}^2\xi_{12}) = 0$$

So roots are

$${}^2\lambda_1 = \frac{({}^2\xi_{11} + {}^2\xi_{22}) + \sqrt{({}^2\xi_{11} + {}^2\xi_{22})^2 - 4({}^2\xi_{11}\,{}^2\xi_{22} - {}^2\xi_{12}\,{}^2\xi_{21}}}{2} \qquad \ldots(3.5)$$

And

$${}^2\lambda_2 = \frac{({}^2\xi_{11} + {}^2\xi_{22}) - \sqrt{({}^2\xi_{11} + {}^2\xi_{22})^2 - 4({}^2\xi_{11}{}^2\xi_{22} - {}^2\xi_{12}{}^2\xi_{21})}}{2} \qquad \ldots(3.6)$$

Then by linear combination of these four roots we can get the roots of bicomplex matrix A i.e. If μ_1, μ_2, μ_3, μ_4 are roots of A then

$\mu_1 = {}^1\lambda_1 e_1 + {}^2\lambda_1 e_2$

$\mu_2 = {}^1\lambda_1 e_1 + {}^2\lambda_2 e_2$

$\mu_3 = {}^1\lambda_2 e_1 + {}^2\lambda_1 e_2$

$\mu_4 = {}^1\lambda_2 e_1 + {}^2\lambda_2 e_2$

So according to fundamental theorem of bicomplex algebra we get 2^2(=4) roots in C_2 for the bicomplex polynomial P(λ) of degree 2 in C_2.
Now if no two roots of P(λ) are equal i.e. all four roots are distinct then P(λ) can be factored into linear factors in $\lfloor n$ essentially different ways i.e. in 2 ways. In The above case, we have

$(l - \mu_1)(l - \mu_4) = P(\lambda)$

$(l - \mu_2)(l - \mu_3) = P(\lambda)$

Note: If A is non singular i.e. $|A| \notin O_2$ then $|^1A| \neq 0$ & $|^2A| \neq 0$ so all the four eigen values are non zero and by linear combination of these 4 eigen values we get non singular eigen values of A.

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