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A Study of Weyl Fractional Calculus Operators and Generalized Multivariable Mittag-Leffler Function

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Abstract: *In this paper we study the further properties of multivariable generalized Mittag-Leffler function* $E^{\gamma_j,l_j,q_j}_{\rho_j,\beta,p_j}(z)$ *associated with Weyl fractional integral and differential operators.A new integral operator* $\xi^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j,w,\infty}(z)$ *depending on Weyl fractional integral operator and containing* $E^{\gamma_j,l_j,q_j}_{\rho_j,\beta,p_j}(z)$ *in its kernel is defined and studied, namely,its boundedness. Also, composition of Weyl fractional integral and differential operators with the new operator* $\xi^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j,w,\infty}(z)$ *is established.*

I. Introduction

The Swedish mathematician Gosta Mittag- Leffler [1] in 1903, introduced the function $E_\rho(z)$, defined as

$$E_\rho(z)=\sum_{n=0}^{\infty}\frac{(z)^n}{\Gamma(\rho n+1)}\ \{\rho,z\in C;\mathrm{Re}(\rho)>0\} \tag{1}$$

where Γ(z) is the familiar Gamma function.The Mittag- Leffler function (1) reduces immediately to the exponential function e^z = $E_1(z)$ when α=1. Mittag- Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.

During the last century and due to its involvement in the problems of physics,engineering, and applied sciences, many authors defined and studied in their research papers different generalizations of Mittag-leffler type function,namely $E_{\rho,\beta}(z)$ as Wiman[2] function, $E^{\gamma}_{\rho,\beta}(z)$ stated by Prabhakar [3] , $E^{\gamma,p}_{\rho,\beta}(z)$ defined and studied by Shukla and Prajapati [4] ,and $E^{\gamma,l,p}_{\rho,\beta,q}(z)$ investigated by Salim and Faraj[5].

The multivariate analogue of generalized Mittag-Leffler function $E^{\gamma}_{\rho,\beta}(z)$ is setup and studied by Saxena et al. [6] in the following form

$$E^{\gamma_j}_{\rho_j,\beta}(z_1,\ldots\ldots,z_m)=E^{(\gamma_1,\ldots,\gamma_m)}_{(\rho_1,\ldots,\rho_m),\beta}(z_1,\ldots,z_m)$$
$$=\sum_{r_1,\ldots,r_m=0}^{\infty}\frac{(\gamma_1)_{r_1}\cdots(\gamma_m)_{r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}\frac{(z_1)^{r_1}\ldots(z_m)^{r_m}}{r_1!\ldots r_m!} \tag{2}$$

where

$\beta,\rho_j,\gamma_j\in C\, and\ \mathrm{Re}(\rho_j)>0;\ j=0,1,2,\ldots m.$

A further generalization of multivariate analogue of generalized Mittag-Leffler function $E^{\gamma,l,p}_{\rho,\beta,q}(z)$ was also mentioned,Saxena et al.[6]in terms of the following multiple series:

$$E_{(\rho_1,\ldots,\rho_m),\beta}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m)}(z_1,\ldots,z_m) = \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{l_1 r_1}\ldots(\gamma_m)_{l_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)} \frac{(z_1)^{r_1}\ldots(z_m)^{r_m}}{r_1!\ldots r_m!} \tag{3}$$

where

$\beta,\rho_j,\gamma_j,l_j \in C\, and\, (\{\mathrm{Re}(\rho_j),\mathrm{Re}(l_j)\}) > 0;\ j = 0,1,2,\ldots m.$

Recently Meena et al [7] introduced a multivariable generalization of Mittag- Leffler function as

$$E_{(\rho_j;\beta;q_j)}^{(\gamma_j;l_j;p_j)}(z_1,\ldots z_m) = \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)} \frac{(z_1)^{r_1}\ldots(z_m)^{r_m}}{(l_1)_{q_1 r_1}\ldots\ldots(l_m)_{q_m r_m}} \tag{4}$$

where

$$\beta,\rho_j,\gamma_j,l_j \in C;\ \min_{1\le j\le m}\{\mathrm{Re}(\beta),\mathrm{Re}(\rho_j),\mathrm{Re}(\gamma_j),\mathrm{Re}(l_j)\} > 0$$

$$and\ p_j, q_j > 0;\ p_j < q_j + \mathrm{Re}(\rho_j);\ j = 0,1,2,\ldots m. \tag{5}$$

Equation (4) is just a multivariable generalized formula of Mittag-Leffler function ; its various properties including differentiation,Laplace ,Beta, and Mellin transforms,and generalized hypergeometric series form and its relationship with other type of special functions were investigated and established.

This paper is devoted for the study the properties of multivariable generalized Mittag-Leffler function $E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}(z)$ defined in (4) with another type of fractional calculus operators called Weyl fractional integral and differential operators written as

$$(I_-^{\lambda}\varphi)(x) = \frac{1}{\Gamma\lambda}\int_x^{\infty}(t-x)^{\lambda-1}\varphi(t)\,dt \tag{6}$$

$$(D_-^{\lambda}\varphi)(x) = (-1)^m\left(\frac{d}{dx}\right)^m \frac{1}{\Gamma(m-\lambda)}\int_x^{\infty}(t-x)^{m-\lambda-1}\varphi(t)\,dt \tag{7}$$

The last definition can be written as

$$(D_-^{\lambda}\varphi)(x) = (-1)^m\left(\frac{d}{dx}\right)^m (I_-^{m-\lambda}\varphi)(x) \tag{8}$$

The basic properties of Weyl fractional integral operator and differential operator with multivariable generalized Mittag-Leffler function $E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}(z)$ were investigated; moreover a new integral operator depending on Weyl fractional integral operator and containing $E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}(z)$ in its kernel is established as

$$(\xi_{\rho_j;\beta;q_j;w,\infty}^{\gamma_j;l_j;p_j}\varphi)(x) = \int_x^{\infty}(t-x)^{\beta-1}E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}[w_1(t-x)^{\rho_1}\ldots,w_m(t-x)^{\rho_m}]\varphi(t)dt \tag{9}$$

The condition of boundedness of the integral operator (9) is discussed and stated in the space L(a,∞) of Lebesgue-measurable functions on (a,∞)

$$L(a,\infty) = \left\{ g(x) : \|g\|_1 = \int_a^{\infty}|g(x)|\,dx < \infty \right. \tag{10}$$

Also , compostion of Weyl fractional integration and differentiation with the operator defined in (9) is established.

To study,various properties ,we need the following well-known definations and results:
(i) Fubini's theorem (Dirichlet formula) [8]

$$\int_a^b dx \int_a^x f(x,t)dt = \int_a^b dt \int_t^b f(x,t)\,dx, \tag{11}$$

$$\frac{d}{dx}\int_a^x h(x,t)\,dt = \left[\int_a^x \frac{\partial}{\partial x} h(x,t)\ dt\right] + h(x,x) \tag{12}$$

(ii) The Riemann-Liouville fractional integral [8]

$$\left(I_{a^+}^{\lambda}\varphi\right)(x) = \frac{1}{\Gamma\lambda}\int_a^x (x-t)^{\lambda-1}\varphi(t)\,dt, \quad (\alpha \in C, \mathrm{Re}(\alpha) > 0 \tag{13}$$

(iii) The Riemann-Liouville fractional derivative [8]

$$\left(D_{a^+}^{\lambda}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\lambda}\varphi\right)(x)\ , \qquad n = \left[\mathrm{Re}(\alpha)\ \right] + 1 \tag{14}$$

(iv) Beta transform (Sneddon [9])

$$B\{f(z); a,b\} = \int_0^1 z^{a-1}(1-z)^b\ f(z)\,dz, \quad (\mathrm{Re}(a), \mathrm{Re}(b) > 0) \tag{15}$$

(v)The Beta function is written as:

$$\beta(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{16}$$

(vi) The difference property of the Gamma function is

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \tag{17}$$

II. Properties of Weyl Fractional Integral Related to Mittag-Leffler Function.

In this section ,we consider Weyl fractional integral and derivate (6) and (7) multivariable generalized Mittag-Leffler function $E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}(z)$ defined in (4)

Theorem 1
Let

$\rho_j, \beta_j, \gamma_j, l_j, \lambda, w \in C;\ \min\{\mathrm{Re}(\rho_j), \mathrm{Re}(\beta_j), \mathrm{Re}(\gamma_j), \mathrm{Re}(l_j), \mathrm{Re}(\lambda)\} > 0$ *and* $p_j, q_j > 0,$ *then*

$$I_-^{\lambda}\left[t^{-\lambda-\beta} E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j}(w_1 t^{-\rho_1}......w_m t^{-\rho_m})\ \right](x) = x^{-\beta} E_{\rho_j,\beta+\lambda,\ q_j}^{\gamma_j l_j,,p_j}(w_1 x^{-\rho_1}......w_m x^{-\rho_m}) \tag{18}$$

Proof:

$$I_{-}^{\lambda}\left[t^{-\lambda-\beta}\, E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j}(w_1 t^{-\rho_1}......w_m t^{-\rho_m})\right](x)$$

$$=\frac{1}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}\,t^{-\lambda-\beta}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}(t)^{-\sum_{j=1}^{m}\rho_j r_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_m r_m}}dt \qquad (19)$$

$$=\frac{1}{\Gamma(\lambda)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}}{(l_1)_{q_1r_1}.........(l_m)_{q_m r_m}}\int_x^{\infty}(t-x)^{\lambda-1}\,t^{-\lambda-\beta}(t)^{-\sum_{j=1}^{m}\rho_j r_j}dt$$

$Let\ \ u=\dfrac{(t-x)}{t},then$

$$I_{-}^{\lambda}\left[t^{-\lambda-\beta}\, E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j}(w_1 t^{-\rho_1}......w_m t^{-\rho_m})\right](x)$$

$$=\frac{1}{\Gamma(\lambda)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}(x)^{-\beta-\sum_{j=1}^{m}\rho_j r_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_m r_m}}$$

$$\times\int_0^1(u)^{\lambda-1}(1-u)^{\beta+\sum_{j=1}^{m}\rho_j r_j-1}du \qquad (20)$$

$$=\frac{1}{\Gamma(\lambda)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}(x)^{-\beta-\sum_{j=1}^{m}\rho_j r_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_m r_m}}B(\lambda,\beta+\sum_{j=1}^{m}\rho_j\, r_j)$$

$$=x^{-\beta}\, E_{\rho_j,\beta+\lambda,\,q_j}^{\gamma_j l_j,,p_j}(w_1 x^{-\rho_1}......w_m x^{-\rho_m})$$

Theorem 2

$Let\ \rho_j,\beta_j,\gamma_j,l_j,\lambda,w\in C;\ \min\{\mathrm{Re}(\rho_j),\mathrm{Re}(\beta_j),\mathrm{Re}(\gamma_j),\mathrm{Re}(l_j),\mathrm{Re}(\lambda)\}>0,$

$\mathrm{Re}(\beta)>\left[\mathrm{Re}(\lambda)\right]+1 and\ \ p_j,q_j>0,\ then$

$$D_{-}^{\lambda}\left[t^{-\lambda-\beta}\, E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j}(w_1 t^{-\rho_1}......w_m t^{-\rho_m})\right](x)=x^{-\beta}\, E_{\rho_j,\beta-\lambda,\,q_j}^{\gamma_j l_j,,p_j}(w_1 x^{-\rho_1}......w_m x^{-\rho_m}) \qquad (21)$$

Proof:

By using (7) we get

$$D_{-}^{\lambda}\left[t^{\lambda-\beta}E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j}(w_1t^{-\rho_1}......w_mt^{-\rho_m})\right](x)$$

$$=(-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\lambda)}\int_x^\infty(t-x)^{m-\lambda-1}t^{\lambda-\beta}$$

$$\times\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}(t)^{-\sum_{j=1}^{m}\rho_jr_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}dt \qquad (22)$$

$$=(-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\lambda)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}$$

$$\times\int_x^\infty(t-x)^{m-\lambda-1}t^{\lambda-\beta}(t)^{-\sum_{j=1}^{m}\rho_jr_j}dt$$

$$Let\ \ u=\frac{(t-x)}{t},then$$

$$D_{-}^{\lambda}\left[t^{\lambda-\beta}E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j}(w_1t^{-\rho_1}......w_mt^{-\rho_m})\right](x)$$

$$=(-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\lambda)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}(x)^{m-\beta-\sum_{j=1}^{m}\rho_jr_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}$$

$$\times\int_0^1(u)^{m-\lambda-1}(1-u)^{\beta-m+\sum_{j=1}^{m}\rho_jr_j-1}du$$

$$=(-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\lambda)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}(x)^{m-\beta-\sum_{j=1}^{m}\rho_jr_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}$$

$$\times B(m-\lambda,\beta-m+\sum_{j=1}^{m}\rho_jr_j)$$

$$= (-1)^m \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)} \frac{(w_1)^{r_1}\ldots(w_m)^{r_m}\,\Gamma(\beta-m+\sum_{j=1}^{m}\rho_j r_j)}{(l_1)_{q_1 r_1}\ldots(l_m)_{q_m r_m}\,\Gamma(\beta-\lambda+\sum_{j=1}^{m}\rho_j r_j)}$$

$$\times (m-\beta-\sum_{j=1}^{m}\rho_j r_j)\ldots(m-\beta-\sum_{j=1}^{m}\rho_j r_j-m+1)(x)^{m-\beta-\sum_{j=1}^{m}\rho_j r_j-m}$$

$$= (-1)^m \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)} \frac{(w_1)^{r_1}\ldots(w_m)^{r_m}\,\Gamma(\beta-m+\sum_{j=1}^{m}\rho_j r_j)}{(l_1)_{q_1 r_1}\ldots(l_m)_{q_m r_m}\,\Gamma(\beta-\lambda+\sum_{j=1}^{m}\rho_j r_j)}$$

$$\times (-1)^m \ (\beta-m+\sum_{j=1}^{m}\rho_j r_j)_m \ (x)^{-\beta-\sum_{j=1}^{m}\rho_j r_j}$$

$$= (x)^{-\beta} \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}(w_1)^{r_1}\ldots(w_m)^{r_m}}{(l_1)_{q_1 r_1}\ldots(l_m)_{q_m r_m}\,\Gamma(\beta-\lambda+\sum_{j=1}^{m}\rho_j r_j)}(x)^{-\sum_{j=1}^{m}\rho_j r_j}$$

$$= x^{-\beta}\, E^{\gamma_j\, l_j,\, p_j}_{\rho_j,\beta-\lambda,\, q_j}\,(w_1 x^{-\rho_1}\ldots w_m x^{-\rho_m})$$

(23)

If we reduce the generalized Mittag-Leffler Function $E^{\gamma,l,p}_{\rho,\beta,q}(z)$ in the right hand side of equation (18) and (21) we get known result due to Faraj et al [10, p.2, eq.18 and p.3, eq.21].

III. Weyl Integral operator with Generalized Mittag-Leffler function in the Kernel

First of all ,we will prove that the operator $\xi^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j,w,\infty}$ is bounded on $L(a,\infty)$, considering the Weyl integral operator defined in (9) containing $E^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j}(z)$ in the kernel.

Theorem 3

Let $\rho_j, \beta_j, \gamma_j, l_j, \lambda, w \in C;\ \min\{\mathrm{Re}(\rho_j), \mathrm{Re}(\beta_j), \mathrm{Re}(\gamma_j), \mathrm{Re}(l_j), \mathrm{Re}(\lambda)\} > 0$ *and* $p_j, q_j > 0,$

then the operator $\xi^{\gamma_j l_j,\, p_j}_{\rho_j \beta, q_{j,w,\infty}}$ *is bounded on* $L(a,\infty)$ *and* $b > a$

$$\left\| \xi^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j,w,\infty}\varphi \right\|_1 \le \beta \|\varphi\|_1, \tag{24}$$

where

$$\beta = (t-a)^{\mathrm{Re}(\beta)} \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}(w_1(t-a)^{\rho_1})^{r_1}\ldots(w_m(t-a)^{\rho_m})^{r_m}}{(l_1)_{q_1 r_1}\ldots(l_m)_{q_m r_m}\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)\left|\mathrm{Re}(\sum_{j=1}^{m}\rho_j r_j)+\mathrm{Re}(\beta)\right|} \quad (25)$$

Proof:

Let C_n denote the n^{th} term of (25),then

$$\prod_{j=1}^{m}\left|\frac{C_{r_j+1}}{C_{r_j}}\right| = \left[\prod_{j=1}^{m}\left|\frac{(\gamma_j)_{p_j r_j \rho_j}}{(\gamma_j)_{p_j r_j}}\right|\left|\frac{(l_j)_{q_j r_j}}{(l_j)_{q_j r_+ + q_j}}\right|\left|\frac{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j+\rho_j)}\right|\left|w_j(t-a)^{\rho_j}\right|\right]$$

$$\approx \left|\prod_{j=1}^{m}\frac{(p_j)^{p_j}}{(q_j)^{q_j}(\rho_j)^{\rho_j}} w_j(t-a)^{\rho_j}(r_j)^{p_j-(q_j+\rho_j)}\right| \quad as \quad r_j \to \infty \quad (26)$$

Hence $\left|\frac{C_{r_J+1}}{C_{r_j}}\right| \to 0$ as $r_j \to \infty$ and $p_j < q_j + \mathrm{Re}(\rho_j)$ which means that the right-hand of (25) is convergent and finite under the given conditions.

Now, according to (9), (10) and (11), we get

$$\left\|\xi^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j,w,\infty}\varphi\right\|_1$$

$$= \int_a^\infty \left|\int_x^\infty (t-x)^{\beta-1} E^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j}\left[w_1(t-x)^{\rho_1},\ldots\ldots w_m(t-x)^{\rho_m}\right]\varphi(t)dt\right| dx$$

$$\le \int_a^\infty \left[\int_a^t (t-x)^{\beta-1}\left|E^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j}\left[w_1(t-x)^{\rho_1},\ldots\ldots w_m(t-x)^{\rho_m}\right]\right| dx\right]\left|\varphi(t)\right| dt$$

$$Let \quad u = (t-x), then$$

$$= \int_a^\infty \left[\int_0^{t-a} (u)^{\beta-1}\left|E^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j}\left[w_1(u)^{\rho_1},\ldots\ldots w_m(u)^{\rho_m}\right]\right| du\right]\left|\varphi(t)\right| dt$$

$$\le \int_a^\infty \left[\int_a^{t-a} (u)^{\beta-1}\left|E^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j}\left[w_1(u)^{\rho_1},\ldots\ldots w_m(u)^{\rho_m}\right]\right| du\right]\left|\varphi(t)\right| dt \quad (27)$$

$$Let \quad \int_0^{t-a} (u)^{\beta-1}\left|E^{\gamma_j,l_j,p_j}_{\rho_j,\beta,q_j}\left[w_1(u)^{\rho_1},\ldots\ldots w_m(u)^{\rho_m}\right]\right| du = \beta \quad (28)$$

then

$$\beta = \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)} \frac{(w_1)^{r_1}\ldots(w_m)^{r_m}}{(l_1)_{q_1 r_1}\ldots(l_m)_{q_m r_m}} \int_0^{t-a} (u)^{(\beta+\sum_{j=1}^{m}\rho_j r_j)-1} du$$

$$\beta = (t-a)^{\mathrm{Re}(\beta)} \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1}\ldots(\gamma_m)_{p_m r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)} \frac{\left[(w_1(t-a)^{\rho_1})^{r_1}\ldots(w_m(t-a)^{\rho_m})^{r_m}\right]}{(l_1)_{q_1 r_1}\ldots(l_m)_{q_m r_m}\left|\mathrm{Re}(\beta)+\mathrm{Re}(\sum_{jj-1}^{m}\rho_j r_j)\right|} \quad (29$$

Hence

$$\left\| \xi^{\gamma_j, l_j, q_j}_{\rho_j,\beta,q_j,w,\infty}\varphi \right\|_1 \le \int_0^{\infty} \beta\left|\varphi(t)dt\right| \le \beta\|\varphi\|_1, \quad (30)$$

We consider now composition of Weyl fractional integration and differentiation $I_-^{\lambda}, D_-^{\lambda}$ with the operator $\xi^{\gamma_j, l_j, p_j}_{\rho_j,\beta,q_j,w,\infty}$ defined in (9) contained in the next two theorems.

Lemma 5

Let $a \in R_+, \rho_j, \beta_j, \gamma_j, l_j, \lambda, w \in C;\ \min\{\mathrm{Re}(\rho_j), \mathrm{Re}(\beta_j), \mathrm{Re}(\gamma_j), \mathrm{Re}(l_j), \mathrm{Re}(\lambda)\} > 0$ *and* $p_j, q_j > 0$, *then for* $x > a$, *one has*

$$I_-^{\lambda}\left[(t-a)^{\beta-1} E^{\gamma_j l_j, p_j}_{\rho_j,\beta,q_j}\left[w_1(t-a)^{-\rho_1}\ldots w_m(t-a)^{-\rho_m}\right]\right](x)$$
$$= (x-a)^{\beta+\lambda-1} E^{\gamma_j l_j, p_j}_{\rho_j,\beta+\lambda,q_j}\left[w_1(x-a)^{-\rho_1}\ldots w_m(x-a)^{-\rho_m}\right] \quad (31)$$

Theorem 6

Let $\rho_j, \beta_j, \gamma_j, l_j, \lambda, w \in C;\ \min\{\mathrm{Re}(\rho_j), \mathrm{Re}(\beta_j), \mathrm{Re}(\gamma_j), \mathrm{Re}(l_j), \mathrm{Re}(\lambda)\} > 0$ *and* $p_j, q_j > 0$, *then*

$$\left(I_-^{\lambda}\ \xi^{\gamma_j l_j, p_j}_{\rho_j,\beta,q_j,w,\infty}\varphi\right)(x) = \left(\xi^{\gamma_j l_j, p_j}_{\rho_j,\beta+\lambda,q_j,w,\infty}\varphi\right)(x) = \left(\xi^{\gamma_j l_j, p_j}_{\rho_j,\beta,q_j,w,\infty} I_-^{\lambda}\varphi\right)(x)$$

(32)

Proof:

Applying (6) and (9), and by using Dirichlet formula (11) yields

$$\left(I_-^{\lambda}\ \xi^{\gamma_j l_j, p_j}_{\rho_j,\beta,q_j,w,\infty}\varphi\right)(x) =$$

$$= \frac{1}{\Gamma(\lambda)}\left(\int_x^{\infty}(u-x)^{\lambda-1}\left[\int_u^{\infty}(t-u)^{\beta-1}E^{\gamma_j l_j, p_j}_{\rho_j,\beta,q_j}\left[w_1(t-u)^{\rho_1}\ldots w_m(t-u)^{\rho_m}\right]\varphi(t)dt\right]du\right)$$
$$= \left(\int_x^{\infty}\frac{1}{\Gamma(\lambda)}\left[\int_x^{t}(u-x)^{\lambda-1}(t-u)^{\beta-1}E^{\gamma_j l_j, p_j}_{\rho_j,\beta,q_j}\left[w_1(t-u)^{\rho_1}\ldots w_m(t-u)^{\rho_m}\right]du\right]\right)\times\varphi(t)dt$$

(33)

$Let \quad \tau = (t-u), then$

(34)

$$\left(I_{-}^{\lambda} \ \xi_{\rho_j,\beta,q_{j,w,\infty}}^{\gamma_j l_j,,p_j} \varphi \right)(x)$$

$$= \left(\int_x^{\infty} \frac{1}{\Gamma(\lambda)} \left[\int_0^{t-x} (t-x-\tau)^{\lambda-1} (\tau)^{\beta-1} E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j} \left[w_1(\tau)^{\rho_1} ......w_m(\tau)^{\rho_m} \right] d\tau \right] \right) \times \varphi(t) dt \tag{35}$$

Applying (13) and result of Lemma 5 ,we get

$$\left(I_{-}^{\lambda} \ \xi_{\rho_j,\beta,q_{j,w,\infty}}^{\gamma_j l_j,,p_j} \varphi \right)(x) =$$

$$= \int_x^{\infty} I_0^{\lambda} \left[(\tau)^{\beta-1} E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j} \left[w_1(\tau)^{\rho_1} ......w_m(\tau)^{\rho_m} \right] \right] (t-x) \varphi(t) dt \tag{36}$$

$$= \int_x^{\infty} (t-x)^{\beta+\lambda-1} E_{\rho_j,\beta+\lambda,q_j}^{\gamma_j l_j,,p_j} \left[w_1(t-x)^{-\rho_1} ......w_m(t-x)^{-\rho_m} \right] \varphi(t) dt$$

$$= \left(\xi_{\rho_j,\beta+\lambda,q_{j,w,\infty}}^{\gamma_j l_j,,p_j} \varphi \right)(x)$$

On the other hand

$$\left(\xi_{\rho_j,\beta,q_{j,w,\infty}}^{\gamma_j l_j,,p_j} I_{-}^{\lambda} \varphi \right)(x)$$

$$= \int_x^{\infty} (t-x)^{\beta-1} E_{\rho_j,\beta+\lambda,q_j}^{\gamma_j l_j,,p_j} \left[w_1(t-x)^{-\rho_1} ......w_m(t-x)^{-\rho_m} \right] \frac{1}{\Gamma\lambda} \times \left[\int_t^{\infty} (u-t)^{\lambda-1} \varphi(u)\, du \right] dt$$

$$= \int_x^{\infty} \frac{1}{\Gamma\lambda} \left[\int_x^{u} (u-t)^{\lambda-1} (t-x)^{\beta-1} E_{\rho_j,\beta+\lambda,q_j}^{\gamma_j l_j,,p_j} \left[w_1(t-x)^{-\rho_1} ......w_m(t-x)^{-\rho_m} \right] dt \right] \times \varphi(u)\, du \tag{37}$$

$Let \quad \tau = (t-x), then$

$$\left(\xi_{\rho_j,\beta,q_{j,w,\infty}}^{\gamma_j l_j,,p_j} I_{-}^{\lambda} \varphi \right)(x)$$

$$= \left(\int_x^{\infty} \frac{1}{\Gamma(\lambda)} \left[\int_0^{u-x} (u-x-\tau)^{\lambda-1} (\tau)^{\beta-1} E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j} \left[w_1(\tau)^{\rho_1} ......w_m(\tau)^{\rho_m} \right] d\tau \right] \right) \times \varphi(u)\, du \tag{38}$$

Returning to (13) and Lemma 5,we have

$$\left(\xi_{\rho_j,\beta,q_{j,w,\infty}}^{\gamma_j l_j,,p_j} I_{-}^{\lambda} \varphi \right)(x)$$

$$= \int_x^{\infty} I_0^{\lambda} \left[(\tau)^{\beta-1} E_{\rho_j,\beta,q_j}^{\gamma_j l_j,,p_j} \left[w_1(\tau)^{\rho_1} ......w_m(\tau)^{\rho_m} \right] \right] (u-x) \varphi(u)\, du \tag{39}$$

$$= \int_x^{\infty} (u-x)^{\beta+\lambda-1} E_{\rho_j,\beta+\lambda,q_j}^{\gamma_j l_j,,p_j} \left[w_1(u-x)^{-\rho_1} ......w_m(u-x)^{-\rho_m} \right] \varphi(u) du$$

$$=\left(\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta+\lambda,q_{j,w,\infty}}\varphi\right)(x)$$

In the following theorem a similar result concerning the Weyl fractional differentiation is stated.

Theorem 7

If the condition of the Theorem 6 is satisfied, then

$$\left(D^{\lambda}_{-}\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_{j,w,\infty}}\varphi\right)(x)=\left(\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta-\lambda,q_{j,w,\infty}}\varphi\right)(x)$$

(40)

Proof:

Making use of (7), we get

$$(D^{\lambda}_{-}\,\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_{j,w,\infty}}\varphi)(x)=(-1)^n\left(\frac{d}{dx}\right)^n(I^{n-\lambda}_{-}\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_{j,w,\infty}}\varphi)(x)$$

(41)

and applying Theorem 6 yields

$$(D^{\lambda}_{-}\,\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_{j,w,\infty}}\varphi)(x)$$

$$=(-1)^n\left(\frac{d}{dx}\right)^n$$

$$\times\int_x^\infty(t-x)^{\beta+n-\lambda-1}E^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_j}\left[w_1(t-x)^{\rho_1}......w_m(t-x)^{\rho_m}\right]\varphi(t)dt \qquad (42)$$

By using Dirichlet formula (12), we get

$$(D^{\lambda}_{-}\,\xi^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_{j,w,\infty}}\varphi)(x)$$

$$=(-1)^n\left(\frac{d}{dx}\right)^n$$

$$\times\int_x^\infty\frac{\partial}{\partial x}(t-x)^{\beta+n-\lambda-1}E^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_j}\left[w_1(t-x)^{\rho_1}......w_m(t-x)^{\rho_m}\right]\varphi(t)dt$$

$$+\lim_{t\to x^+}(t-x)^{\beta+n-\lambda-1}E^{\gamma_j l_j,,p_j}_{\rho_j,\beta,q_j}\left[w_1(t-x)^{\rho_1}......w_m(t-x)^{\rho_m}\right]\varphi(t)$$

$$=(-1)^n\left(\frac{d}{dx}\right)^{n-1}\times\int_x^\infty\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(w_1)^{r_1}....(w_m)^{r_m}\Gamma(\beta+n-\lambda+\sum_{j=1}^{m}\rho_jr_j-1)}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}\Gamma(\beta+n-\lambda+\sum_{j=1}^{m}\rho_jr_j)}$$

$$\times(t-x)^{\beta+n-\lambda+\sum_{j=1}^{m}\rho_jr_j-2}\varphi(t)\,dt$$

$$= (-1)^n \left(\frac{d}{dx}\right)^{n-1} \int_x^{\infty} (-1)(t-x)^{\beta+n+\lambda-2} E^{\gamma_j l_j, , p_j}_{\rho_j, \beta+n-\lambda-1, q_j} \left[w_1(t-x)^{\rho_1} .....w_m(t-x)^{\rho_m}\right]\varphi(t)dt \tag{43}$$

Repeating this process n-1 times,we get

$$(D^{\lambda}_{-} \, \xi^{\gamma_j l_j, , p_j}_{\rho_j, \beta, q_{j,w,\infty}} \varphi)(x)$$

$$= (-1)^n \,)(-1)^n \int_x^{\infty} (-1)(t-x)^{\beta-\lambda-1} E^{\gamma_j l_j, , p_j}_{\rho_j, \beta-\lambda, q_j} \left[w_1(t-x)^{\rho_1} ......w_m(t-x)^{\rho_m}\right]\varphi(t)dt$$

$$= \left(\xi^{\gamma_j l_j, , p_j}_{\rho_j, \beta-\lambda, q_{j,w,\infty}} \varphi\right)(x)$$

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References

[1] Mittag-Leffler G. M., Sur la nouvelle function $E_\alpha(x)$, C. R. Acad. Sci. Paris No **137** (1903), 554-558.

[2] Wiman A., Uber de fundamental satz in der theorie der funktionen $E_\alpha(x)$, Acta Math. No. **29** (1905), 191-201.MR1555014.JFM36.0471.01.

[3] Prabhakar T. R.,A singular integral equation with a generalized Mittag-Leffler function in the Kernel, Yokohama Math. J. No **19** (1971) 7-15. MR0293349 (45#2426).Zbl 0221.45003.

[4] Shukla A.K.and Prajapati J.C.,On a generalization of Mittag-Leffler function and its properties, Math. Anal. Appl., **336** (2007), 797-811.

[5] Salim T.O. and Faraj A.W., A generalization of Mittag-Leffler function and integral operator associated with fractional calculus, J. of Fract. Calc. and Appl., **3** (2012), 1-13.

[6] Kilbas A.A., Siago M. and Saxena R.K., Generalized Mittag-Leffler function and generalized fractional calculus operators,Integral Transforms Spec. Funct. , Vol. **15**, (2004), pp. 31-49.

[7] Gujar M.K., Prajapati J.C. and Gupta K., A study of generalized Mittag Leffler function via Fractional Calculus, Journal Of Inequalities And Special Functions, **5, No**. (2014), 6-13.

[8] Samko S.G. ,Kilbas A.A. and Marichev O.I. ,Fracional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel,"Applied Mathematics and computation, Vol. 211,no. 1,pp. 198-210,2009.

[9] Sneddon I.N.,The Use of integral Transforms,Tata McGraw Hill,New Delhi,India,1979.

[10] Faraj A.,Salim T.,Sadek S. and Ismail J., Journal of Mathematics,(2013),1-5.