IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 4 Ver. I (Jul. - Aug.2016), PP 01-06
www.iosrjournals.org

A Typical Sequence of + and – signs, and an Application of the Powers of Twos in the Expression of a Positive Integer in Binary Scale

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Abstract: *The paper describes a special recurrence relation whose expansion involve with a typical sequence of plus (+) and minus (-) signs by a process of recursive substitution. The kind of a sign at any k-th place of the expansion depends on the powers of twos in the expression of k in binary scale.*
Keywords: *recurrence; sequence; binomial coefficient; compositions; positive integer in binary scale.*

I. Introduction

We define a recurrence function by an alternating signs recurrence relation such that the solution of the function is a binomial coefficient. The recurrence relation can generate an expression of 2^n terms by a process of recursive substitution. The type of sign at any k^{th} place of the expression depends on the powers of twos in the expression of k in a binary scale: $k = 2^{h_1} + 2^{h_2} + \dots$, $h_1 > h_2 > \dots$ The rule for the kind of sign at any k^{th} place is as shown.

The lowest power	Number of powers	Sign at k^{th} place
even	odd	+
even	even	–
odd	odd	–
odd	even	+

Example: Recurrence relation of each order yields an identity. From the 4^{th} order, we find:

$$\binom{k+3}{4} = \binom{k+3}{1}\binom{k+2}{1}\binom{k+1}{1}\binom{k}{1} - \binom{k+3}{1}\binom{k+2}{1}\binom{k+1}{2} - \binom{k+3}{1}\binom{k+2}{2}\binom{k}{1}$$
$$+\binom{k+3}{1}\binom{k+2}{3} - \binom{k+3}{2}\binom{k+1}{1}\binom{k}{1} + \binom{k+3}{2}\binom{k+1}{2} + \binom{k+3}{3}\binom{k}{1} - \binom{k+3}{4}.$$

The sequence of 8 signs in 8 terms on the right is: + – – + – + + –. Binary expressions of the first 8 natural numbers are: $1= 2^0$; $2 =2^1$; $3 = 2^1 + 2^0$; $4 = 2^2$; $5 = 2^2 + 2^0$; $6 = 2^2 + 2$; $7 = 2^2 + 2^1 + 2^0$; $8 = 2^3$. For $k =7 = 2^2 + 2^1 + 2^0$, the lowest power among the powers of 2s is 0 (even) and the number of powers is 3 (odd); and hence + sign occurs at 7^{th} place by the above rule in tabular form. In this way one can determine the sign of any $k^{\text{-th}}$ place.

II. Recurrence Relation

Letting the initial condition: $F(1, k) = \binom{k}{1}$, we define an $(n + 1)$-th order recurrence function $F(n + 1, k)$ by Recurrence relation 1:

$$F(n+1,\ k) = \binom{n+k}{1}F(n,\ k) - \binom{n+k}{2}F(n-1,\ k) + \dots + (-1)^{n-1}\binom{n+k}{n}F(1,\ k)$$
$$+ (-1)^n\binom{n+k}{n+1}.$$

(a) Solution of the function.

The solution of the function is Theorem 3. The proof of Theorem 3 depends on the proofs of Theorem 1 and Theorem 2.

Theorem 1: *For all* $n \in \mathbb{N}$, $F(n, 1) = 1$.
Proof: The proof is short and simple. We have:

$$F(1, 1) = 1;$$

$$F(2, 1) = \binom{2}{1} F(1, 1) - \binom{2}{2} = 2\cdot 1 - 1 = 1.$$

Hence the theorem holds for $n = 1$ and for $n = 2$. To complete the proof, we assume that the theorem holds for all $n \in \mathbb{N}$ with $1 \le n \le m$. By our induction hypothesis, we have:
$F(m, 1) = F(m-1, 1) = \ldots = F(1, 1) = 1.$
Then we deduce that

$$F(m+1,\ 1) = \binom{m+1}{1}F(m,\ 1) - \ldots + (-1)^{m-1}\binom{m+1}{m}F(1,\ 1) + (-1)^m\binom{m+1}{m+1}$$
$$= \binom{m+1}{1}.1 - \ldots + (-1)^{m-1}\binom{m+1}{m}.1 + (-1)^m\binom{m+1}{m+1}$$
$$= 1.$$

The theorem follows.▮

Theorem 2: *For all $n, k \in \mathbb{N}$, $F(n+1,\ k+1) = F(n+1,\ k) + F(n,\ k+1)$.*

Proof: From Recurrence relation 1, we have:

$$F(2,\ k+1) = \binom{k+2}{1}F(1,\ k+1) - \binom{k+2}{2}$$
$$= \left[\binom{k+1}{1} + 1\right]F(1,\ k+1) - \left[\binom{k+1}{2} + \binom{k+1}{1}\right]$$
$$= \binom{k+1}{1}F(1,\ k+1) + F(1,\ k+1) - \binom{k+1}{2} - \binom{k+1}{1}$$
$$= \binom{k+1}{1}[F(1,\ k) + 1] + F(1,\ k+1) - \binom{k+1}{2} - \binom{k+1}{1}$$
$$= \binom{k+1}{1}F(1,\ k) - \binom{k+1}{2} + F(1,\ k+1)$$
$$= F(2,\ k) + F(1,\ k+1).$$

It follows that the theorem is true for $n = 1$ and a fixed positive integer k. We assume that the theorem is true for all $n \in \mathbb{N}$ with $1 \le n \le m$ and a fixed k. Then we deduce that

$$F(m+2,\ k+1) = \binom{m+k+2}{1}F(m+1,\ k+1) - \binom{m+k+2}{2}F(m,\ k+1) + \ldots + (-1)^m\binom{m+k+2}{m+1}F(1,\ k+1) + (-1)^{m+1}\binom{m+k+2}{m+2}.$$

$$= \left[\binom{m+k+1}{1} + 1\right]F(m+1, k+1) - \left[\binom{m+k+1}{2} + \binom{m+k+1}{1}\right]F(m, k+1) + \ldots + (-1)^m\left[\binom{m+k+1}{m+1} + \binom{m+k+1}{m}\right]F(1,\ k+1) + (-1)^{m+1}\left[\binom{m+k+1}{m+2} + \binom{m+k+1}{m+1}\right].$$

$$= \binom{m+k+1}{1}F(m+1,\ k+1) - \binom{m+k+1}{2}F(m,\ k+1) + \ldots + (-1)^m\binom{m+k+1}{m+1}F(1,\ k+1) + (-1)^{m+1}\binom{m+k+1}{m+2} + F(m+1,\ k+1) - F(m+1,\ k+1).$$

$$= \binom{m+k+1}{1}[F(m+1,\ k) + F(m,\ k+1)] - \binom{m+k+1}{2}[F(m,\ k) + F(m-1,\ k+1)] + \ldots + (-1)^m\binom{m+k+1}{m+1}[F(1,\ k) + 1] + (-1)^{m+1}\binom{m+k+1}{m+2}.$$

[By induction hypothesis and initial condition of Recurrence relation 1]

$$= F(m+2,\ k) + F(m+1,\ k+1).$$

Thus we have the theorem by induction on n. Yet k can be given any positive integer-value to obtain the above result. It follows that the theorem holds for all $n, k \in \mathbb{N}$. ▮

Theorem 3: For all $n, k \in \mathbb{N}$, $F(n,\ k) = \binom{n+k-1}{n}$.

Proof: From Theorem 2, we have:

$$\textstyle\sum_{i=1}^{k}[F(n+1,\ i+1) - F(n+1,\ i)] = \sum_{i=1}^{k}F(n,\ i+1)$$
$$\Rightarrow F(n+1,\ k+1) - F(n+1,\ 1) = \textstyle\sum_{i=1}^{k}F(n,\ i+1).$$

Immediately by Theorem 1,

$$F(n+1,\ k+1) = \textstyle\sum_{i=1}^{k+1}F(n,\ i).$$

Then

$$F(2, k+1) = F(1, k+1) + \ldots + F(1, 1)$$
$$= (k+1) + \ldots + 1 = \binom{k+2}{2};$$
$$F(3, k+1) = F(2, k+1) + \ldots + F(2, 1)$$
$$= \binom{k+2}{2} + \ldots + \binom{2}{2} = \binom{k+3}{3};$$

… …

Proceeding thus we get: For all $n, k \in \mathbb{N}$,

$$F(n,\ k+1) = \binom{n+k}{n}.$$

Then by Theorem 1, we have: For all $n, k \in \mathbb{N}$,

$$F(n,\ k) = \binom{n+k-1}{n}.$$

This completes the proof. ▌

(b) A binomial coefficient identity

From Recurrence relation 1, its initial condition and Theorem 3, we get:

$$\binom{n+k-1}{n} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n+k-1}{i}\binom{n+k-1-i}{n-i}.$$

$$\Rightarrow \text{For } m \geq n \geq 1,\ \binom{m}{n} = \sum_{i=1}^{n} (-1)^{i-1} \binom{m}{i}\binom{m-i}{n-i}. \qquad (1)$$

III. A Typical Sequence of + and – signs

(i) The initial condition of Recurrence relation 1is:

$$F(1, k) = \binom{k}{1}. \qquad (2.1)$$

Now we plan to carry out a process of recursive substitution involving Recurrence relation 1 and Theorem 3 as shown.

(ii) From Recurrence relation 1 for $n = 1$, Theorem 3 and (2.1), we get the expression of 2 terms for $F(2, k)$:

$$F(2, k) = \binom{k+1}{2} = \binom{k+1}{1}\binom{k}{1} - \binom{k+1}{2}. \qquad (2.2)$$

(iii) From Recurrence relation 1 for $n = 2$, Theorem 3, (2.1) and (2.2), we get the expression of (2 + 1 + 1) or 4 terms for $F(3,k)$:

$$F(3,\ k) = \binom{k+2}{3} = \binom{k+2}{1}\binom{k+1}{1}\binom{k}{1} - \binom{k+2}{1}\binom{k+1}{2} - \binom{k+2}{2}\binom{k}{1} + \binom{k+2}{3}. \qquad (2.3)$$

(iv) Similarly from Recurrence relation 1for $n = 2$, Theorem 3, (2.1), (2.2) and (2.3), we get the expression of (4 + 2 + 1 + 1) or 8 terms for $F(4, k)$:

$$F(4,k) = \binom{k+3}{4} = \binom{k+3}{1}\binom{k+2}{1}\binom{k+1}{1}\binom{k}{1} - \binom{k+3}{1}\binom{k+2}{1}\binom{k+1}{2} - \binom{k+3}{1}\binom{k+2}{2}\binom{k}{1}$$

$$+ \binom{k+3}{1}\binom{k+2}{3} - \binom{k+3}{2}\binom{k+1}{1}\binom{k}{1} + \binom{k+3}{2}\binom{k+1}{2} + \binom{k+3}{3}\binom{k}{1} - \binom{k+3}{4} \qquad (2.4)$$

… …

The sequence of two signs in (2.2) is: + –; this of four signs in (2.3) is: + – – +; this of eight signs in (2.4) is: + – – + – + + –; … Now the problem is: What is the general rule for the above sequences of signs? Confining our attention to the sequences of signs, we can define a simple and reduced form of Recurrence relation 1in the following way.

Letting $R_1 = C_1$ as the initial condition, we define a recurrence function R_{n+1} by Recurrence relation 2:

$$R_{n+1} = R_n - R_{n-1} + \ldots + (-1)^{n-1} R_1 + (-1)^n C_{n+1}. \qquad (3)$$

We have:

$$R_1 = C_1. \qquad (3.1)$$

Then by the process recursive substitution, we get:

$$R_2 = C_1 - C_2. \qquad (3.2)$$

$$R_3 = C_1 - C_2 - C_1 + C3 \qquad (3.3)$$

$$R_4 = (C_1 - C_2 - C_1 + C_3) - (C_1 - C_2) + C_1 - C_4$$

$$= C_1 - C_2 - C_1 + C_3 - C_1 + C_2 + C_1 - C_4 \qquad (3.4)$$

… …

The number of terms of the expressions (3.2), (3.3), (3.4), ..., are: (1+1), (2+1+1), (2^2+2+1+1), …, or 1, 2, 2^2, 2^3, … in succession. The terms of (3.2), (3.3) ... are composed of C_1, C_2, …; and for convenience, we name these expressions as C_k - expressions. Then C_k - expression of R_n is (3.n) which has 2^{n-1}terms. On the other hand R_n has an alternating signs expression of n terms according to (3).

We have: $R_1 = C_1$; $R_2 = R_1 - C_2$; and then for $n \geq 3$,

$$R_n = R_{n-1} - \{R_{n-2} - \ldots + (-1)^{n-3} R_1 + (-1)^{n-2} C_n\}$$

$$= \{R_{n-2} - \ldots + (-1)^{n-3} R_1 + (-1)^{n-2} C_{n-1}\} - \{R_{n-2} - \ldots + (-1)^{n-3} R_1 + (-1)^{n-2} C_n\}$$

$$= A - B, \text{ say.} \qquad (4)$$

Each of two parts of (4) is the alternating signs expression of $n - 1$ terms such that the first $n - 2$ terms contain R_{n-2}, …, R_1 in succession. C_k-expression of R_{n-1} is the C_k- expression of part *A* and has 2^{n-2} terms. From the forms of *A* and *B*, it follows that C_k-expression of *B* has also 2^{n-2}terms such that the sequences of 2^{n-2} signs in the C_k- expressions of both *A* and *B* are same. The successive sequences of signs are as shown.

(i) One sign in (3.1) is +.

(ii) Sequence of 2 signs in (3.2) is: + –.

(iii) Sequence of 4 signs in (3.3)

= [Sequence of 2signs in (3.2)] – [Sequence of 2 signs in (3.2)]

= [+ –] – [+ –]

= + – – + .

(iv) Sequence of 8 signs in (3.4)

= Sequence of 4 signs in (3.3)] – [Sequence of 4 signs in (3.3)]

= [+ – – +] – [+ – – +]

= + – – + – + + – .

… …

The general form of the sequences of signs can be stated in the following way.

Rule for the sequence of 2^n ***signs:*** *When* $n \geq 1$ and $0 \leq m \leq n - 1$ *then in the sequence of* 2^n *signs starting with + sign, the sequence of signs obtained by the multiplication of each of first* 2^m *signs by – sign in succession is the sequence of second* 2^m *signs*.

Now the problem is, 'What is the sign of any k^{th} term of (3.n)?' We give a solution of the problem below.

(i) One sign in (3.1) is +.

(ii) The sequence of all 2^{n-2} signs in (3.n–1) is the sequence of the 1st 2^{n-2} signs in (3.n); and the sequence of 2^{n-2} signs obtained by the multiplication of – sign with each of 2^{n-2} signs in (3.n–1) in succession is the sequence of the 2nd or last 2^{n-2} signs in (3.n). Hence j^{th} and $(2^{n-2} + j)^{\text{th}}$ terms in (3.n) have the opposite signs when $1 \leq j \leq 2^{n-2}$.

(iii) The sequence of all 2^{n-3} signs in (3.n–2) is the sequence of the 1st 2^{n-3} signs in (3.n–1); and the sequence of 2^{n-3} signs obtained by the multiplication of – sign with each of 2^{n-3} signs in (3.n–2) in succession is the sequence of the 2nd 2^{n-3} signs in (3.n–1).

It follows from point (iii) and point (ii) that the sequence of all 2^{n-3} signs in (3.n–2) is the sequence of the 1st 2^{n-3} signs in (3.n); and the sequence of 2^{n-3} signs obtained by the multiplication of – sign with each of 2^{n-3} signs in (3.n–2) in succession is the sequence of the 2nd 2^{n-3} signs in (3.n). Hence *j*-th and $(2^{n-3} + j)$-th terms of (3.n) have the opposite signs when $1 \leq j \leq 2^{n-3}$.

… …

In general one sign in (3.1), the sequences of 2 signs in (3.2), 2^2 signs in (3.3), … , 2^{n-2} signs in (3.n–1) appear as the 1st one, the sequences of the 1st 2 signs, 1st 2^2 signs, …, 1st 2^{n-2} signs respectively in the sequence of all 2^{n-1} signs in (3.n) such that the sequence of *j* signs obtained by the multiplication of each the 1st *j* signs with – sign in (3.n) is the sequence of *j* signs which appears next to the 2^mth sign in (3.n) when $1 \leq j \leq 2^m$, $0 \leq m \leq n - 2$. Hence we have the following conclusion.

IV. Conclusion

j^{th} and $(2^m + j)$-th terms of (3.n) for $1 \leq j \leq 2^m$, $0 \leq m \leq n - 2$ have the opposite signs.

Case 1: When $j = 2^m$.

It follows from the conclusion that if $j = 2^m$ then 2^mth and 2^{m+1}th terms of (3.n) have the opposite signs. Hence in (3.n), 1st or 2^0th, 2nd or 2^1th, 2^2th, 2^3th term … in succession have + and – signs alternately starting with + sign. This implies that in 2^m th term of (3.n), + sign appears when *m* is even and – sign appears when *m* is odd.

Case 2: When $j < 2^m$.

Let *e* be 0 or a positive even integer; *d* be a positive odd integer; and $m_1, m_2, \ldots, m_t$ are the positive integers such that $e, d < m_1 < m_2 < \ldots < m_t \leq n - 2$. Then the greatest values of $m_t, m_{t-1}, m_{t-2}, \ldots$ are: $n - 2$, $n - 3$, $n - 4$, … in succession. We have the inequality: $2^{n-1} + \ldots + 2 + 1 < 2^n$. Consequently we find: $2^e < 2^{m_1}$; $2^{m_1} + 2^e < 2^{m_2}$; $2^{m_2} + 2^{m_1} + 2^e < 2^{m_3}$; ...; $2^{m_{t-1}} + \ldots + 2^{m_1} + 2^e < 2^{m_t}$; and similarly $2^d < 2^{m_1}$; $2^{m_1} + 2^d < 2^{m_2}$; $2^{m_2} + 2^{m_1} + 2^d < 2^{m_3}$; ... ; $2^{m_{t-1}} + \ldots + 2^{m_1} + 2^d < 2^{m_t}$ such that the smaller and bigger integers in two sides of the inequalities are the values of *j* and 2^m respectively. It then follows from the above conclusion that (i) 2^eth, $(2^{m_1} + 2^e)$th, ... , $(2^{m_t} + \ldots + 2^{m_1} + 2^e)$th terms in succession have + and – signs alternately starting with + sign; and (ii) 2^dth, $(2^{m_1} + 2^d)$th, … , $(2^{m_t} + \ldots + 2^{m_1} + 2^d)$-th terms in succession have – and + signs alternately starting with – sign.

Thus we find a general rule to determine the sign of k^{th} term of (3.n) for $1 \le k \le 2^{n-1}$. We name the rule as 'The lowest power rule in binary scale' due to an important role of the lowest power of 2 in the expression of k in binary scale: $k = 2^{h_1} + 2^{h_2} + \dots$, $h_1 > h_2 > \dots$

The lowest power rule in binary scale: The rule is shown in tabular form (Table 1):

Table 1

The lowest power	Number of powers	Sign at k^{th} place
even	odd	+
even	even	–
odd	odd	–
odd	even	+

Sequence of 64 signs with their ordinal numbers is shown in Table 2.

Table 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
+	–	–	+	–	+	+	–	–	+	+	–	+	–	–	+
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
–	+	+	–	+	–	–	+	+	–	–	+	–	+	+	–
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
–	+	+	–	+	–	–	+	+	–	–	+	–	+	+	–
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
+	–	–	+	–	+	+	–	–	+	+	–	+	–	–	+

Starting with the preliminary sequence: +, –, a verbal statement of the 'the lowest power rule in binary scale' in a general form can be given by considering + and – signs as two elements A and B respectively in the following way.

Starting with A and then B, if a permutation of 2^n elements for $n \ge 2$ by some repetitions of A and B is obtained in such a way that the sequence of the elements obtained by keeping A in place of B and B in place of A in the sequence of first 2^m elements for $1 \le m \le n - 1$ is the sequence of second 2^m elements, then the type of element at k^{th} place for $1 \le k \le 2^n$ depends on the lowest one among the powers of 2s and number of powers of 2s in the expression of k in binary scale: $k = 2^{h_1} + 2^{h_2} + \dots$, $h_1 > h_2 > \dots$ When one between the lowest power of 2 and number of powers of 2 is even and another is odd, then A appears at k^{th} place. When both of them are either even or odd, then B appears at k^{th} place.

The permutation is: *A B B A B A A B B A A B A B B A B A A B A B B A* …This is a permutation of two elements *A* and *B* by their typical repetitions in powers of 2.

We can get the greatest power rule from the lowest power rule. Indeed the greatest power rule is a particular case of the lowest power rule.

The greatest power rule in binary scale:

In the expression of k in binary scale, if the successive powers are even and odd alternately starting with the even greatest power then the sign at k^{th} place is +; and if the successive powers are odd and even alternately starting with the odd greatest power then the sign at k^{th} place is – . Obviously the greatest power rule is applicable when the powers are consecutive integers.

Remark 1: Intervals between two consecutive As or two Bs

The special permutation by two elements *A* and *B* has a property that if two successive *A*s or two successive *B*s appear at k_1^{th} and k_2^{th} places then $|k_1 - k_2| \in (1, 2, 3)$.

Remark 2: Another quality of the lowest power in Binary scale

In the context of the lowest power of 2, we mention an interesting property of the lowest power in Conjecture 1.

Conjecture 1: *The last bottom index in any m^{th} term among 2^{n-1} terms of the special expression for $\binom{k+n-1}{n}$ is z + 1 if the lowest one among the powers of 2s in the expression of m in binary scale is z.*

Example: In (2.4), the last bottom indices in 8 terms of the special expression for $\binom{k+3}{4}$ are: 0 + 1, 1 + 1, 0 + 1, 2 + 1, 0 + 1, 1 + 1, 0 + 1 and 3 + 1 respectively, where $1 = 2^0$; $2 = 2^1$; $3 = 2^1 + 2^0$; $4 = 2^2$; $5 = 2^2 + 2^0$; $6 = 2^2 + 2^1$; $7 = 2^2 + 2^1 + 2^0$; and $8 = 2^3$.

Remark 3: Connation of the recurrence with ordered compositions

Two sets of bottom indices in two terms of (2.2) are: (1, 1) and 2 such that 1 + 1 = 2. Four sets of bottom indices in four terms of (2.3) are: (1, 1, 1), (1, 2), (2, 1) and 3 such that 1 + 1 + 1 = 1 + 2 = 2 + 1 = 3. Eight sets of bottom indices in eight terms of (2.4) are: (1, 1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 3), (2, 1, 1), (2, 2), (3, 1) and 4 such that 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 1 + 3 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4. In fact 1, 2, 4 and 8 sets of bottom indices in 1, 2, 4 and 8 terms of (2.1), (2.2), (2.3) and (2.4) involve with 1, 2, 4 and 8 compositions of 1, 2, 3 and 4 respectively in a definite order. The rule of ordered compositions is demonstrated in the paper: Bera Soumendra, Relationships between Ordered Compositions and Fibonacci Numbers, Journal of Mathematics Research, Canadian Center of Science and Education, Vol.7, No.3, 2015.

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