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On Generalized Inverses, Group Inverses And Reverse Order Law For Range Quaternion Hermitian Matrices (Q-Ep)

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Abstract: *In this paper we discuss theGeneralized Inverses, Group Inverses And Reverse Order Law For Range Quaternion Hermitian Matrices (q-EP).*
Keywords :*Moore-Penrose inverse , q-EP matrix, Generalized Inverses for q-EP, Group Inverses for q-EP, Reverse Order Law for q-EP.*

I. Introduction

Throught we shall deal with nxn quaternion matrices[7]. Let A* denote the conjugate transpose of A. Let A^- be the generalized inverse of A satisfying AA^-A and z be the Moore-Penrose of A[6]. Any matrix $A \in H_{nXn}$ is called q-EP(2) if $R(A)=R(A^*)$ and his called q-EP_r, if A is q-EP and rk(A)=r, where N(A), R(A) and rk(A) denote the null space, range space and rank of A respectively. It is well known that sum and sum of parallel summable q-EP matrices are q-EP[3].In this paper we discuss theGeneralized Inverses, Group Inverses And Reverse Order Law For Range Quaternion Hermitian Matrices (q-EP).In this section, equivalent conditions for various generalized inverses of a q-EP_r matrix to be q-EP_r are determined. Generalized inverses belonging to the sets A{1,2}, A{1,2,3} and A{1,2,4} of a q-EP_r matrix A are characterized.A generalized inverse $A \in A\{1,2\}$ is shown to be q-EP_r whenever A is q-EP_r under certain conditions in the following way.

Theorem 1.1
Let $A \in H_{nXn}$, $X \in A\{1,2\}$ and XA, AX are q-EP_r matrices. Then A is q-$EP_r \Leftrightarrow$ X is q-EP_r

Proof
Since AX and XA are q-EP_r, by theorem([2],11), we have $R(AX) = R((AX)^*)$ and $R(XA) = R((XA^*))$.
Since $X \in \{1,2\}$ we have AXA = A, XAX = X
Now,

$$\begin{aligned} R(A) &= R(AX) \\ &= R((AX)^*) \\ &= R(X^*A^*) \\ &= R(X^*) \\ R(A^*) &= R(A^*X^*) \\ &= R((XA)^*) \\ &= R(XA) \\ &= R(X) \end{aligned}$$

Now, A is q-$EP_r \Leftrightarrow R(A) = R(A^*)$ and rk(A) = r
$\Leftrightarrow R(X^*) = R(X)$ and rk(A) = rk(X) = r
$\Leftrightarrow$ X is q-EP_r
Hence the theorem

Remark 1.2
In the above theorem, the conditions that both AX and XA to be q-EP_r are essential.
For instance, let

A = $\begin{pmatrix} 1 & k \\ -k & 1 \end{pmatrix}$, A is q-$EP_1$

X = $A^{=}$= $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A\{1,2\}$

AX = $\begin{pmatrix} 1 & 0 \\ -k & 0 \end{pmatrix}$

XA = $\begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}$

AX and XA are not q-EP_1. Also X is not q-EP

Now, we show that generalized inverses belonging to the sets A{1,2,3} and A{1,2,4] of a q-EP_r matrix A is also q-EP_r under certain conditions in the following theorems.

Theorem 1.3

Let $A \in H_{nXn}$, $X \in A\{1,2,3\}$, $R(X) = R(A^*)$. Then A is q-EP_r $\Leftrightarrow$ X is q-EP_r

Proof

Since $X \in A\{1,2,3\}$, we have AXA = A, XAX = X, $(AX)^* = AX$. Therefore,

$$R(A) = R(AX) = R((AX)^*) = R(X^*A^*) = R(X^*)$$

$$R(X) = R(A^*) \Rightarrow XX^{\dagger} = A^*(A^*)^{\dagger} \quad \text{[by[1]]}$$

$$\Rightarrow XX^{\dagger} = A^*(A^{\dagger})^*$$

$$\Rightarrow XX^{\dagger} = (A^{\dagger}A)^*$$

$$\Rightarrow XX^{\dagger} = A^{\dagger}A$$

$$\Rightarrow XX^{\dagger} = (A^*)((A)^*)^{\dagger}$$

$$\Rightarrow X = R((A)^*)$$

$$\Rightarrow R(X) = R(A^*)$$

A is q-EP_r $\Leftrightarrow R(A) = R(A^*)$ and rk(A) = r

$\Leftrightarrow R(X^*) = R(X)$ and rk(A) = rk(X) = r

$\Leftrightarrow$ X is q-EP_r

Hence the theorem.

Theorem 1.4

Let $A \in H_{nxn}$,$X \in \{1,2,4\}$, R(A)=R(X*). Then A is q-EP_r $\Leftrightarrow$ X is q-EP_r

Proof

Since $X \in A\{1,2,4\}$, we have AXA=A, XAX=A, (XA)* = XA.

Also. $R(A) = R(X^*)$. Now

$$R(A^*) = R(A^*X^*) = R((XA)^*) = R(XA) = R(X)$$

A is q-EP_r $\Leftrightarrow R(A) = R(A^*)$ and rk(A) = r

$\Leftrightarrow$ R(X*) = R(X) and rk(A) = rk(X) = r [by[2], 11]

$\Leftrightarrow$ X is q-EP_r

Remarks 1.5

In particular, if X = $A^{\dagger}$ then $R(A^{\dagger})=R(A^*)$ holds. Hence A is q-EP_r is equivalent to $A^{\dagger}$ is q-EP_r.

II. Group Inverse of q-EP matrices

In this section, the existence of the group inverse for q-EP matrices under certain condition is derived. It is well known that, for an EP matrix, group inverse exists and coincides with it Moore-Penrose inverse. However, this is not the case for a q-EP matrix.

For example,

Consider A = $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

A is q-EP, matrix, $A^2 = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$, rk(A) = rk($A^2$)

Therefore, by theorem [p.162[1]], group inverse $A^{\dagger}$ does not exist for A. Here it is proved that for q-EP matrix A, if the group inverse exists, it is also a q-EP matrix.

Theorem 2.1

Let $A \in H_{nxn}$ be q-EPr and rk(A) = rk(A^2). Then $A^{\#}$ exists and is q-EP_r

Proof

Since rk(A) = rk(A^2), by theorem[p.162,[1]$A^{\#}$exists for A. To show that $A^{\#}$is q-EP_r, it is enough to show that

$R(A^{\#}) = R((A^{\#})^*)$

Since $AA^{\#} = A^{\#}A$

We have R(A)= R($AA^{\#}$)

= R($A^{\neq}A$)

= R($A^{\neq}$)

$AA^{\neq}A = A \Rightarrow A^* = A^*(A^{\neq})^* A^*$

Therefore R(A^*) = R($A^*(A^{\neq})^* A^*$)

= R($A^*(A^{\neq})^*$)

= R(($A^{\neq}A$)*)

= R(($AA^{\#}$)*)

= R(($A^{\neq}$)* A^*)

= R(($A^{\neq}$)*)

Now,

A is q-$EP_r \Rightarrow$ R(A) = R(A^*) and rk(A) = r

$\Rightarrow$ R(A^*) = R(($A^{\neq}$)*) and

Rk(A) = rk($A^{\neq}$) = r

$\Rightarrow$ $A^{\neq}$ is q-EP_r

Hence the Theorem.

Remark 2.2

In the above theorem the condition that rk(A)=rk(A^2) is essential. Therefore, $A^{\neq}$ does not exist for a q-EP matrix A. Thus, for a q-EP matrix A, if $A^{\neq}$ exists then it is also q-EPr.

Theorem 2.3

For at H_{nxn}, if $A^{\neq}$ exists then, A is q-EP $\Leftrightarrow$ $A^{\neq}$ = $A^{\dagger}$

Proof

A is q-EP $\Leftrightarrow$ A is Ep [By Theorem11,[2]]

$\Leftrightarrow$ $A^{\neq}$ =$A^{\dagger}$ [p.164[8]]

Hence the theorem.

Theorem 2.5

For $A \in H_{nxn}$, A is q-EPr $\Leftrightarrow$ $A^{\dagger}$ = polynomial in A

Proof

It is clear that if $A^{\dagger}$ =f(A) for some polynomial f(X), then A commutes with $(A)^{\dagger}$ for some polynomial f(X), then A commutes with $(A)^{\dagger}$

$\Rightarrow AA^{\dagger} = A^{\dagger}A$

$\Rightarrow$ A is q-EP_r [By [2],11]

Conversely,

Let A be q-EP_r, then $AA^{\dagger} = A^{\dagger}A$ and $A^{\dagger}A = AA^{\dagger}$.

Now, we will prove the $A^{\dagger}$ can be expressed as polynomial in A.

Let $(A)^s+\lambda_1(A)^{s+1}+\lambda_2(A)^{s+2}+\ldots\ldots+\lambda_q(A)^{s+q} = 0$,

Be the minimum polynomial of A. Then s=0 or s=1.

For suppose that $s \geq 2$, then

$A^{\dagger}[(A)^s+\lambda_1(A)^{s+1}+\ldots\ldots\ldots+\lambda_q(A)^{s+q}] = 0$,

$[AA^{\dagger}A]A^{s-2}+\lambda_1[AA^{\dagger}A]A^{s-1}+\ldots\ldots\ldots+\lambda_9[AA^{\dagger}A]A^{s+q-2}=0$

That is $(A)^{s-1}+\lambda_1(A)^s+\ldots\ldots\ldots\lambda_9(A)^{s+q-1}=0$

Which is contradiction.

If s=0 then

$(A^{\dagger}) = A^{-1}=-\lambda_1 I-\lambda_2(A)-\ldots\ldots\ldots\ldots\ldots\ldots\ \lambda_q(A)^{q-1}$

$A^{\dagger} = A^{-1}=-\lambda_1-\lambda_2(A)-\ldots\ldots\ldots\ldots\ldots\ldots-\lambda_q(A)^{q-1}$

$= [-\lambda_1 I-\lambda_2 A-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots-\lambda_q(A)^{q-1}]$

$A^{\dagger}$ = polynomial in A

If s=1, then $(A^{\dagger})[A+\lambda_1(A)^2+\ldots\ldots+\lambda_q(A)^{q+1}] = 0$

and it follows that

$A^{\dagger}A=-\lambda_1(A)-\lambda_2(A^2)-\ldots\ldots\ldots\ldots-\lambda q(A)^q$ is a polynomial in A.

However, $A^{\dagger}=[A^{\dagger}A]A^{\dagger}$

$= -\lambda_1(A)^{\dagger}(A)-\lambda_2(A)-\ldots\ldots\ldots\ldots\ldots\ldots-\lambda_q(A)^{q-1}$

$= [-\lambda_1 I-\lambda_2(A)-\ldots\ldots\ldots\ldots-\lambda_q(A)^{q-1}]$

$A^{\dagger}$ = polynomial in A.

Hence the theorem.

III. Reverse order law for q-EP matrices

For any two non singular matrices $A,B \in H_{nxn}$ $(AB)^{-1}=B^{-1}A^{-1}$ holds. However, it is not true for generalized inverses of matrices [15]. In general, $(AR)^{\dagger} \neq B^{\dagger}A^{\dagger}$ for any two matricesa and B. we say that reverse order law holds for Moore-Penrose inverse of the product of A and B, if $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$. It is well known that [P.181,[1]], $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$ if and only if $R(BB^*A) \subseteq R(A^*)$ and $R(A^*AB) \subseteq R(B)$.In this section, for a pair of q-EPr matrices A and B, necessary and sufficient condition for $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$ given.

Theorem

If A and B are q-EPr matrices with $R(A)=R(B^*)$ then $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$

Proof

Since A is q-EPr,

$\Rightarrow R(A) = R(A^*)$

$\Rightarrow R(B^*) = R(A)$ (B is q-EPr)

$\Rightarrow R(B) = R(A^*)$
$\Rightarrow R(B) = R(A^\dagger)$ [by[8]]

That is, given $x \in C_{nxn}$, there exists $y \in C_n$ such that $Bx=A^\dagger y$

Now, $Bx = A^\dagger y \Rightarrow (B^\dagger A^\dagger A)\, Bx = (B^\dagger A^\dagger A)A^\dagger y$

$\Rightarrow B^\dagger A^\dagger ABx = B^\dagger A^\dagger AA^\dagger y$

$\Rightarrow B^\dagger A^\dagger ABx = B^\dagger A^\dagger y$

$\Rightarrow B^\dagger A^\dagger ABx = B^\dagger Bx$

Since $B^\dagger B$ is hermitian, it follows that $B^\dagger A^\dagger AB$ is hermitian.
Similarly,

$A^\dagger y = Bx \Rightarrow (ABB^\dagger)A^\dagger y = (ABB^\dagger B)x$

$\Rightarrow ABB^\dagger A^\dagger y = A(BB^\dagger B)x$

$\Rightarrow ABB^\dagger A^\dagger y = A(Bx)$

$\Rightarrow ABB^\dagger A^\dagger \quad y = A(A^\dagger y)$

$\Rightarrow ABB^\dagger A^\dagger y = AA^\dagger y$

Since $AA^\dagger$ is hermitian, it follows that $ABB^\dagger A^\dagger$ is hermitian. Further, by theorem [8]

$R(A)=R(B) \Rightarrow AA^\dagger=BB^\dagger$

$R(A^\dagger)=R(B) \Rightarrow A^\dagger (A^\dagger)^\dagger=BB^\dagger$

$\Rightarrow A^\dagger A = BB^\dagger$

Hence $(AB)\,(B^\dagger A^\dagger)\,(AB) = ABB^\dagger\,(A^\dagger A)B$

$= ABB^\dagger\,(B^\dagger BB^\dagger)B$

$= (AB)(B^\dagger)B$

$= A(BB^\dagger B)$

$= A(B)$

$= AB$

$lll^{ly}\,(B^\dagger A^\dagger)\,(AB)\,(B^\dagger A^\dagger) = B^\dagger A^\dagger$.

Thus, $B^\dagger A^\dagger$ satisfies the definition of the Moore-Penrose inverse, that is $(AB)^\dagger = B^\dagger A^\dagger$
Hence the theorem.

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