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G- Frame Operator in C* Algebra

P. Kalyani
Department of Mathematics, Aurora's Technological and Research Institute, Uppal, Hyderabad

Abstract: *The g-frame operator for g-frames in C* algebra is introduced. The results on g-frame operators are proved. Frame identities are shown. Result on direct sum of g-frame operators on direct sum of Hilbert Spaces is presented.*

I. Introduction

Frames are generalization of bases .D. Han and D. R. Larson have developed a number of basic concepts of operator theoretic approach to frame theory in C* algebra. Peter G Casazza presented a tutorial on frame theory and he suggested the major directions of research in frame theory. Radu V. Balan and Peter G. Casazza have analyzed decomposition of a normalized tight frame and obtained identities for frames. A. Najati and A. Rahimi have developed the generalized frame theory and introduced methods for generating g-frames of a C* algebra.

1.1. Banach Algebra:- A Banach Algebra is a complex Banach space A together with an associative and distributive multiplication such that λ(ab)= λ(a) λ(b) ,
$\|ab\| \leq \|a\|\|b\|$ ∀a,b∈ A and $\lambda \in C$
For any x,$x^1, y, y^1 \in A$ we have $\|xy - x^1y^1\| \leq$
$\|x\|\|y - y^1\|$+$\|x - x^1\|$
The algebra A is said to be commutative if ab=ba∀$a, b \in A$

1.2. Definition (Involution of an algebra):- Let A be a Banach algebra .An involution on A is a map *: A→A such that
1. a^{**}=a
2. $(\lambda a + \mu b)^*$ =$\bar{\lambda}$a*+$\bar{\mu}$b*
3. $(ab)^*$=b*a*
1.3. Definition:- (C* algebra) If A is a Banach algebra with involution and also $\|aa*\|$=$\|a\|^2$ then A is called a c* algebra.
Example:- C(X) let X be a compact space and C(X) is a Banach space of all complex valued functions on X with norm $\|f\|$ =$sup_{x\in X}|f(x)|$ Multiplication on C(X) is defined as pointwise ief.g(x)=f(x)g(x)
And involution by complex conjugation f*(x)=$\overline{f(x)}$

II. G- Frame And G-Frame Operator

Throughout this paper {A_j j∈J} will denote a sequence of C* algebras .Let L (A,A_j) be a collection of bounded linear operators from A toA_j and {Δ_j∈ L (A,A_j); j ∈ J} we obtain some characterization of g-frame operator. They are the generalizations of results of frame operator.

2.1. Definition: - A sequence of operators$\{\Delta_j\}_{j \in J}$ is said to be g-frame for C* algebra A with respect to sequence of C* algebras {A_j, j ∈ J} if there exists two constants 0 < A ≤ B < ∞ for any vector f ∈ H,

$$A \|\bar{f}\|^2 \leq \sum_{j\in J} \|\Delta_j \bar{f}\|^2 \leq B\|f^2\| \text{ where } \bar{f}(x) = f^*(x)$$

The above inequality is called a g-frame inequality. The numbers A, B are called the lower frame bound and upper frame bound respectively.

2.2. Definition: -A g-frame for $\{\Delta_j\}_{j \in J}$ is said to be g-tight frame if A=B, then we have

$$A \|\bar{f}\|^2 = \sum_{j\in J} \|\Delta_j \bar{f}\|^2 \text{ for all } f^* \in A$$

2.3. Definition: - A g-frame $\{\Delta_j\}_{j \in J}$for A is said to be a g-normalized tight frame for A if A = B = 1. Then we have $\|\bar{f}\|^2 = \sum_{j\in J}\|\Delta_j \bar{f}\|^2$ for all f ∈ H

2.4. Definition: -Let $\{\Delta_j\}_{j \in J}$ be a g-frame for c* algebra. G-frame operator

$$S^g : A \rightarrow A \text{ is defined as}$$
$$S^g f = \sum_{j\in J} \Delta_j^* \Delta_j \, f^* \text{ for all } f^* \in A$$

By using above definitions we have the following theorems.

III. Main Result

3.1. Theorem: If S^g is a g-frame operator, then we have

1) $< S^g$ f, f$>$ = $\sum_{j\in J} \|\Delta_j f\|^2$ for all f ∈ A
2) S^g is a positive operator
3) S^g is a self-adjoint operator.

Proof: -S^g is a g-frame operator means

$S^g f = \sum_{j\in J} \Delta_j^* \Delta_j$ f* for all f^*∈ A

1) $< S^g$ f, f$>$ = $< \sum_{j\in J} \Delta_j^* \Delta_j$ f*f, f$>$

$= \sum < \Delta_j^* \Delta_j f, \Delta_j^* \Delta_j f >$

$= \sum \| \Delta_j$ f$\|^2$[$\because <$ x, x $>$ = $\|x^2\|$]

(2) Clearly S^g is a positive operator by definition

(3) It is left to the reader

3.2. Theorem: Suppose $\{\Delta_j\}_{j\in J}$ is a g- frame if and only if AI≤ S^g ≤BI and $\{\Delta_j\}_{j\in J}$ is a g-normalized tight frame if and only if S^g =I where I is an identity operator on A.

Proof: Since $\{\Delta_j\}_{j\in J}$ is a g-frame so we have

A$\|\bar{f}^2\| \leq \|\Delta_j \bar{f}\|^2$ ≤B$\|\bar{f}^2\|$ for all $\bar{f}$ ∈A

Consider$\langle AI\bar{f},\bar{f}\rangle$ =A$\langle \bar{f},\bar{f}\rangle$ =A$\|\bar{f}^2\| \leq \sum_{j\in J} [\![\Delta_j \bar{f}]\!]^2 \leq B\|f\|^2$

= B$\langle \bar{f},\bar{f}\rangle$=$\langle BI\bar{f},\bar{f}\rangle$

Conversely suppose AI≤ $S^g \leq BI$

$\Rightarrow \langle AI\bar{f},\bar{f}\rangle \leq \langle S^g\bar{f},\bar{f}\rangle \leq \langle IB\bar{f},\bar{f}\rangle$for all $\bar{f}$ ∈A

$\Rightarrow$ A$\|\bar{f}^2\| \leq \sum_{j\in J} \|\Delta_j \bar{f}\|^2 \leq B\|\bar{f}\|^2$

Which implies $\{\Delta_j\}_{j\in J}$ is a g-frame for A.

Suppose $\{\Delta_j\}_{j\in J}$ is a g-normalized tight frame for A

$\Leftrightarrow \sum_{j\in J} \|\Delta_j \bar{f}\|^2 = \|\bar{f}\|^2$ for all $\bar{f}$ ∈A

If and only if $\langle s^g f,f\rangle$=$\langle If,f\rangle$

If and only if S^g=I

Note: We can easily see that frame operator S^g is invertible and $S^{g^{-1}}$ is a positive operator.

3.3. Theorem: Let S^g be a g- frame operator for the g-frame $\{\Delta_j\}_{j\in J}$ with frame bounds A,B in the C* algebra A. Then B^{-1} I≤ $S^{g^{-1}} \leq A^{-1}$I

Proof: Since $\{\Delta_j\}_{j\in J}$ is a g-frame for C* algebra A ,we have AI≤ S^g ≤BI

Since AI≤ $S^g \Rightarrow$ 0≤(S^g-AI)$S^{g^{-1}} \Rightarrow$0≤I-A$S^{g^{-1}}$

$\Rightarrow S^{g^{-1}} \leq A^{-1}$I ____________________(1)

Similarly, we can prove B^{-1}I≤ $S^{g^{-1}}$ ____________________(2)

Hence from (1) and (2) B^{-1} I≤ $S^{g^{-1}} \leq A^{-1}$I

3.4. Theorem: A sequence of operators $\{\Delta_j\}_{j\in J}$ where $\bar{\Delta}_j$=$\Delta_j S^{g^{-1}}$ is a G-frame for C* algebra with frame bounds 1/B and 1/A

Proof: Consider $\sum_{j\in J} \|\Delta^*{}_j f\|^2$=$\sum_{j\in J} \|\bar{\Delta}_j f\|^2$

$=\sum_{j\in J} \|\Delta_j S^{g^{-1}} f\|^2$

$=\sum_{j\in J} \langle \Delta_j{}^* \Delta_j S^{g^{-1}} f, S^{g^{-1}} f\rangle$

$= \langle \sum_{j\in J} \Delta_j{}^* \Delta_j S^{g^{-1}} f, S^{g^{-1}} f\rangle$

$=\langle S^g S^{g^{-1}f}, S^{g^{-1}} f\rangle$

$= \langle f, S^{g^{-1}} f\rangle \leq \frac{1}{A} \|f\|^2$

$$\Rightarrow \sum_{j\in J} \|\bar{\Delta}_j f\|^2 \leq \frac{1}{A} \|f\|^2$$

$\|f\|^2 = \langle \sum_{j\in J} \bar{\Delta}_j{}^* \Delta_j f, f\rangle = \sum_{j\in J} \langle \Delta_j{}^* \Delta_j f, f\rangle$

$$=\sum_{j\in J}\langle \Delta_j f\ \bar{\Delta}_j f\rangle$$

$$\leq \left[\sum_{j\in J}\left\|\Delta_j f\right\|^2\right]^{\frac{1}{2}}\left[\sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2\right]^{\frac{1}{2}}$$

$$\leq \sqrt{B}\|f\|\left[\sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2\right]^{1/2}$$

$$\Rightarrow \|f\|^2 \leq \sqrt{B}\|f\|\left[\sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2\right]^{1/2}$$

$$\Rightarrow \frac{1}{B}\|f\|^2 \leq \sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2$$

Hence $\Rightarrow \frac{1}{B}\|f\|^2 \leq \sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2 \leq \frac{1}{A}\|f\|^2$

Which shows that the sequence of operators $\{\bar{\Delta}_j\}_{j\in J}$ is a g-frame for the C* algebra A with frame bounds $\frac{1}{B}$ *and* $\frac{1}{A}$

3.5. Theorem: - Let $\{\Delta_j\}_{j\in J}$ be a g-frame for C* algebra A with respect to $\{A_j$, j∈J} and V∈B (H) be an invertible operator. Then $\{\Delta_j \mathrm{V}\}_{j\in J}$ is a G-frame for A with respect to $\{A_j$, jϵJ} and its g-frame operator is V^*S^g V.

Proof: - Since VϵB (H), ∀fϵH, we have Vf ∈H given that $\{\Delta_j\}_{j\in J}$ is a G-frame for H, so for all VfϵH we have $\|\mathrm{V}f\|^2 \leq \sum_{j\in J}\|\Delta_j Vf\|^2 \leq \mathrm{B}\,\|\mathrm{V}f\|^2$

Since V is invertible operator, therefore we have

$$\|\mathrm{V}f\|^2 \leq \|\mathrm{V}\|^2\|f\|^2 \text{ and } \|V^{-1}\|^{-2}\|f\|^2 \leq \|\mathrm{V}f\|^2$$

By above inequalities, the equation become

$$\mathrm{A}\|V^{-1}\|^{-2}\|f\|^2 \leq \sum_{j\in J}\|\Delta_j Vf\|^2 \leq \mathrm{B}\|\mathrm{V}\|^2\|f\|^2,\ \forall f\in H$$

$\Rightarrow \{\Delta_j V\}_{j\in J}$ is a g-frame for A.

For each, fϵA. We have S^gV f= $\sum_{j\in J}\Delta_j^*\Delta_j$Vf

$\Rightarrow V^*S^g$Vf = $\sum_{j\in J}V^*\Delta_j^*\Delta_j$Vf

$\Rightarrow V^*S^g$V is a g-frame operator for the frame $\{\Delta_j V\}_{j\in J}$

Frame Identities for g-frames

3.6. Proposition: If, T_1 and T_2 are two operators in a C* algebra A satisfying $T_1 + T_2 = I$, Then $T_1 - T_2 = T_1^2 - T_2^2$

Proof: Consider $T_1 - T_2 = T_1 - (I - T_1) = T_1^2 - (I - 2T_1 + T_1^2)$

$$= T_1^2 - (I - T_1)^2$$

$$= T_1^2 - T_2^2$$

3.7. Theorem: - Let $\{\Delta_j\}_{j\in J}$ be a g-normalize d tight frame for A for ICJ, then

$$\sum_{j\in J}\|\Delta_j f\|^2 = \|\sum_{j\in J}\Delta_j^*\Delta_j\|^2 \Leftrightarrow S_I^g S_{I^c}^g = 0$$

Proof: Consider $\sum_{j\in J}\|\Delta_j f\|^2 = \|\sum_{j\in J}\Delta_j^*\Delta_j\|^2 \Leftrightarrow \sum_{j\in J}\|\Delta_j f\|^2$

$$\Leftrightarrow \|\sum_{j\in J}\Delta_j^*\Delta_j\|^2 = 0$$

$$\Leftrightarrow \sum_{j\in J}\|\Delta_j f\|^2 - <\sum_{j\in J}\Delta_j^*\Delta_j f, \sum_{j\in J}\Delta_j^*\Delta_j f> = 0$$

$\Leftrightarrow < S_I^g f, f > - < S_I^g f, S_I^g f > = 0$

$\Leftrightarrow < (S_I^g - S_I^g)^2 f, f > = 0$

$\Leftrightarrow S_I^g (I - S_I^g) f = 0$

$\Leftrightarrow S_I^g S_{I^c}^g = 0$ for all f∈H

3.8. Theorem: Let $\{\Delta_j\}_{j\in J}$ be a g-normalized tight frame for H, for ICJ with respect to [H_j, j∈J]. Then for ICJ and for all fϵH.

$$\sum_{j\in J}\|\Delta_j f\|^2 + \|S_{I^c}^g f\|^2 = \sum_{j\in J}\|\Delta_j f\|^2 + \|S_I^g f\|^2$$

Proof: Science $\{\Delta_j\}_{j\in J}$ is a g-normalized tight frame for H,

Therefore S^g = I and $S_I^g + S_{I^c}^g$ = I, for all f∈H.

Consider $\sum_{j\in J} \|\Delta_j f\|^2 + \|S_{I^c}^g f\|^2 = < S_I^g f, f > + < S_{I^c}^g f, S_{I^c}^g f >$

$= < (S_I^g + S_{I^c}^g)^2 f, f >$

$= < (S_I^g + (I - S_I^g)^2 f, f >$

$= < (I - S_I^g + S_I^g)^2 f, f >$

$= < (S_{I^c}^g + S_I^g)^2 f, f >$

$= < S_{I^c}^g f, f > + < S_I^g f, S_I^g f >$

$= \sum_{j\in I^c} \|\Delta_j f\|^2 + \|S_I^g f\|^2$

References

[1]. M.R.Abdollahpour ,M.H. Faroughi and A.Rahimi " PG-Frames in Banach spaces" Methods of Functional analysis and topology
[2]. Vol.13 no.3(2007),pp.201-210
[3]. R.Balan P.G.Casazza, D.Edidin and G.Kutynoik " Decomopostion of frames and a new frame identity " Wavelet XI (San
[4]. Diego),CA(2005) pp.379-388 SPIE Proc 5914 ,SPIE Bellinginam WA.
[5]. P.G.Casazza,, "The art of Frame theory" ,Taiwanese Journal of Mathematics Vol.4,No.2(2000) pp 129-201
[6]. D.Han and D.R.Larson "Frames Bases and Group Representations" ,Memories Ams Nov 7 (2000) Providence RI
[7]. A.Najati and A.Rahimi "Generalized frames in Hilbert spaces" ,Bulletin of the Iranian Mathematical society Vol.35,No.1(2009) pp.97- 109.
[8]. A.Najati and M.H.Faroughi " P- frames of subspaces of separable Hilbert spaces" ,South east Asian Bulletin of Mathematics 31(2007) pp.713-726.
[9]. G. Upender Reddy and N.Gopal Reddy, "Some results of Frame operator in Hilbert space", Journal of Mathematical Education, Volume XLV, No..3 September 2011.