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Certain Results for Laguerre Polynomials of Several Variables

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Abstract: *The aim of this research paper is to derive some hypergeometric formulas of Laguerre polynomials of two, three and several variables. Also we apply this formulas to derive some integral formulas involving Laguerre polynomials of several variables.*
Keywords: *Generalized Lauricella function, Hypergeometric formulas, Integral formulas, Laguerre polynomials.*

I. Introduction

In 1991,Ragab[7] defined Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows :

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r!\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)} \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomials of one variable [8]

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}\ {}_1F_1\left[\begin{matrix}-n\ ;\\ 1+\alpha\ ;\end{matrix}x\right]. \tag{1.2}$$

In 1996, Khan and Shukla [5] defined the Laguerre polynomials of r -variables $L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ as follows :

$$L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=\frac{\prod_{j=1}^{r}(1+\alpha_j)_n}{(n!)^r}\Psi_2^{(r)}\left[-n;1+\alpha_1,\cdots,1+\alpha_r;x_1,\cdots,x_r\right] \tag{1.3}$$

where $\Psi_2^{(r)}$ is a confluent hypergeometric function of r-variables [10]

$$\Psi_2^{(r)}\left[a;c_1,\cdots,c_r;x_1,\cdots,x_r\right]=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(a)_{m_1+\cdots+m_r}}{(c_1)_{m_1}\cdots(c_r)_{m_r}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{1.4}$$

Srivastava and Daoust [9] defined extremely generalized hypergeometric function of n -variables (which is referred to in the literature as the generalized Lauricella's function of several variables) defined by [10]

$$F\begin{matrix}A:B';\cdots;B^{(n)}\\ C:D';\cdots;D^{(n)}\end{matrix}\left[z_1,\cdots,z_n\right]$$

$$\equiv F\begin{matrix}A:B';\cdots;B^{(n)}\\ C:D';\cdots;D^{(n)}\end{matrix}\left(\begin{matrix}\left[(a):\theta',\cdots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\cdots;\left[(b^{(n)}):\phi^{(n)}\right];\\ \left[(c):\psi',\cdots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\cdots;\left[(d^{(n)}):\delta^{(n)}\right];\end{matrix}z_1,\cdots,z_n\right)$$

$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\Omega(m_1,\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!}, \tag{1.5}$$

where

$$\Omega(m_1,\cdots,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \tag{1.6}$$

and the coefficients $\theta_j^{(k)}$, $j=1,2\cdots,A$; $\phi_j^{(k)}$, $j=1,2,\cdots,B^{(k)}$; $\psi_j^{(k)}$, $j=1,2,\cdots,C$; $\delta_j^{(k)}$, $j=1,2,\cdots,D^{(k)}$; for all $k\in\{1,2,\cdots,n\}$ are real and positive, (a) abbreviates the array of A parameters $a_1,\cdots a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}, j=1,2,\cdots,B^{(k)}$ for all $k\in\{1,2,\cdots,n\}$ with similar interpretations for (c) and $(d^{(k)})$ $k\in 1,2,\cdots,n$; *et cetera*. Note that, when the coefficients in equation (1.5) equal to 1, the generalized Lauricella function (1.5) reduces to the following multivariable extension of the Kamp´e de F´eriet function [10]:

$$F\begin{matrix} p:q_1;\cdots;q_n \\ l:m_1;\cdots;m_n \end{matrix}[z_1,\cdots,z_n] \equiv F\begin{matrix} p:q_1;\cdots;q_n \\ l:m_1;\cdots;m_n \end{matrix}\left(\begin{matrix} (a_p):(b'_{q_1});\cdots;(b^{(n)}_{q_n}); \\ (c_l):(d'_{m_1});\cdots;(d^{(n)}_{m_n}); \end{matrix} z_1,\cdots,z_n\right)$$

$$=\sum_{s_1,\cdots,s_n=0}^{\infty}\Omega(s_1,\cdots,s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}, \tag{1.7}$$

where

$$\Omega(s_1,\cdots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(c_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(d'_j)_{s_1}\cdots\prod_{j=1}^{m_n}(d_j^{(n)})_{s_n}}. \tag{1.8}$$

II. Hypergeometric Formulas of $L_n^{(\alpha_1,\cdots,\alpha_r)}[x_1,\cdots,x_r]$

Two interesting generating functions of Laguerre polynomials of two and several variables are given by Chatterje [2] and Khan and Shukla [5]

$$\sum_{n=0}^{\infty}\frac{n!L_n^{(\alpha,\beta)}(x,y)t^n}{(\alpha+1)_n(\beta+1)_n}=e^t\,{}_0F_1(-;1+\alpha;-xt)\,{}_0F_1(-;1+\beta;-yt) \tag{2.1}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{s-1}t^n}{\prod_{j=1}^{s}(\alpha_j+1)_n}L_n^{(\alpha_1,\cdots,\alpha_s)}(x_1,\cdots x_s)=e^t\prod_{j-1}^{s}{}_0F_1(-;1+\alpha_j;-x_jt) \tag{2.2}$$

In (2.1) if we replace y and β by $-x$ and α and use the result [3]

$${}_0F_1(-;a;x)\,{}_0F_1(-;a;-x)={}_0F_3\left(-;a,\frac{a}{2},\frac{a}{2}+\frac{1}{2};\frac{-x^2}{4}\right), \tag{2.3}$$

we get

$$\sum_{n=o}^{\infty}\frac{n!L_n^{(\alpha,\alpha)}(x,-x)t^n}{(\alpha+1)_n(\alpha+1)_n}=e^t\,{}_0F_3\left[\begin{matrix} ---------- & ;-(xt)^2 \\ \alpha+1,\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+2); & 4 \end{matrix}\right] \tag{2.4}$$

On expressing e^t and ${}_0F_3$ in series forms, we get

$$\sum_{n=0}^{\infty}\frac{(n!)^2L_n^{(\alpha,\alpha)}(x,-x)t^n}{(\alpha+1)_n(\alpha+1)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-x^2/4)^k\,t^n}{(\alpha+1)_k((\alpha+1)/2)_k((\alpha+2)/2)_k(n-2k)!\,k!} \tag{2.5}$$

Comparing the coefficient of t^n on both sides of (2.5) and using the following identities [10]:

$$(n-k)!=\frac{(-1)^k n!}{(-n)_k},0\le k\le n \tag{2.6}$$

and

$$(\lambda)_{2n} = 2^{2n}\left(\tfrac{1}{2}\lambda\right)_n \left(\tfrac{1}{2}\lambda+\tfrac{1}{2}\right)_n, \quad n=0,1,2,\cdots, \tag{2.7}$$

we get

$$L_n^{(\alpha,\alpha)}(x,-x) = \frac{(\alpha+1)_n(\alpha+1)_n}{(n!)^2}\ {}_2F_3\left[\begin{matrix} -\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2} & ; \\ 1+\alpha, \tfrac{1}{2}(1+\alpha), \tfrac{1}{2}(2+\alpha); \end{matrix} -x^2\right], \tag{2.8}$$

where ${}_pF_q$ is the generalized hypergeometric function of one variable[10]

$${}_pF_q\left[\begin{matrix} a_1, a_2\cdots, a_p; \\ b_1, b_2, \cdots, b_q; \end{matrix} x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n\, x^n}{(b_1)_n(b_2)_n\cdots(b_q)_n\, n!}. \tag{2.9}$$

Now, we consider (2.2) with $n=3, x_1=x, x_2=-x, x_3=y, \alpha_1=\alpha_2=\alpha, \alpha_3=\beta$, using (2.1) and comparing the coefficient of t^n, we get

$$\frac{(n!)^2 L_n^{(\alpha,\alpha,\beta)}(x,-x,y)}{(\alpha+1)_n(\alpha+1)_n(\beta+1)_n} = \sum_{k=0}^{n} \frac{(n-k)!(-y)^k}{(\alpha+1)_{n-k}(\alpha+1)_{n-k}(\beta+1)_k\, k!} L_{n-k}^{(\alpha,\alpha)}(x,-x) \tag{2.10}$$

In (2.10) if we apply (2.6) and (2.8), we arrive to the following result :

$$L_n^{(\alpha,\alpha,\beta)}(x,-x,y) = \frac{(\alpha+1)_n(\alpha+1)_n(\beta+1)_n}{(n!)^3}$$

$$\times X^{1:\,0;\,0}_{0:\,3;1}\left[\begin{matrix} -n & : ------------ & ; -- & ; \\ -- & : \alpha+1, \tfrac{1}{2}(\alpha+1), \tfrac{1}{2}(\alpha+2) & ; \beta+1; & \end{matrix} \frac{-x^2}{4}, y\right] \tag{2.11}$$

where $X^{A:B;\,B'}_{C:D;\,D'}$ is Exton double hypergeometric series [4]

$$X^{A:\,B;\,B'}_{C:\,D;D'}\left[\begin{matrix} (a):(b);(b'); \\ (c):(d);(d'); \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{((a))_{2m+n}((b))_m((b'))_n\, x^m\, y^n}{((c))_{2m+n}((d))_m((d'))_n\, m!\, n!}. \tag{2.12}$$

Similarly, in (2.2) if we put $n=4, x_1=x, x_2=-x, x_3=y, x_4=-y, \alpha_1=\alpha_2=\alpha, \alpha_3=\alpha_4=\gamma$ and use (2.1), (2.3) and (2.6), we get

$$L_n^{(\alpha,\alpha,\gamma,\gamma)}(x,-x,y,-y) = \frac{(\alpha+1)_n(\alpha+1)_n(\gamma+1)_n(\gamma+1)_n}{(n!)^4} F^{\,2:0;0}_{\,0:3;3}\left[\begin{matrix} -\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}: \\ ---- \quad : \end{matrix}\right.$$

$$\left.\begin{matrix} --------- & ; & -------- & ; \\ \alpha+1, \tfrac{1}{2}(\alpha+1), \tfrac{1}{2}(\alpha+2) & ; & \gamma+1, \tfrac{1}{2}(\gamma+1), \tfrac{1}{2}(\gamma+2); & \end{matrix} -x^2, -y^2\right], \tag{2.13}$$

where $F^{\,p:q;\,k}_{\,l:\,m;n}[x,y]$ denotes the Kampé de Fériet function of two variables[10]

$$F^{\,p:q;k}_{\,l:m;n}\left[\begin{matrix} (a_p);(b_q);(c_k); \\ (\alpha_l);(\beta_m);(\gamma_n); \end{matrix} x, y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!}\frac{y^s}{s!}. \tag{2.14}$$

Further, in (2.2), if we put $n=4,\ x_1=x_2=x,\ x_3=y,\ x_4=-y,\ \alpha_1=\alpha,\ \alpha_2=\beta, \alpha_3=\alpha_4=\gamma$ and use (2.1) and (2.3), we get

$$\sum_{n=0}^{\infty}\frac{(n!)^3 L_n^{(\alpha,\beta,\gamma,\gamma)}(x,x,y,-y)t^n}{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n(\gamma+1)_n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n!\,L_n^{(\alpha,\beta)}(x,x)\,t^n}{(\alpha+1)_n(\beta+1)_n}\frac{((-y^2t^2)/4)^k}{(\gamma+1)_k((\gamma+1)/2)_k\,((\gamma+2)/2)_k\,k!} \qquad (2.15)$$

Now, using the identity (2.6) and comparing the coefficient of t^n, we get

$$\frac{(n!)^3 L_n^{(\alpha,\beta,\gamma,\gamma)}(x,x,y,-y)}{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n(\gamma+1)_n}$$

$$=\sum_{k=0}^{[n/2]}\frac{(n-2k)!\,((-y^2)/4)^k}{(\alpha+1)_{n-2k}(\beta+1)_{n-2k}(\gamma+1)_k((\gamma+1)/2)_k\,((\gamma+2)/2)_k\,k!}L_{n-2k}^{(\alpha,\beta)}(x,x) \qquad (2.16)$$

In (2.16), if we use the result [7]

$$L_n^{(\alpha,\beta)}(x,x)=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\,{}_3F_3\left[\begin{matrix}-n,\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2); \\ \alpha+1,\ \beta+1,\alpha+\beta+1\quad ;\end{matrix}4x\right], \qquad (2.17)$$

we arrive after some simplification to the following result :

$$L_n^{(\alpha,\beta,\gamma,\gamma)}(x,x,y,-y)=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n(\gamma+1)_n}{(n!)^4}$$

$$\times X^{1:\,0;\,2}_{0:\,3;3}\left[\begin{matrix}-n: & ---------- & ;\frac{1}{2}(\alpha+\beta+2),\frac{1}{2}(\alpha+\beta+1); \\ -: & \gamma+1,\frac{1}{2}(\gamma+1),\frac{1}{2}(\gamma+2); & \alpha+1,\beta+1,\alpha+\beta+1\quad ;\end{matrix}\frac{-y^2}{4},4x\right]. \qquad (2.18)$$

On the same lines of derivation of the results (2.11), (2.13) and (2.18), we have the following formulas for $L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$:

$$L_n^{(\alpha_1,\alpha_1,\cdots,\alpha_r,\alpha_r\ ,\alpha_{r+1})}[x_1,-x_1,\cdots,x_r,-x_r,x_{r+1}]$$

$$=\frac{\prod_{j=1}^{r}((1+\alpha_j)_n(1+\alpha_j)_n)(1+\alpha_{r+1})_n}{(n!)^{2r+1}}$$

$$F^{1:0;\,\cdots;\,0;0}_{0:3;\,\cdots;\,3;1}\left[\begin{matrix}(-n:2,\cdots,2,1): & ---------------- & ;\cdots \\ - & :(1+\alpha_1:1),(\frac{1}{2}(1+\alpha_1):1),(\frac{1}{2}(2+\alpha_1):1); & \cdots\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots; & -------------------- ; & ---- & ; \\ \cdots;(1+\alpha_r:1),(\frac{1}{2}(1+\alpha_r):1),(\frac{1}{2}(2+\alpha_r):1); & (1+\alpha_{r+1}:1); & & \end{matrix}\frac{-x_1^2}{4},\cdots,\frac{-x_r^2}{4},x_{r+1}\right], \qquad (2.19)$$

$$L_n^{(\alpha_1,\alpha_1,\cdots,\alpha_r,\alpha_r)}[x_1,-x_1,\cdots,x_r,-x_r]=\frac{\prod_{j=1}^{r}\left[(1+\alpha_j)_n(1+\alpha_j)_n\right]}{(n!)^{2r}}$$

$$F^{2:\,0;\cdots;\,0}_{0\ :3;\cdots;\,3}\left[\begin{matrix}-\frac{1}{2}n,-\frac{1}{2}n+\frac{1}{2}\ : & ------- & ;\cdots \\ ---- & :1+\alpha_1,\frac{1}{2}(1+\alpha_1),\frac{1}{2}(2+\alpha_1) & ;\cdots\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots; & ---------- & ; \\ \cdots;1+\alpha_r,\frac{1}{2}(1+\alpha_r),\frac{1}{2}(2+\alpha_r); & & \end{matrix}-x_1^2,-x_2^2,\cdots,-x_r^2\right] \qquad (2.20)$$

and

$$L_n^{(\alpha_1,\beta_1,\cdots,\alpha_r,\beta_r,\gamma_1,\gamma_1,\cdots,\gamma_s,\gamma_s)}[x_1,x_1,\cdots,x_r,x_r,y_1,-y_1,\cdots,y_s,-y_s]$$

$$=\frac{\prod_{j=1}^{r}[(1+\alpha_j)_n(1+\beta_j)_n]\prod_{j=1}^{s}[(1+\gamma_j)_n(1+\gamma_j)_n]}{(n!)^{2(r+s)}}$$

$$F\begin{matrix}1:0;\cdots;0;2;\cdots;2\\0:3;\cdots;3;3;\cdots;3\end{matrix}\left[\begin{matrix}(-n:2,\cdots,2,1,\cdots,1) & : & ------------- & ;\cdots\\ ---------- & :(1+\gamma_1:1),\left(\tfrac{1}{2}(1+\gamma_1):1\right),\left(\tfrac{1}{2}(2+\gamma_1):1\right); & & \cdots\end{matrix}\right.$$

$$\begin{matrix}\cdots; & ------------- & ;\left(\tfrac{1}{2}(2+\alpha_1+\beta_1):1\right),\left(\tfrac{1}{2}(1+\alpha_1+\beta_1):1\right);\cdots\\ \cdots;(1+\gamma_s:1),\left(\tfrac{1}{2}(1+\gamma_s):1\right),\left(\tfrac{1}{2}(2+\gamma_s):1\right);(1+\alpha_1:1),(1+\beta_1:1),(1+\alpha_1+\beta_1:1);\cdots\end{matrix}$$

$$\left.\begin{matrix}\cdots;\left(\tfrac{1}{2}(2+\alpha_r+\beta_r):1\right),\ \left(\tfrac{1}{2}(1+\alpha_r+\beta_r):1\right);\\ \cdots;(1+\alpha_r:1),\ (1+\beta_r:1),\ (1+\alpha_r+\beta_r:1);\end{matrix}\ \frac{-y_1^2}{4},\cdots,\frac{-y_s^2}{4},4x_1,\cdots,4x_r\right] \tag{2.21}$$

respectively .

III. Integral Formulas Involving $L_n^{(\alpha_1,\cdots,\alpha_r)}[x_1,\cdots,x_r]$

In this section, we use the results obtained in the section II to obtain the following integral formulas involving Laguerre polynomials of several variables :

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_{m_1}^{(\alpha_1,\alpha_1)}(\gamma_1 x,-\gamma_1 x)\cdots L_{m_r}^{(\alpha_r,\alpha_r)}(\gamma_r x,-\gamma_r x)\,dx$$

$$=\frac{\Gamma(\lambda)\prod_{j=1}^{r}[(1+\alpha_j)_{m_j}(1+\alpha_j)_{m_j}]}{\sigma^\lambda\prod_{j=1}^{r}(m_j\,!)^2}F\begin{matrix}2:2;\cdots;2\\0:3;\cdots;3\end{matrix}\left[\begin{matrix}\tfrac{1}{2}\lambda,\tfrac{1}{2}\lambda+\tfrac{1}{2}: & -\tfrac{1}{2}m_1,-\tfrac{1}{2}m_1+\tfrac{1}{2} & ;\cdots\\ - & :1+\alpha_1,\tfrac{1}{2}(1+\alpha_1),\tfrac{1}{2}(2+\alpha_1) & ;\cdots\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots; & -\tfrac{1}{2}m_r,-\tfrac{1}{2}m_r+\tfrac{1}{2} & ;\\ \cdots;1+\alpha_r & ,\tfrac{1}{2}(1+\alpha_r),\tfrac{1}{2}(2+\alpha_r) & ;\end{matrix}\frac{-4\gamma_1^2}{\sigma^2},\cdots,\frac{-4\gamma_r^2}{\sigma^2}\right], \tag{3.1}$$

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_{m_1}^{(\alpha_1,\beta_1)}(\gamma_1 x,\gamma_1 x)\cdots L_{m_r}^{(\alpha_r,\beta_r)}(\gamma_r x,\gamma_r x)\,dx$$

$$=\frac{\Gamma(\lambda)\prod_{j=1}^{r}[(1+\alpha_j)_{m_j}(1+\beta_j)_{m_j}]}{\sigma^\lambda\prod_{j=1}^{r}(m_j!)^2}F\begin{matrix}1:3;\cdots;3\\0:3;\cdots;3\end{matrix}\left[\begin{matrix}\lambda:-m_1,\tfrac{1}{2}(1+\alpha_1+\beta_1),\tfrac{1}{2}(2+\alpha_1+\beta_1);\cdots\\ -: \quad 1+\alpha_1,1+\beta_1,1+\alpha_1+\beta_1 \quad ;\cdots\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots;-m_r,\tfrac{1}{2}(1+\alpha_r+\beta_r),\tfrac{1}{2}(2+\alpha_r+\beta_r);\\ \cdots; \quad 1+\alpha_r,1+\beta_r,1+\alpha_r+\beta_r \quad ;\end{matrix}\frac{4\gamma_1}{\sigma},\cdots,\frac{4\gamma_r}{\sigma}\right], \tag{3.2}$$

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_n^{(\alpha_1,\beta_1,\cdots,\alpha_r,\beta_r,\rho_1,\rho_1,\cdots,\rho_s,\rho_s)}[\gamma_1 x,\gamma_1 x,\cdots,\gamma_r x,\gamma_r x,\delta_1 x,-\delta_1 x,\cdots,\delta_s x,-\delta_s x]dx$$

$$=\frac{\Gamma(\lambda)\prod_{j=1}^{r}[(1+\alpha_j)_n(1+\beta_j)_n]\prod_{j=1}^{s}[(1+\rho_j)_n(1+\rho_j)_n]}{\sigma^\lambda(n!)^{2(r+s)}}$$

$$F^{2:0;\cdots;0;2;\cdots;2}_{0:3;\cdots;3;3;\cdots;3}\left[\begin{array}{l}(\lambda:2,\cdots,2,1,\cdots,1),(-n:2,\cdots,2,1,\cdots,1)\;:\\ \text{---------------}\;:\end{array}\right.$$

$$\begin{array}{l}\text{---------------}\;;\cdots;\;\text{-------------}\;;\\ (1+\rho_1:1),\left(\tfrac{1}{2}(1+\rho_1):1\right),\left(\tfrac{1}{2}(2+\rho_1):1\right);\cdots;(1+\rho_s:1),\left(\tfrac{1}{2}(1+\rho_s):1\right),\left(\tfrac{1}{2}(2+\rho_s):1\right);\\ \left(\tfrac{1}{2}(2+\alpha_1+\beta_1):1\right),\left(\tfrac{1}{2}(1+\alpha_1+\beta_1):1\right);\cdots;\left(\tfrac{1}{2}(2+\alpha_r+\beta_r):1\right),\left(\tfrac{1}{2}(1+\alpha_r+\beta_r):1\right);\\ (1+\alpha_1:1),(1+\beta_1:1),(1+\alpha_1+\beta_1:1);\cdots;(1+\alpha_r:1),(1+\beta_r:1),(1+\alpha_r+\beta_r:1)\;;\end{array}$$

$$\left.\frac{-\delta_1^{\,2}}{4\sigma^2},\cdots,\frac{-\delta_s^{\,2}}{4\sigma^2},\frac{4\gamma_1}{\sigma},\cdots,\frac{4\gamma_r}{\sigma}\right], \tag{3.3}$$

where $\mathrm{Re}(\lambda)>0;\mathrm{Re}(\sigma)>0$ and Γ is the will known Gamma function defined by [10]

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\ \ \mathrm{Re}(z)>0\ . \tag{3.4}$$

Proof of (3.1):

To obtain the main integral formula (3.1), we consider the left-hand side of (3.1),using (2.8), expressing ${}_2F_3$ in series forms and changing the order of integration and summation we get

$$L.H.S=\frac{\prod_{j=1}^{r}[(1+\alpha_j)_{m_j}(1+\alpha_j)_{m_j}]}{\prod_{j=1}^{r}(m_j!)^2}$$

$$\sum_{p_1,\cdots,p_r=0}^{\infty}\frac{\left(-\tfrac{1}{2}m_1\right)_{p_1}\left(-\tfrac{1}{2}m_1+\tfrac{1}{2}\right)_{p_1}(\gamma_1^2)^{p_1}}{(\alpha_1+1)_{p_1}\,\tfrac{1}{2}(\alpha_1+1)_{p_1}\,\tfrac{1}{2}(\alpha_1+2)_{p_1}\,p_1!}\times\cdots\times\frac{\left(-\tfrac{1}{2}m_r\right)_{p_r}\left(-\tfrac{1}{2}m_r+\tfrac{1}{2}\right)_{p_r}(\gamma_r^2)^{p_r}}{(\alpha_r+1)_{p_r}\,\tfrac{1}{2}(\alpha_r+1)_{p_r}\,\tfrac{1}{2}(\alpha_r+2)_{p_r}\,p_r!}$$

$$\times\int_0^\infty e^{-\sigma x}x^{\lambda+2p_1+\cdots+2p_r-1}\,dx \tag{3.5}$$

In (3.5) using the definition of Gamma function (3.4) and finally considering the definition (1.7),we obtain the right –hand side of (3.1). This completes the of (3.1).The integrals (3.2) and (3.3) can be proved in the same manner.

Following two integrals can be obtained readily from (3.3) as follows:

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}\,L_n^{(\alpha_1,\beta_1,\cdots,\alpha_r,\beta_r)}\,[\gamma_1 x\;,\;\gamma_1 x,\cdots,\gamma_r x,\gamma_r x]\,dx$$

$$=\frac{\Gamma(\lambda)\prod_{j=1}^{r}[(1+\alpha_j)_n(1+\beta_j)_n]}{\sigma^\lambda(n!)^r}F^{2:2;\cdots;2}_{0:3;\cdots;3}\left[\begin{array}{l}\lambda\;,-n\;\;:\tfrac{1}{2}(2+\alpha_1+\beta_1)\;,\tfrac{1}{2}(1+\alpha_1+\beta_1)\;;\cdots\\ \text{---}\;\;:\;1+\alpha_1\;,1+\beta_1,1+\alpha_1+\beta_1\;;\cdots\end{array}\right.$$

$$\left.\begin{array}{l}\cdots;\;\tfrac{1}{2}(2+\alpha_r+\beta_r),\tfrac{1}{2}(1+\alpha_r+\beta_r)\;;\\ \cdots;\;1+\alpha_r\;,\;1+\beta_r\;,1+\alpha_r+\beta_r\;;\end{array}\frac{4\gamma_1}{\sigma},\cdots,\frac{4\gamma_r}{\sigma}\right] \tag{3.6}$$

and

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}\,L_n^{(\alpha_1,\alpha_1,\cdots,\alpha_r,\alpha_r)}\,[\gamma_1 x,-\gamma_1 x\;,\cdots,\gamma_r x,-\gamma_r x]dx=\frac{\Gamma(\lambda)\;\prod_{j=1}^{r}(1+\alpha_j)_n\,(1+\alpha_j)_n}{\sigma^\lambda(n!)^{2r}}$$

$$F^{4:0;\cdots;0}_{0:3;\cdots;3}\left[\begin{array}{l}-\tfrac{1}{2}n,-\tfrac{1}{2}n+\tfrac{1}{2},\tfrac{1}{2}\lambda,\tfrac{1}{2}\lambda+\tfrac{1}{2}\;:\;\text{---------}\;;\cdots\\ \text{-----------}\;:1+\alpha_1,\tfrac{1}{2}(1+\alpha_1),\tfrac{1}{2}(2+\alpha_1)\;;\cdots\end{array}\right.$$

$$\left. \begin{matrix} \cdots; & ---------- & ; \\ \cdots;1+\alpha_r & ,\tfrac{1}{2}(1+\alpha_r),\tfrac{1}{2}(2+\alpha_r) & ; \end{matrix} \frac{-4\gamma_1^2}{\sigma^2},\cdots,\frac{-4\gamma_r^2}{\sigma^2} \right]. \tag{3.7}$$

Now, we mentioned the following special cases :
On setting $r = s = 1$ and $\gamma = \delta$ in (3.1) and using the result [1]

$$K_5(a,a,a,a\,;b,b,d,d\,;e,e,h,h\,;y,-y,z,-z)$$
$$= F\begin{matrix} 2:2;2 \\ 0:3;3 \end{matrix}\left[\begin{matrix} \tfrac{1}{2}a,\tfrac{1}{2}(a+1): & \tfrac{1}{2}b,\tfrac{1}{2}(b+1) & ; & \tfrac{1}{2}d,\tfrac{1}{2}(d+1) & ; \\ - & :e,\tfrac{1}{2}e,\tfrac{1}{2}(e+1) & ; & h,\tfrac{1}{2}h,\tfrac{1}{2}(h+1) & ; \end{matrix} -4y^2,-4z^2\right], \tag{3.8}$$

we get

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_{m_1}^{(\alpha_1,\alpha_1)}(\gamma_1 x,-\gamma_1 x)L_{m_2}^{(\alpha_2,\alpha_2)}(\gamma_2 x,-\gamma_2 x)\,dx = \frac{\Gamma(\lambda)[(1+\alpha_1)_{m_1}(1+\alpha_2)_{m_2}]^2}{\sigma^\lambda\,(m_1\,!\,m_2\,!)^2}$$

$$K_5\left[\lambda,\lambda,\lambda,\lambda;-m_1,-m_1,-m_2,-m_2;1+\alpha_1,1+\alpha_1,1+\alpha_2,1+\alpha_2;\frac{\gamma_1}{\sigma},-\frac{\gamma_1}{\sigma},\frac{\gamma_2}{\sigma},-\frac{\gamma_2}{\sigma}\right], \tag{3.9}$$

where K_5 is the Exton's quadruple hypergeometric series[3]

$$K_5\left(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$= \sum_{p,q,r,s=0}^{\infty} \frac{(a)_{p+q+r+s}\,(b_1)_{p+q}\,(b_2)_{r+s}\,x^p\,y^q\,z^r\,t^s}{(c_1)_p\,(c_2)_q\,(c_3)_r\,(c_4)_s\;p!q!r!s!} \tag{3.10}$$

On setting $r = 2$, integral (3.2) reduces to a known result [6]

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}L_{m_1}^{(\alpha_1,\beta_1)}(\gamma_1 x,\gamma_1 x)L_{m_2}^{(\alpha_2,\beta_2)}(\gamma_2 x,\gamma_2 x)\,dx$$
$$= \frac{\Gamma(\lambda)(1+\alpha_1)_{m_1}(1+\beta_1)_{m_1}(1+\alpha_2)_{m_2}(1+\beta_2)_{m_2}}{\sigma^\lambda\,(m_1!)^2(m_2!)^2}$$
$$F\begin{matrix} 1:3;3 \\ 0:3;3 \end{matrix}\left[\begin{matrix} \lambda:-m_1,\tfrac{1}{2}(1+\alpha_1+\beta_1),\tfrac{1}{2}(2+\alpha_1+\beta_1); \\ -:\quad 1+\alpha_1,1+\beta_1,1+\alpha_1+\beta_1 \quad ; \end{matrix}\right.$$
$$\left.\begin{matrix} -m_2,\tfrac{1}{2}(1+\alpha_2+\beta_2),\tfrac{1}{2}(2+\alpha_2+\beta_2); \\ 1+\alpha_2,1+\beta_2,1+\alpha_2+\beta_2 \quad ; \end{matrix} \frac{4\gamma_1}{\sigma},\frac{4\gamma_2}{\sigma}\right] \tag{3.11}$$

On setting $r = s = 1$ and $\alpha = \beta$, integral (3.3) reduces to

$$\int_0^\infty e^{-\sigma x}x^{\lambda-1}\,L_n^{(\alpha,\alpha,\rho,\rho)}\left[\gamma x,\gamma x,\delta x,-\delta x\right]dx = \frac{\Gamma(\lambda)[(1+\alpha)_n(1+\rho)_n]^2}{\sigma^\lambda\,(n!)^4}$$
$$X\begin{matrix} 1:0;1 \\ 0:3;2 \end{matrix}\left[\begin{matrix} \lambda,-n: & ----- & ; & \tfrac{1}{2}(1+2\alpha) & ; \\ -- & :1+\rho,\tfrac{1}{2}(1+\rho),\tfrac{1}{2}(2+\rho) & ; & 1+\alpha,1+2\alpha & ; \end{matrix} \frac{-\delta^2}{4\sigma^2},\frac{4\gamma}{\sigma}\right]. \tag{3.12}$$

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