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Projective Flat Finsler Space with Special (α,β)-Metrics

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Abstract: *In this article, we devoted to study about the n-dimensional Finsler $F^n = (M^n, L)$ with an (α,β)-metric $L(\alpha,\beta)$ to be projectively flat, where α is Riemannian metric and β is differential 1-form under some geometric conditions on the basis of Matsumoto results.*
Keywords: *Finsler space, (α,β)-metrics, Projective flatness.*

I. Introduction

The concept of an (α,β)-*metric* $L(\alpha,\beta)$ was introduced in 1972 by M. Matsumoto [1]. An (α,β)-metric is of the form $F = \alpha\phi(s); s = \frac{\beta}{\alpha}$ where $\alpha^2 = a_{ij}(x)y^iy^j$ is Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form with $\|\beta_x\| < b_0, x \in M$. The function $\phi(s)$ is a c^∞ positive function on an open interval $(-b_0, b_0)$ satisfying:
$\phi(s) - s\phi'(s) + (b^2 - s^2) > 0.$
In this case, the fundamental form of the metric tensor induced by F is positive definite.
An n-dimensional Finsler space $F^n = (M^n, L)$ equipped with the fundamental function $L(x,y)$ is called an (α,β)-metric if L is a positively homogeneous function of degree one two variables α and β.
A Finsler space $F^n = (M^n, L)$ is called a locally minkowskian space [2], if M^n is covered by co-ordinate neighborhood system (x^i) in each of which L is a function of (y^i) only. A Finsler space $F^n = (M^n, L)$ is called projective flat if F^n is projective to a locally minkowskian space. The condition for a Finsler space to be projectively flat was studied by L. Berwlad [3], in tensorial form and completed by M. Matsumoto [4]. Later on many authors worked on projective flatness of (α,β)-metric ([1], [5], [6], [7], [8], [9], [10], [11], [12]).

The purpose of the present article is devoted to studying the condition for a Finsler space with certain special (α,β)-metrics to be projective flat.

II. Preliminaries

A Finsler metric on a manifold M is a function $F: TM \to [0,\infty)$ which has the following properties:
(i) F is a c^∞ on TM_0,
$(ii) F(x,\lambda y) = \lambda F(x,y), \lambda > 0,$
(iii) For any tangent vector $y \in T_xM$, the vertical–Hessian $\frac{1}{2}F^2$ given by $g_{ij}(x,y) = \frac{1}{2}[F^2]y^iy^j$, is positive definite.
The canonical spray of F denoted by $G = y^i\left\{\frac{\partial}{\partial x^i}\right\} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ and it is defined as
$G^i(x,y) = \frac{1}{4}g^{il}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\}y^jy^k,$
where the matrix (g^{ij}) means the inverse of the matrix (g_{ij}).

Let us consider an n-dimesional Finsler space $F^n = (M^n, L)$ with an (α,β)-metric $L(\alpha,\beta)$. The space $R^n = (M^n, \alpha)$ is called the associated Riemannian space. Let $\gamma^i_{jk}(x)$ be the christoffel symbols constructed from α and we denote the covariant differentiation with respect to $\gamma^i_{jk}(x)$ by $(|)$. From the differential 1-form $\beta(x,y) = b_i(x)y^i$, we define

$$2r_{ij} = b_{i|j} + b_{j|i}, \quad 2s_{ij} = b_{i|j} - b_{j|i},$$
$$s^i_j = a^{ih}s_{hj},\ s_j = b_is^i_j,\ b^i = a^{ih}b_h,\ b^2 = b^ib_i.$$

According to [1], a Finsler space $F^n = (M^n, L)$ with an (α,β)-metric $L(\alpha,\beta)$ is projectively flat if and only if for any point of space M there exist local coordinate neighborhoods containing the point such that $\gamma^i_{jk}(x)$ satisfies:

$$\frac{\left(\frac{\gamma^i_{00} - \gamma_{000}y^i}{\alpha^2}\right)}{2} + \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i_0 + \left(\frac{L_{\alpha\alpha}}{L_\alpha}\right)\left(C + \frac{\alpha r_{00}}{2\beta}\right)\left(\frac{\alpha^2 b^i}{\beta} - y^i\right) = 0, \quad (2.1)$$

where a subscript 0 means a contraction by y^i, $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, and C is given by

$$C + \left(\frac{\alpha^2 L_\beta}{\beta L_\alpha}\right) s_0 + \left(\frac{\alpha L_{\alpha\alpha}}{\beta^2 L_\alpha}\right)(\alpha^2 b^2 - \beta^2)\left(C + \frac{\alpha r_{00}}{2\beta}\right) = 0. \tag{2.2}$$

By the homogeneity of L we known $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$, so that (2.2) can be written as

$$\left\{1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2)\right\}\left(C + \frac{\alpha r_{00}}{2\beta}\right) = \left(\frac{\alpha}{2\beta}\right)\left\{r_{00} - \left(\frac{2\alpha L_\beta}{L_\alpha}\right)s_0\right\}. \tag{2.3}$$

If $1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $\left(C + \frac{\alpha r_{00}}{2\beta}\right)$ in (2.1) and it is written in the form,

$$\left\{1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2)\right\}\left\{\frac{\left(\frac{\gamma^i_{00} - \gamma_{000} y^i}{\alpha^2}\right)}{2} + \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i_0\right\} + \left(\frac{L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\alpha}{2\beta}\right)\left\{r_{00} - \left(\frac{2\alpha L_\beta}{L_\alpha}\right)s_0\right\}\left(\frac{\alpha^2 b^i}{\beta} - y^i\right) = 0. \tag{2.4}$$

Thus we state that

Theorem 2.1: [12] If $1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2) \neq 0$, then a Finsler space F^n with an (α,β)-metric is projectively flat if and only if (2.4) is satisfied.

According to [13], It is known that if α^2 contains β as a factor, then the dimension is equal to two and $b^2 = 0$. So throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, that is, $\alpha^2 \not\equiv 0(mod\ \beta)$.

III. Projective Flat Finsler Space with (α,β)-metric $L^2 = \alpha^2 + \epsilon\alpha\beta + k\beta^2$

Let F^n be a Finsler space with an (α,β)-metric is given by

$$L^2 = \alpha^2 + \epsilon\alpha\beta + k\beta^2;\ \epsilon, k \neq 0. \tag{3.1}$$

The partial derivatives with respect to α and β of (3.1) are given by

$$L_\alpha = \frac{2\alpha + \epsilon\beta}{2\sqrt{\alpha^2 + \epsilon\alpha\beta + k\beta^2}},\quad L_{\alpha\alpha} = \frac{(4k - \epsilon^2)\beta^2}{4\sqrt{\alpha^2 + \epsilon\alpha\beta + k\beta^2}(\alpha^2 + \epsilon\alpha\beta + k\beta^2)},$$

$$L_\beta = \frac{\epsilon\alpha + 2k\beta}{2\sqrt{\alpha^2 + \epsilon\alpha\beta + k\beta^2}},\quad L_{\beta\beta} = \frac{(4k - \epsilon^2)\alpha^2}{4\sqrt{\alpha^2 + \epsilon\alpha\beta + k\beta^2}(\alpha^2 + \epsilon\alpha\beta + k\beta^2)}. \tag{3.2}$$

If $1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2) = 0$, then we have $[\{4 + (4k - \epsilon^2)b^2\}\alpha^3 + 6\epsilon\alpha^2\beta + 3\epsilon^2\alpha\beta^2 + 2\epsilon k\beta^3] = 0$ which leads to contradiction. Thus $1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2) \neq 0$ and hence theorem (2.1) can be applied.

Substituting (3.2) into (2.4), we get

$\{(2\alpha^2 + 2\epsilon\alpha\beta + 2k\beta^2)(2\alpha + \epsilon\beta) + (4k - \epsilon^2)(\alpha^3 b^2 - \alpha\beta^2)\}\{(\alpha^2\gamma^i_{00} - \gamma_{000} y^i)(2\alpha + \epsilon\beta) + 2\alpha^3(\epsilon\alpha +$
2kβs0i+4k−ϵ2α3α2bi−βyir002α+ϵβ−2αϵα+2kβs0=0. (3.3)

The terms of (3.3) can be written as

$$(p_7\alpha^6 + p_5\alpha^4 + p_3\alpha^2 + p_1)\alpha + (p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0) = 0. \tag{3.4}$$

Where

$p_7 = \{8\epsilon + (8\epsilon k - 2\epsilon^3)b^2\}s^i_0 - 2\epsilon(4k - \epsilon^2)s_0 b^i$,
$p_6 = 2\{4 + (4k - \epsilon^2)b^2\}\gamma^i_{00} + \{12\epsilon^2 + 16k + (16k^2 - 4\epsilon^2 k)b^2\}s^i_0\beta + 2(4k - \epsilon^2)b^i r_{00} - 4k(4k - \epsilon^2)\beta s_0 b^i$,
$p_5 = \{16\epsilon + \epsilon(4k - \epsilon^2)b^2\}\beta\gamma^i_{00} + (24\epsilon k + 6\epsilon^3)\beta^2 s^i_0 + (4k\epsilon - \epsilon^3)\beta r_{00} b^i + (8k\epsilon - 2\epsilon^3)\beta s_0 y^i$,
$p_4 = 12\epsilon^2\beta^2\gamma^i_{00} - \{8 + (8k - 2\epsilon^2)b^2\}\gamma_{000} y^i + 16k\epsilon^2\beta^3 s^i_0 - (8k - 2\epsilon^2)\beta y^i r_{00} + (16k^2 - 4\epsilon^2 k)\beta^2 s_0 y^i$,
$p_3 = (3\epsilon^3 + 4\epsilon k)\beta^3\gamma^i_{00} - \{16\epsilon + (4k\epsilon - \epsilon^3)b^2\}\gamma_{000} y^i\beta + 8k^2\epsilon\beta^4 s^i_0 - (4k\epsilon - \epsilon^3)\beta^2 y^i r_{00}$,
$p_2 = -12\epsilon^2\beta^2\gamma_{000} y^i + 2\epsilon^2 k\beta^4\gamma^i_{00}$,
$p_1 = -4\epsilon k\beta^3\gamma_{000} y^i$,
$p_0 = -2\epsilon^2 k\beta^4\gamma_{000} y^i$.

Since $(p_7\alpha^6 + p_5\alpha^4 + p_3\alpha^2 + p_1)$ and $(p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0)$ are rational and α irrational in y^i, we have

$$(p_7\alpha^6 + p_5\alpha^4 + p_3\alpha^2 + p_1) = 0, \tag{3.5}$$

$$(p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0) = 0. \tag{3.6}$$

The term which does not contain β in (3.5) is $p_7\alpha^6$. Therefore there exist a homogeneous polynomial v_6 of degree six in y^i such that

$$[\{8\epsilon + (8\epsilon k - 2\epsilon^3)b^2\}s^i_0 - 2\epsilon(4k - \epsilon^2)s_0 b^i]\alpha^6 = \beta v^i_6.$$

Since $\alpha^2 \not\equiv 0(mod\beta)$, we have a function $u^i = u^i(x)$ satisfying

$$\{8\epsilon + (8\epsilon k - 2\epsilon^3)b^2\}s^i_0 - 2\epsilon(4k - \epsilon^2)s_0 b^i = \beta u^i. \tag{3.7}$$

Contracting above by b_i, we have

$$8\epsilon s_0 = u^i\beta b_i, \tag{3.8}$$

implies $8\epsilon s_j = u^i b_j b_i = 0$. Again transvecting by b^j, we have $u^i b_i = 0$. Plugging $u^i b_i = 0$ in (3.8), we have $s_0 = 0$. Thus from (3.7), we have

$$\{8\epsilon + (8\epsilon k - 2\epsilon^3)b^2\}s_{ij} = u_i b_j, \tag{3.9}$$

which implies $u_i b_j + u_j + b_i = 0$. Contracting this by b^j, we have $u_i b^2 = 0$ by virtue of $u_j b^j = 0$. Therefore we get $u_i = 0$. Hence, from (3.9), we have $s_{ij} = 0$, provided $8\epsilon + (8\epsilon k - 2\epsilon^3)b^2 \neq 0$.

Again, From (3.6), we observe that the terms $-2\epsilon^2 k\beta^4\gamma_{000}y^i$ must have a factor α^2. Therefore there exist a 1-form $v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \tag{3.10}$$

Plugging $s_0 = 0, s_0^i = 0$ and (3.10) in to (3.3) which yields,

$$\{(2\alpha^2 + 2\epsilon\alpha\beta + 2k\beta^2)(2\alpha + \epsilon\beta) + (4k - \epsilon^2)(\alpha^3 b^2 - \alpha\beta^2)\}(\gamma_{00}^i - v_0 y^i)$$
$$+(4k - \epsilon^2)\alpha(\alpha^2 b^i - \beta y^i)r_{00} = 0. \tag{3.11}$$

Terms of (3.11) can be written as

$$\left[\{(4 + 4k - \epsilon^2 b^2)\alpha^2 + 3\epsilon^2\beta^2\}(\gamma_{00}^i - v_0 y^i) + (4k - \epsilon^2)(\alpha^2 b^i - \beta y^i)r_{00}\right]\alpha$$
$$+[6\epsilon\alpha^2\beta + 2\epsilon k\beta^3](\gamma_{00}^i - v_0 y^i) = 0. \tag{3.12}$$

Again (3.12) written in the form $P\alpha + Q = 0$, where

$$P = \{(4 + 4k - \epsilon^2 b^2)\alpha^2 + 3\epsilon^2\beta^2\}(\gamma_{00}^i - v_0 y^i) + (4k - \epsilon^2)(\alpha^2 b^i - \beta y^i)r_{00},$$
$$Q = [6\epsilon\alpha^2\beta + 2\epsilon k\beta^3](\gamma_{00}^i - v_0 y^i).$$

Since P and Q are rational and α irrational in (y^i), we have $P = 0$ *and* $Q = 0$.

By the term $Q = 0$, we have

$$(\gamma_{00}^i - v_0 y^i) = 0. \tag{3.13}$$

which yields

$$2\gamma_{jk}^i = v_j\delta_k^i + v_k\delta_j^i, \tag{3.14}$$

which shows that associated Riemannian space (M,α) is projectively flat.

Again from $P = 0$ and (3.13) we have

$$(4k - \epsilon^2)(\alpha^2 b^i - \beta y^i)r_{00} = 0. \tag{3.15}$$

Transvecting (3.15) by b^i, we have $(4k - \epsilon^2)(\alpha^2 b^2 - \beta^2)r_{00} = 0$ implies $r_{00} = 0$ provided that $\epsilon, k \neq 0$. i.e., $r_{ij} = 0$. By summarizing up the above results, i.e., by using $s_{ij} = r_{ij} = 0$ we conclude that $b_{i|j} = 0$.

Conversely, if $b_{i|j} = 0$, then we have $r_{00} = s_0^i = s_0 = 0$. So (3.3) is a consequence of (3.13).

Thus we state that,

Theorem-3.1: *A Finsler space F^n with an (α,β)-metric $L(\alpha,\beta)$ given by (3.1) provided that $\epsilon, k \neq 0$ is projectively flat, if and only if we have $b_{i|j} = 0$ and the associated Riemannian space (M^n,α) is projectively flat.*

IV. Projective Flat Finsler Space with (α,β)-metric $L = \alpha + \beta + \frac{\alpha^2}{\alpha-\beta}$

Let F^n be a Finsler space with an (α,β)-metric is given by

$$L = \alpha + \beta + \frac{\alpha^2}{\alpha-\beta}. \tag{4.1}$$

The partial derivatives with respect to α and β of (4.1) are given by

$$L_\alpha = \frac{2\alpha^2+\beta^2-4\alpha\beta}{(\alpha-\beta)^2}, \quad L_\beta = \frac{2\alpha^2+\beta^2-2\alpha\beta}{(\alpha-\beta)^2},$$
$$L_{\alpha\alpha} = \frac{2\beta^2}{(\alpha-\beta)^3}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}. \tag{4.2}$$

If $1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2) = 0$, then we have $\{\alpha^3(2 + 2b^2) - 6\alpha^2\beta + 3\alpha\beta^2 - \beta^3\} = 0$ which leads to contradiction. Thus $1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2 b^2 - \beta^2) \neq 0$ and hence theorem (2.1) can be applied.

Substituting (4.2) into (2.4), we get

$$\{\alpha^3(2 + 2b^2) - 6\alpha^2\beta + 3\alpha\beta^2 - \beta^3\}\{(\alpha^2\gamma_{00}^i - \gamma_{000}y^i)(2\alpha^2 + \beta^2 - 4\alpha\beta) + 2\alpha^3(2\alpha^2 + \beta^2 - 2\alpha\beta)s_0^i\}$$
$$+2\alpha^3\{(2\alpha^2 + \beta^2 - 4\alpha\beta)r_{00} - 2\alpha(2\alpha^2 + \beta^2 - 2\alpha\beta)s_0\}(\alpha^2 b^i - \beta y^i) = 0. \tag{4.3}$$

The terms of (4.3) can be written as,

$$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 + \alpha(p_5\alpha^4 + p_3\alpha^2 + p_1) = 0, \tag{4.4}$$

Where

$$p_8 = 4\{(2 + 2b^2)s_0^i - 2b^i s_0\},$$
$$p_7 = (4 + 4b^2)\gamma_{00}^i - (32 + 8b^2)s_0^i\beta + 4b^i r_{00} + 8\beta s_0^i b^i,$$
$$p_6 = -(20 + 8b^2)\beta\gamma_{00}^i + (40 + 4b^2)\beta^2 s_0^i - 8\beta r_{00}b^i + 8\beta s_0 y^i - 4\beta^2 s_0 b^i,$$

$p_5 = \beta^2(32 + 2b^2)\gamma_{00}^i - (4 + 4b^2)\gamma_{000}y^i - 28\beta^3 s_0^i + 2\beta^2 b^i r_{00} - 4\beta y^i r_{00} - 8\beta^2 s_0 y^i,$
$p_4 = \beta(20 + 8b^2)\gamma_{000}y^i - 20\beta^3\gamma_{00}^i + 10\beta^4 s_0^i + 8\beta^2 r_{00}y^i + 4\beta^7 s_0 y^i,$
$p_3 = -(32 + 2b^2)\beta^2\gamma_{000}y^i + 7\beta^4\gamma_{00}^i - 2\beta^5 s_0^i - 2\beta^3 r_{00}y^i,$
$p_2 = 20\beta^3\gamma_{000}y^i - \beta^5\gamma_{00}^i,$
$p_1 = -7\beta^4\gamma_{000}y^i,$
$p_0 = \beta^5\gamma_{000}y^i.$

Since $(p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0)$ and $(p_5\alpha^4 + p_3\alpha^2 + p_1)$ are rational and α is irrational in y^i, we have,

$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 = 0,$ (4.5)

$p_5\alpha^4 + p_3\alpha^2 + p_1 = 0.$ (4.6)

The term which does not contain β in (4.5) is $p_8\alpha^8$. Therefore there exists a homogeneous polynomial v_8 of degree eight in y^i such that

$4\{(2 + 2b^2)s_0^i - 2b^i s_0\}\alpha^8 = \beta v_8^i.$

Since $\alpha^2 \not\equiv 0(mod\beta)$, we have a function $u^i = u^i(x)$ satisfying

$4\{(2 + 2b^2)s_0^i - 2b^i s_0\} = \beta u^i.$ (4.7)

Contracting above by b_i, we have

$8s_0 = u^i\beta b_i,$ (4.8)

implies $8s_j = u^i b_j b_i = 0$. Again transvecting (4.8) by b^j, we have $u^i b_i = 0$. Plugging $u^i b_i = 0$ in (4.8), we have $s_0 = 0$. Thus from (4.7), we have

$4(2 + 2b^2)s_{ij} = u_i b_j,$ (4.9)

Which implies $u_i b_j + u_j b_i = 0$. Contracting this by b^j, we have $u_i b^2 = 0$ by virtue of $u_j b^j = 0$. Therefore we get $u_i = 0$. Hence, from (4.9), we have $s_{ij} = 0$, provided $(2 + 2b^2) \neq 0$.

Again, From (4.6), we observe that the terms $-7\beta^4\gamma_{000}y^i$ must have a factor α^2. Therefore there exist a 1-form $v_0 = v_i(x)y^i$ such that

$\gamma_{000} = v_0\alpha^2.$ (4.10)

Plugging $s_0 = 0, s_0^i = 0$ and (4.10) in to (4.3) which yields,

$\{\alpha^2(2 + 2b^2) - 6\alpha^2\beta + 3\alpha\beta^2 - \beta^3\}(\gamma_{00}^i - v_0 y^i) + 2\alpha(\alpha^2 b^i - \beta y^i)r_{00} = 0.$ (4.11)

Terms of (4.11) can be written as

$[\{(2 + 2b^2)\alpha^2 + 3\beta^2\}(\gamma_{00}^i - v_0 y^i) + 2(\alpha^2 b^i - \beta y^i)r_{00}]\alpha - \beta[6\alpha^2 + \beta](\gamma_{00}^i - v_0 y^i) = 0.$ (4.12)

The terms in (4.12) are rational and irrational in y^i, which yields

$$\{(2 + 2b^2)\alpha^2 + 3\beta^2\}(\gamma_{00}^i - v_0 y^i) + 2(\alpha^2 b^i - \beta y^i)r_{00} = 0, \quad (4.13)$$

And $[6\alpha^2 + \beta](\gamma_{00}^i - v_0 y^i) = 0.$ (4.14)

From (4.14), it follows that

$(\gamma_{00}^i - v_0 y^i) = 0.$ (4.15)

which yields

$2\gamma_{jk}^i = v_j\delta_k^i + v_k\delta_j^i,$ (4.16)

which shows that associated Riemannian space (M, α) is projectively flat.

Again from (4.13) and (4.15), we have

$r_{00}(\alpha^2 b^i - \beta y^i) = 0.$ (4.17)

Implies $r_{ij} = 0$. By studying the above results i.e., using $s_{ij} = r_{ij} = 0$, we conclude that $b_{i|j} = 0$.

Conversely, if $b_{i|j} = 0$, then we have $r_{00} = s_0^i = s_0 = 0$. So (4.3) is a consequence of (4.10).

Thus we state that,

Theorem-4.1: *A Finsler space F^n with an (α, β)-metric $L(\alpha, \beta)$ given by (4.1) is projectively flat, if and only if we have $b_{i|j} = 0$ and the associated Riemannian space (M^n, α) is projectively flat.*

V. Projective Flat Finsler Space with (α, β)-metric $L = \frac{\beta^{m+1}}{\alpha^m}$

Let F^n be a Finsler space with an (α, β)-metric is given by

$L = \frac{\beta^{m+1}}{\alpha^m}.$ (5.1)

The partial derivatives with respect to α and β of (5.1) are given by

$$L_\alpha = -m\frac{\beta^{m+1}}{\alpha^{m+1}}, \quad L_{\alpha\alpha} = m(m+1)\frac{\beta^{m+1}}{\alpha^{m+2}},$$
$$L_\beta = (m+1)\frac{\beta^m}{\alpha^m}, \quad L_{\beta\beta} = m(m+1)\frac{\beta^{m-1}}{\alpha^m}. \quad (5.2)$$

If $1+\left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2b^2-\beta^2)=0$, then we have $\{\beta^2(m+2)-(m+1)\alpha^2b^2\}=0$ which leads to contradiction. Thus $1+\left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)(\alpha^2b^2-\beta^2)\neq 0$ and hence theorem (2.1) can be applied.
Substituting (5.2) into (2.4), we get

$$\{(1+m\lambda)\beta^2-m\lambda\alpha^2b^2\}\{(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)\beta-2\lambda\alpha^4s^i_0\}-m\lambda\alpha^2\{\beta r_{00}+2\lambda\alpha^2s_0\}(\alpha^2b^i-\beta y^i)=0. \quad (5.3)$$

where $\lambda=\frac{m+1}{m}$.
Only the terms $-\beta^3(1+m\lambda)\gamma_{000}y^i$ of (5.3) seemingly does not contain α^2 as a factor and hence we must have $hp(5)v^i_5$ satisfying $-\beta^3(1+m\lambda)\gamma_{000}y^i=\alpha^2v^i_5$.
For sake of brevity, we suppose $\alpha^2\not\equiv 0(mod\beta)$, then we have

$$\gamma_{000}=v_0\alpha^2. \quad (5.4)$$

Where v_0 is hp(1).
Plugging (5.4) in to (5.3), we have

$$\{(1+m\lambda)\beta^2-m\lambda\alpha^2b^2\}\{(\gamma^i_{00}-v_0y^i)\beta-2\lambda\alpha^2s^i_0\}-m\lambda\{\beta r_{00}+2\lambda\alpha^2s_0\}(\alpha^2b^i-\beta y^i)=0. \quad (5.5)$$

The terms of (5.5) which seemingly does not contain α^2 are $(1+m\lambda)\beta^3(\gamma i_{00}-v_0y^i)+m\lambda\beta^2r_{00}y^i$. Consequently we must have $hp(1)u^i_0$ such that the above is equal to $\alpha^2\beta^2u^i_0$.
Thus we come by

$$(1+m\lambda)\beta(\gamma^i_{00}-v_0y^i)+m\lambda r_{00}y^i=\alpha^2u^i_0. \quad (5.6)$$

Contracting (5.6) by $a_{ir}y^r$, leads to

$$m\lambda r_{00}=u^i_0y_i. \quad (5.7)$$

Substituting (5.7) in (5.6), we get

$$\gamma^i_{00}=v_0y^i, \quad (5.8)$$

which yields

$$2\gamma^i_{jk}=v_j\delta^i_k+v_k\delta^i_j, \quad (5.9)$$

Consequently (5.9) shows that associated Riemannian space is projectively flat.
Again substituting (5.8) in (5.5), we have

$$-2\lambda\alpha^2\{(1+m\lambda)\beta^2-m\lambda\alpha^2b^2\}s^i_0-m\lambda\{\beta r_{00}+2\lambda\alpha^2s_0\}(\alpha^2b^i-\beta y^i)=0. \quad (5.10)$$

Contracting (5.10) by b_i, we have,

$$(-2\beta s_0-mb^2r_{00})\alpha^2+m\beta^2r_{00}=0. \quad (5.11)$$

Then there exists a function $k(x)$ such that

$$-2\beta s_0-mb^2r_{00}=k\beta^2,\ and\quad mr_{00}=k\alpha^2. \quad (5.12)$$

By eliminating r_{00} from the above, we have

$$2\beta s_0=k(\beta^2-\alpha^2b^2). \quad (5.13)$$

Implies

$$(s_ib_j+s_jb_i)=k(b_ib_j-b^2a_{ij}). \quad (5.14)$$

Contracting the above by a^{ij}, we have $k=0$.
From (5.13), we have $s_0=0$ and hence from (5.12), we obtain $r_{00}=0$.
Again from $s_i=0$ and $r_{00}=0$, (5.10) implies $s^i_0=0$ implies $s_{ij}=0$.
Since $r_{ij}=s_{ij}=s^i_0=0$, we have $b_{i|j}=0$.
Conversely, if $b_{i|j}=0$, then we have $r_{00}=s^i_0=s_0=0$. So (5.3) is a consequence of (5.8).
Thus we state that,

Theorem-5.1: *A Finsler space F^n with an (α,β)-metric $L(\alpha,\beta)$ given by (5.1) is projectively flat, if and only if we have $b_{i|j}=0$ and the associated Riemannian space (M^n,α) is projectively flat.*

VI. Conclusion

A Finsler metric being projectively equivalent on a manifold means their geodesics are same up to a parametrization

$$G^i=\bar{G}^i+Py^i,$$

where $P=P(x,y)$ is a positively y-homogeneous of degree one. If a quantity does not change between two projectively equivalent Finsler metrics, then it is called as a projectively invariant.

We have a two essential projective invariants, namely Weyl tensor W and the other is the Douglas tensor D. A Finsler space where both of these tensors vanish is characterized as a projecitvely flat Finsler space which can be projectively mapped to a locally minkowskian space. A Locally minkowskian space with (α,β)-metric is flat parallel if α is locally flat and β is parallel with respect to α.

A Finsler space is called projectively flat, or with rectilinear gedesic, if the space is covered by cordinate neighborhoods in which the geodesic can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called rectilinear.

Still now it is an open problem to classify the projectively flat (α, β)- metrics in dimension $n = 2$. In this article we are discussing about the condition for Finsler space F^n of dimension $n > 2$ of the above mentioned metrics are projectively flat if and only if $b_{i|j} = 0$ and F^n is covered by coordinate neighborhoods on which the Christoffel symbol of the associated Riemannian space with the metric α are written as $\gamma_{jk}^i = v_k \delta_j^i + v_j \delta_k^i$.

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