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Solution of a Variational inequality Problem for Accretive Operators in Banach Spaces

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Abstract: *This paper introduces a two-step iterative process for finding a solution of a variational inequality problem for accretive operators in Banach spaces. The result obtained in this paper is motivated by the result given by Koji Aoyama et al [3]. Further, we consider the problem of finding a fixed point of a strictly pseudocontractive mapping in a Banach space.*

Keywords*: Accretive operators, sunny non-expansive retractions, Banach spaces, variational inequality problem.*

I. Introduction

Let E be any smooth Banach space with $\|.\|$. Let E^* denote the dual of E and < x, f > denote the value of $f \in E^*$ at $x \in E$. Let C be a nonempty closed convex subset of E and let A be an accretive operator of C into E. The generalized variational inequality problem in Banach space is to find an element $u \in C$ such that
$\langle Au, J(v - u)\rangle \geq 0\ \forall\ v \in C$, where J is the duality mapping of E into E^*.

Definiton 1.1 A Banach space E is called uniformly convex iff for any ε, $0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply there exists a $\delta > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

Definition 1.2 Let E be any smooth Banach space. Then a function $\rho_E : R^+ \to R^+$ is said to be modulus of smoothness of E if

$$\rho_E(t) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1; \|x\| = 1, \|y\| = t\right\}.$$

Definition 1.3 A Banach space E is said to be uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0$$

Remark 1.4 Let q > 1. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant c > 0 such that $\rho_E(t) = ct^q$ for all t > 0. For more details, see [4, 11]. It is obvious that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Definition 1.5 Let J be any mapping from E into E^* satisfying
$J(x) = \{f \varepsilon E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$. Then J is called the normalized duality mapping of E.

Definition 1.6 Let C be a non-empty subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive [10] if
$\|Tx - Ty\| = \|x - y\| \quad \forall\ x, y \varepsilon C$.
T is called η-strictly pseudo-contractive if there exists a constant $\eta \varepsilon (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \eta\|(I - T)x - (I - T)y\|^2 \quad (1.1)$$

for every $x, y \varepsilon C$ and for some $j(x - y) \varepsilon J(x - y)$.
It is obvious that (1.1) is equivalent to

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \eta\|(I - T)x - (I - T)y\|^2 \quad (1.2)$$

Definition 1.7 A Banach space E is said to be smooth if the limit

$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \varepsilon U$, where $U = \{x \varepsilon E : \|x\| = 1\}$.

Remark 1.8 It is known that $J_q(x) = \|x\|^{q-2} J(x)$ for all x ε E. If E is a Hilbert space, then J = I. The normalized duality mapping J has the following properties:

1. If E is smooth, then J is single valued.
2. If E is strictly convex, then J is one-one and $< x - y, x^* - y^* > > 0$ for all (x, x^*), (y, y^*) ε J with x ≠ y.
3. If E is reflexive, then J is surjective.
4. If E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of E.
5. It is also known that $q < y - x, j_x > \leq \|y\|^q - \|x\|^q$ for all x, y ε E and j_x ε $J_q(x)$.

In 2006, Aoyama et al [3] obtained a weak convergence theorem.

Theorem 1.9 [3] Let E be a uniformly convex and 2-uniformly smooth Banach space with best smooth constant K and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be α-inverse strongly accretive operator of C into E. Let S(C, A) ≠ φ and the sequence $\{x_n\}$ be generated by

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n)$, $x_1 \in C$, n = 1, 2, 3,………….,

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in [0, 1] and $\lambda_n \in [a, \alpha/K^2]$ for some a > 0 and let $\alpha_n \in [b, c]$, where 0 < b < c < 1, then $\{x_n\}$converges weakly to some element z of S(C, A).

After that for finding a common element of F(S) ∩ VI(C, A), Nadezhkina and Takahashi [5] gave another result. They obtained the following weak convergence theorem.

Theorem 1.2 [5] Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that F (S) ∩ VI(C,A) ≠ ϕ. Let $\{x_n\}$, $\{y_n\}$ be sequences generated by $x_0 = x \in C$,

$y_n = P_C(x_n - \lambda_n A x_n)$,

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n)$, $\forall n \geq 0$, (1.3)

where $\{\lambda_n\} \subset [a, b]$ for some a, b ϵ (0,1/k) and $\{\alpha_n\} \subset [c, d]$ for some c, d ϵ (0,1). Then the sequences $\{x_n\}$, $\{y_n\}$ generated generated by (1.3) converge weakly to some z ϵ F(S) ∩ VI(C, A).

Motivated by above results, we provide the following iterative process for an accretive operator A in a Banach space E,

$x_1 = x$ ε C,

$y_n = Q_C(x_n - \lambda_n A x_n)$,

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(y_n - \lambda_n A y_n)$, for n = 1, 2…………..,

where Q_C is sunny nonexpansive retraction from E onto C. Using this iterative process, we shall obtain a weak convergence theorem.

II. Preliminaries

Let D be a subset of C and Q be a mapping from C to D. Then Q is said to be sunny if Q(Qx + t(x - Qx)) = Qx, whenever Qx + t(x - Qx) ε C for x ε C and t ≥ 0. A mapping Q : C → C is called retraction if $Q^2 = Q$. If Q is any retraction, then Qz = z for every z ε R(Q), where R(Q) is the range set of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Now we collect some results.

Lemma 2.1 [7] Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with F(T) ≠ φ. Then the set F(T) is a sunny nonexpansive retract of C.

Lemma 2.2 [6, 8] Let C be a nonempty closed convex subset of a smooth Banach space E and let Q_C be a retraction of E onto C.Then the following are equivalent

(i). Q_C is both sunny and nonexpansive.

(ii). $< x - Q_C x, J(y - Q_C x) > \leq 0$ for all x ε E, y ε C.

Also it is well known that if E is a Hilbert space, then sunny nonexpansive retraction is coincident with metric projection.

Also Q_C satisfies

$x_0 = Q_C x$ iff $< x - x_0, J(y - x_0) > \leq 0$ for all y ε C.

Let E be a Banach space and let C be a nonempty closed convex subset of E. An operator A of C into E is said to accretive if there exists j(x - y) ε J(x - y) such that

$< Ax - Ay, j(x - y) > \geq 0$ for all x, y ε C.

Lemma 2.3 [3] Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,
$S(C, A) = F(Q_C(I - \lambda A))$, where
$S(C, A) = \{ u \varepsilon C : < Au, J(v - u) > \geq 0$, for all $v \varepsilon C\}$.
An operator $A : C \to E$ is said to be α-inverse strongly accretive if
$< Ax - Ay, J(x - y) > \geq \alpha \|Ax - Ay\|^2$ for all x, y ε C.
It is obvious from above equation that

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\|.$$

Lemma 2.4 [3] Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let $\alpha > 0$ and let $A : C \to E$ be an α-inverse strongly accretive operator. If $0 < \lambda \leq \frac{\alpha}{K^2}$, then $I - \lambda A$ is a nonexpansive mapping of C into E, where K is the 2-uniformly smoothness constant of E.

Lemma 2.5 [9] Let C be a nonempty closed convex subset of a uniformly convex Banach space with a frechet differentiable norm. Let $\{T_1, T_2, \ldots\}$ be a sequence of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \varphi$. Let x ε C and $S_n = T_n T_{n-1} \ldots T_1$ for all $n \geq 1$. Then the set $\bigcap_{n=1}^{\infty} c\overline{o}\{S_m x : m \geq n\} \bigcap \bigcap_{n=1}^{\infty} F(T_n)$ consists of atmost one point, where $c\overline{o}$ D is the closure of the convex hull of D.

Lemma 2.6 [2] Let q be a given real number with $1 < q \leq 2$ and let E be a q-uniformly smooth Banach space. Then,
$\|x + y\|^q \leq \|x\|^q + q < y, J_q(x) > + 2 \|Ky\|^2$, for all x, y ε E, where J_q is the generalized duality mapping of E and K is the q-uniformly smoothness constant of E.

Theorem 2.7 [1] Let D be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of D into itself. If $\{u_j\}$ is a sequence of D such that $u_j \rightharpoonup u_0$ and let $\lim_{j\to\infty} \|u_j - Tu_j\| = 0$, then u_0 is a fixed point of T.

III. Main Result

In this section, we shall prove our main result.
Theorem 3.1 Let E be a uniformly convex and 2-uniformly smooth Banach space with best smooth constant K and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be α-inverse strongly accretive operator of C into E. Let $S(C, A) \neq \varphi$ and the sequence $\{x_n\}$ be generated by
$y_n = Q_C(x_n - \lambda_n Ax_n)$,
$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(y_n - \lambda_n Ay_n)$, $x_1 \in C$, $n = 1, 2, 3, \ldots$, (3.1)
where $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $\lambda_n \leq \alpha$ and $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and let $\alpha_n \in [b, c]$, where $0 < b < c < 1$, then $\{x_n\}$converges weakly to some element z of S(C, A).

Proof. Let $z_n = Q_C(y_n - \lambda_n Ay_n)$ for $n = 1, 2, \ldots$. Let $u \in S(C, A)$. Now,
$\|y_n - u\| \leq \| Q_C(x_n - \lambda_n Ax_n) - Q_C(u - \lambda_n Au)\|$
$\leq \| x_n - u \|$ (3.2)
Also,
$\|z_n - u\| \leq \| Q_C(y_n - \lambda_n Ay_n) - Q_C(u - \lambda_n Au)\|$
$\leq \|y_n - u\| \leq \|x_n - u\|$ (3.3)
Now, for every $n = 1, 2, \ldots$,
$\|x_{n+1} - u\| = \| \alpha_n (x_n - u) + (1 - \alpha_n)(z_n - u)\|$
$\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \| z_n - u\|$

Using (3.2) and (3.3), $\|x_{n+1} - u\| \le \|x_n - u\|$ (3.4)

(3.4) shows that $\{\|x_n - u\|\}$ is non-increasing sequence.

So, there exists $\lim_{n\to\infty} \|x_n - u\|$ and hence $\{x_n\}$ is a bounded sequence . (3.2) and (3.3) shows that $\{y_n\}$, $\{Ax_n\}$ and $\{z_n\}$ are also bounded .

Next, we shall show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Conversely, let $\lim_{n\to\infty} \|x_n - y_n\| \neq 0$. Then there exists $\in > 0$ and a subsequence $\{x_{n_i} - y_{n_i}\}$ of $\{x_n - y_n\}$ such that $\|x_{n_i} - y_{n_i}\| \ge \in$ for each i = 1, 2…………. Since E is uniformly convex, so the function $\|.\|^2$ is uniformly convex on bounded convex subset B (0, $\|x_1 - u\|$), where B (0, $\|x_1 - u\|$) = $\{x \in E : \|x\| \le \|x_1 - u\|\}$.

So, for any $\in$, there exists $\delta > 0$ such that $\|x - y\| \ge \in$ implies

$$\|\lambda x + (1-\lambda)y\|^2$$

$$\le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\delta,$$

where x, y $\in$ B (0, $\|x_1 - u\|$), $\lambda \in (0, 1)$. So for i = 1, 2,……….,

$$\|x_{n_i+1} - u\|^2 = \|\alpha_{n_i}(x_{n_i} - u) + (1-\alpha_{n_i})(z_{n_i} - u)\|^2$$

$$\le \alpha_{n_i}\|x_{n_i} - u\|^2 + (1-\alpha_{n_i})\|y_{n_i} - u\|^2 - \alpha_{n_i}(1-\alpha_{n_i})\delta$$

$$\le \alpha_{n_i}\|x_{n_i} - u\|^2 + (1-\alpha_{n_i})\|x_{n_i} - u\|^2 - \alpha_{n_i}(1-\alpha_{n_i})\delta$$

$$\le \|x_{n_i} - u\|^2 - \alpha_{n_i}(1-\alpha_{n_i})\delta$$

Therefore,

$$0 < b(1-c)\delta \le \alpha_{n_i}(1-\alpha_{n_i})\delta \le \|x_{n_i} - u\|^2 - \|x_{n_i+1} - u\|^2 \qquad (3.5)$$

Since right hand side of inequality (3.5) converges to 0, so we get a contradiction.

Hence, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (3. 6)

Now, since $\{x_n\}$ is bounded, so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that weakly converges to z. Also $\lambda_{n_i} \in [a, \alpha/K^2]$, so $\{\lambda_{n_i}\}$ is bounded. Hence, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ that weakly converges to $\lambda_0 \in [a, \alpha/K^2]$. Without loss of generality assume that $\lambda_{n_i} \to \lambda_0$. Since Q_C is nonexpansive , so

$y_{n_i} = Q_C(x_{n_i} - \lambda_{n_i}Ax_{n_i})$ implies that

$$\left\|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - x_{n_i}\right\|$$

$$\le \left\|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - y_{n_i}\right\| + \left\|y_{n_i} - x_{n_i}\right\|$$

$$= \left\|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - Q_C(x_{n_i} - \lambda_{n_i}Ax_{n_i})\right\| + \left\|y_{n_i} - x_{n_i}\right\|$$

$$\le \left|\lambda_0 - \lambda_{n_i}\right|\left\|Ax_{n_i}\right\| + \left\|y_{n_i} - x_{n_i}\right\|$$

$$\le M\left|\lambda_0 - \lambda_{n_i}\right| + \left\|y_{n_i} - x_{n_i}\right\| \qquad (3.7)$$

where M = sup $\{\|Ax_n\| : n = 1, 2, 3……………\}$. Equation (3.6), (3.7) and convergence of $\{\lambda_{n_i}\}$ implies that

$$\lim_{i\to\infty} \|Q_C(I - \lambda_0 A)x_{n_i} - x_{n_i}\| = 0 \qquad (3.8)$$

Also, $Q_C(I - \lambda_0 A)$ is nonexpansive, so (3.8), lemma 2.3 and theorem 2.7 implies
$z \in F(Q_C(I - \lambda_0 A)) = S(C, A)$.

Lastly, we shall prove that $\{x_n\}$ is convergent to some element of S(C, A). Let
$T_n = \alpha_n I + (1 - \alpha_n)Q_C(I - \lambda_n A)$, for n = 1, 2,……….. (3.9)

Then, $x_{n+1} = T_n T_{n+1}……………….T_1 x$ and $z \in \bigcap_{n=1}^{\infty} \overline{co}\{x_m : m \ge n\}$. Also from lemma 2.4, T_n is nonexpansive mapping of C into itself. And from lemma 2.3, we have,

$$\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F\left(Q_C\left(I - \lambda_n A\right)\right) = S(C, A).$$

Using theorem (2.5), we obtain

$$\bigcap_{n=1}^{\infty} c\bar{o}\{x_m : m \geq n\} \bigcap S(C, A) = \{z\} \qquad (3.10)$$

Hence, the sequence $\{x_n\}$ is weakly convergent to some element of S(C, A).

IV. Application

Using our main result, we shall prove a result for strongly accretive operator.

Let C be a subset of a smooth Banach space E. Let $\alpha > 0$. An operator A of C into E is said to be α-strongly accretive if

$< Ax - Ay, J(x - y) > \geq \alpha \|x - y\|^2$ for all x, y ε C.

Let $\beta > 0$. An operator A of C into E is said to be β-Lipschitz continuous if

$\|Ax - Ay\| \leq \beta \|x - y\|$, for all x, y ε C.

Theorem 4.1 Let E be a uniformly convex and 2-uniformly smooth Banach space with best smooth constant K and C be a nonempty closed convex subset of E. Let Q_C be be a sunny nonexpansive retraction from E onto C, $\alpha > 0$, $\beta > 0$ and A be α- strongly accretive operator and β-Lipschitz continuous operator of C into E. Let S(C, A) $\neq \varphi$ and the sequence $\{x_n\}$ be generated by

$y_n = Q_C(x_n - \lambda_n Ax_n)$,

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(y_n - \lambda_n Ay_n)$, $x_1 \in C$, n = 1, 2, 3,…………,

where $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $0 \leq \lambda_n \leq 1$ and $\lambda_n \in [a, \alpha/K^2]$ for some a > 0 and let $\alpha_n \in [b, c]$, where 0 < b < c < 1, then $\{x_n\}$converges weakly to a unique element z of S(C, A).

Proof. Since A is an α-strongly accretive and β-Lipschitz continuous operator of C into E, we have

$$< Ax - Ay, J(x - y) > \geq \alpha \|x - y\|^2 \geq \frac{\alpha}{\beta^2} \|Ax - Ay\|^2, \text{ for all } x, y \in C.$$

So A is $\frac{\alpha}{\beta^2}$ - inverse strongly accretive. Since A is strongly accretive and S(C, A) $\neq \varphi$, so the set S(C, A) consists of one point z. Using theorem 3.1, $\{x_n\}$ converges weakly to a unique element z of S(C, A).

References

[1]. F. E. Browder, Nonlinear operators and nonlinear equations of evoluation in Banach spaces, Nonlinear Functional Analsis (proc.Sympos.pure Math., vol. XVIII, Part 2,Chicago, III, 1968), American Mathematical Society, Rhode Island, 1976, pp. 1-308.

[2]. H. K. Xu, Inequalities in Banach Spaces with applications, Nonlinear Analysis, 16(1991), no.12, 1127-1138.

[3]. K. Aoyama, H. Iiduka, W. Takashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed point theory and applications, vol 2006, p 1-13.

[4]. K. Ball, A. Carlen and E. H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms, Inventiones Mathematicae, 115(1994), no. 3, 463-482.

[5]. N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone Mappings, Journal of Optimization Theory and Applications, vol. 128, pp. 191-201, 2006.

[6]. R. E. Bruck Jr., Nonexpansive retracts of Banach spaces, Bulletin of the American mathematical society, 76(1970), 384-386.

[7]. S. Kitahara and W. Takahashi, Image recovery by convex combinations of sunny nonexpansive retractions, Topological methods in Nonlinear analysis, 2(1993), no. 2, 333-342.

[8]. S. Reich, Asymptotic behavior of contractions in Banach spaces, Journal of Mathematical Analysis and Applications, 44(1973), no. 1, 57-70.

[9]. S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, Journal of Mathematical Analysis and Applications, 67(1979), no. 2, 274-276.

[10]. W. Takahashi, Nonlinear functional analysis, Yokohama publisher, Yokohama, Japan, 2000.

[11]. Y. Takahashi, K. Hashimoto and M. Kato, On sharp uniform convexity, smoothness and strong type, cotype inequality, Journal of Nonlinear and Convex Analysis, 3(2002), no. 2, 267-281.