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Growth Estimates of Entire Functions on the Basis of Central Index and (p,q)th Order

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Abstract: *In this paper we discuss (p,q)th order of an entire function in terms of central index and use it to estimate the growth of composite entire functions.*
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I. Introduction, Definitions and Notations.

Let

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

be an entire function. $M(r,f)=\max_{|z|=r}|f(z)|$ denote the maximum modulus of f on $|z|=r$ and $\mu(r,f)=\max_{n\geq 0}|a_n|r^n$ denote the maximum term of f on $|z|=r$. The central index $\nu(r,f)$ is the greatest exponent m such that $|a_m|r^m=\mu(r,f)$. We note that $\nu(r,f)$ is real, non-decreasing function of r.

We do not explain the standard definitions and notations in the theory of entire function as those are available in [5]. In the sequel the following notions are used:

$$\log^{[k]} x=\log\big(\log^{[k-1]} x\big) \qquad \text{for } k=1,2,3,\ldots$$
$$\text{and} \quad \log^{[0]} x=x.$$

To start our paper we just recall the following definitions:

Definition 1: The order ρ_f and lower order λ_f of an entire function f are defined as follows

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2: The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of an entire function f are defined as follows

$$\bar{\rho}_f=\limsup_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \bar{\lambda}_f=\liminf_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 3 ([**4**])**:** Let l be an integer ≥ 1. The generalised order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f are defined as follows

$$\rho_f^{[l]}=\limsup_{r\to\infty}\frac{\log^{[l+1]} M(r,f)}{\log r} \text{ and } \lambda_f^{[l]}=\liminf_{r\to\infty}\frac{\log^{[l+1]} M(r,f)}{\log r}.$$

When $l=1$, Definition 3 coincides with Definition 1 and when $l=2$, Definition 3 coincides with Definition 2.

Juneja, Kapoor and Bajpai [3] defined the (p,q)th order, and (p,q)th lower order of an entire function f respectively as follows:

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} \qquad (1)$$

$$\text{and} \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}, \qquad (2)$$

where p,q are positive integers with $p \geq q$.

For $p = 1$ and $q = 1$ we respectively denote $\rho_f(1,1)$ and $\lambda_f(1,1)$ by ρ_f and λ_f.

In this paper we intend to establish some results relating to the growth properties of composite entire functions on the basis of central index and (p,q)th order, where p,q are positive integers with $p \geq q$.

II. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1] and [2, Theorems 1.9 and 1.10, or 11, Satz 4.3 and 4.4]):
Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function, $\mu(r,f)$ be the maximum term i.e., $\mu(r,f) = \max_{n\geq 0} |a_n| r^n$ and $\nu(r,f)$ be the central index of f. Then

(i) For $a_0 \neq 0$,

$$\log \mu(r,f) = \log|a_0| + \int_0^r \frac{\nu(t,f)}{t}\, dt,$$

(ii) For $r < R$,

$$M(r,f) < \mu(r,f)\left\{\nu(R,f) + \frac{R}{R-r}\right\}.$$

Lemma 2: Let $f(z)$ be an entire function with (p,q)th order $\rho_f(p,q)$, where p,q are positive integers with $p \geq q$ and let $\nu(r,f)$ be the central index of f. Then

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} \nu(r,f)}{\log^{[q]} r}.$$

Proof: Set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Without loss of generality, we can assume that $|a_0| \neq 0$. By (i) of Lemma 1, we have

$$\log \mu(2r,f) = \log|a_0| + \int_0^{2r} \frac{\nu(t,f)}{t}\, dt \geq \nu(r,f)\log 2 .$$

Using the Cauchy inequality, it is easy to see that $\mu(2r,f) \leq M(2r,f)$. Hence

$$\nu(r,f)\log 2 \leq \log M(2r,f) + C,$$

where $C(> 0)$ is a suitable constant. By this and (1), we get

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q]} r} \leq \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} = \rho_f(p,q). \qquad (3)$$

On the other hand, by (ii) of Lemma 1, we have

$$M(r,f) < \mu(r,f)\{\nu(2r,f)+2\} = \left|a_{\nu(r,f)}\right| r^{\nu(r,f)}\{\nu(2r,f)+2\}.$$

Since $\{|a_n|\}$ is a bounded sequence, we have

$$\log M(r,f) \leq \nu(r,f)\log r + \log \nu(2r,f) + C_1$$
$$\Rightarrow \log^{[p+1]} M(r,f) \leq \log^{[p]} \nu(r,f) + \log^{[p+1]} \nu(2r,f) + \log^{[p+1]} r + C_2$$
$$\Rightarrow \log^{[p+1]} M(r,f) \leq \log^{[p]} \nu(2r,f)\left[1 + \frac{\log^{[p+1]} \nu(2r,f)}{\log^{[p]} \nu(2r,f)}\right] + \log^{[p+1]} r + C_3 ,$$

where $C_j(> 0)$ (j = 1, 2, 3) are suitable constants. By this and (1), we get

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} \nu(2r,f)}{\log^{[q]} 2r} = \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q]} r}. \qquad (4)$$

From (3) and (4), Lemma 2 follows.

Lemma 3: Let $f(z)$ be an entire function with (p,q)th lower order $\lambda_f(p,q)$, where p,q are positive integers with $p \geq q$ and let $\nu(r,f)$ be the central index of f. Then

$$\lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} \nu(r,f)}{\log^{[q]} r}.$$

Proof: Set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Without loss of generality, we can assume that $|a_0| \neq 0$. By (i) of Lemma 1, we have

$$\log \mu(2r,f) = \log|a_0| + \int_0^{2r} \frac{\nu(t,f)}{t}\, dt \geq \nu(r,f)\log 2\,.$$

Using the Cauchy inequality, it is easy to see that $\mu(2r,f) \leq M(2r,f)$. Hence

$$\nu(r,f)\log 2 \leq \log M(2r,f) + C,$$

where $C(>0)$ is a suitable constant. By this and (2), we get

$$\liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q]} r} \leq \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} = \lambda_f(p,q) \qquad (5)$$

On the other hand, by (ii) of Lemma 1, we have

$$M(r,f) < \mu(r,f)\{\nu(2r,f)+2\} = |a_{\nu(r,f)}| r^{\nu(r,f)}\{\nu(2r,f)+2\}.$$

Since $\{|a_n|\}$ is a bounded sequence, we have

$$\log M(r,f) \leq \nu(r,f)\log r + \log \nu(2r,f) + C_1$$
$$\Rightarrow \log^{[p+1]} M(r,f) \leq \log^{[p]} \nu(r,f) + \log^{[p+1]} \nu(2r,f) + \log^{[p+1]} r + C_2$$
$$\Rightarrow \log^{[p+1]} M(r,f) \leq \log^{[p]} \nu(2r,f)\left[1 + \frac{\log^{[p+1]} \nu(2r,f)}{\log^{[p]} \nu(2r,f)}\right] + \log^{[p+1]} r + C_3\,,$$

where $C_j(>0)$ (j = 1, 2, 3) are suitable constants. By this and (2), we get

$$\lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$
$$\leq \liminf_{r\to\infty} \frac{\log^{[p]} \nu(2r,f)}{\log^{[q]} 2r} = \liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q]} r}. \qquad (6)$$

From (5) and (6), Lemma 3 follows.

III. Theorems.

In this section we present the main results of the paper.

Theorem 1: Let f and g be entire functions such that $0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ and $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$. Then

$$\frac{\lambda_{fog}(p,q)}{\rho_g(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)}\,,$$

where p,q,m are positive integers with $p \geq q \geq m$.

Proof: Using respectively Lemma 3 for the entire function fog and Lemma 2 for the entire function g, we have for arbitrary positive ε and for all large values of r that

$$\log^{[p]} \nu(r,fog) \geq \left(\lambda_{fog}(p,q) - \varepsilon\right) \log^{[q]} r \tag{7}$$

and

$$\log^{[m]} \nu(r,g) \leq \left(\rho_g(m,q) + \varepsilon\right) \log^{[q]} r. \tag{8}$$

Now from (7) and (8) it follows for all large values of r,

$$\frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \geq \frac{\lambda_{fog}(p,q) - \varepsilon}{\rho_g(m,q) + \varepsilon}.$$

As $\varepsilon\ (> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \geq \frac{\lambda_{fog}(p,q)}{\rho_g(m,q)}. \tag{9}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} \nu(r,fog) \leq \left(\lambda_{fog}(p,q) + \varepsilon\right) \log^{[q]} r \tag{10}$$

and for all large values of r,

$$\log^{[m]} \nu(r,g) \geq \left(\lambda_g(m,q) - \varepsilon\right) \log^{[q]} r. \tag{11}$$

So combining (10) and (11) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\lambda_{fog}(p,q) + \varepsilon}{\lambda_g(m,q) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{12}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} \nu(r,g) \leq \left(\lambda_g(m,q) + \varepsilon\right) \log^{[q]} r. \tag{13}$$

Now from (7) and (13) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \geq \frac{\lambda_{fog}(p,q) - \varepsilon}{\lambda_g(m,q) + \varepsilon}$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \geq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)} \tag{14}$$

Also for all large values of r,

$$\log^{[p]} \nu(r,fog) \leq \left(\rho_{fog}(p,q) + \varepsilon\right) \log^{[q]} r. \tag{15}$$

So from (11) and (15) it follows for all large values of r,

$$\frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\rho_{fog}(p,q) + \varepsilon}{\lambda_g(m,q) - \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)} \tag{16}$$

Thus the theorem follows from (9), (12), (14) and (16).

Theorem 2: Let f and g be entire functions such that $0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ and $0 < \rho_g(m,q) < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r,g)},$$

where p, q, m are positive integers with $p \geq q \geq m$.

Proof. Using Lemma 2 for the entire function g, we get for a sequence of values of r tending to infinity that

$$\log^{[m]}\nu(r,g) \geq \left(\rho_g(m,q)-\varepsilon\right)\log^{[q]}r. \tag{17}$$

Now from (15) and (17) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r,g)} \leq \frac{\rho_{fog}(p,q)+\varepsilon}{\rho_g(m,q)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r,g)} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}. \tag{18}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]}\nu(r,fog) \geq \left(\rho_{fog}(p,q)-\varepsilon\right)\log^{[q]}r. \tag{19}$$

So combining (8) and (19) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r,g)} \geq \frac{\rho_{fog}(p,q)-\varepsilon}{\rho_g(m,q)+\varepsilon}$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r,g)} \geq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}. \tag{20}$$

Thus the theorem follows from (18) and (20).

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3: Let f and g be entire functions such that $0<\lambda_{fog}(p,q)\leq\rho_{fog}(p,q)<\infty$ and $0<\lambda_g(m,q)\leq\rho_g(m,q)<\infty$. Then

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r,g)} &\leq \min\left\{\frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)},\frac{\rho_{fog}(p,q)}{\rho_g(m,q)}\right\}\\ &\leq \max\left\{\frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)},\frac{\rho_{fog}(p,q)}{\rho_g(m,q)}\right\}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r,g)},\end{aligned}$$

where p,q,m are positive integers such that $p\geq q\geq m$.
The proof is omitted.

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