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A new Type of Fuzzy Functions in Fuzzy Topological Spaces

Assist. Prof. Dr. Munir Abdul Khalik Alkhafaji
Al-Mustansiriyah University College of Education Department of Math.

Abstract: *In this paper, we introduce and study some characterization and some properties of fuzzy continuous functions (Fuzzy super continuous functions) from fuzzy topological space into another fuzzy topological space has its bases on the notation of quasi-coincidence , quasi-neighborhood, fuzzy δ -closure and θ- neighborhood.*

I. Introduction

The first publications in fuzzy set theory by Zadeh [11] show the intention of the authors to generalize the classical notion of a set and proposition (Statement) to accommodate fuzziness, Chang [4], Wong [9], [10] and other applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces in 1980, pu and liu [8], introduced the concepts of quasi-coincidence and quasi-neighborhood

II. Preliminaries

We will discuss some fundamental notions and basic concepts related to fuzzy topological space.

Definition 2 – 1

A fuzzy topology is a family $\tilde{T}$ of fuzzy sets X (X be any set of elements) which satisfying the following conditions:

1- O , 1 (i.e. Ø and X) $\in \tilde{T}$,

2- If $\tilde{A}$, $\tilde{B} \in \tilde{T}$, then $\tilde{A} \cap \tilde{B} \in \tilde{T}$

3- If $\tilde{A}_i \in \tilde{T}$ for each $i \in I$ (where I is the index set), then $\bigcup_{i \in I} \tilde{A}_i \in \tilde{T}$.

$\tilde{T}$ is called a fuzzy topology for X , and the pair (X , $\tilde{T}$) is a fuzzy topological space. Every member of $\tilde{T}$ is called $\tilde{T}$- open fuzzy set. A fuzzy set $\tilde{C}$ in X is a $\tilde{T}$ - closed fuzzy set if and only if its complement $\tilde{C}^c$ is a $\tilde{T}$ - open fuzzy set.

Definition 2 – 2

A fuzzy set $\tilde{A}$ in a fuzzy topological space (X , $\tilde{T}$) is said to be quasi –coincident (q – coincident , for short) with fuzzy set $\tilde{B}$, denoted by $\tilde{A} q \tilde{B}$,

If and only if there exist $x \in X$ such that $\mu_{\tilde{A}(x)} + \mu_{\tilde{B}(x)} > 1$.

Definition 2 – 3

A fuzzy set $\tilde{A}$ in a fuzzy topological space (X , $\tilde{T}$) is said to be quasi – neighborhood (q – nbd , for thort) of a fuzzy point x_α (where x is the support and α is the value of the fuzzy point, $0 < \alpha \leq 1$)if and only if there exists open fuzzy set $\tilde{B}$ such that $x_\alpha q \tilde{B} \subseteq \tilde{A}$.

Definition 2 – 4

Two fuzzy sets $\tilde{A}, \tilde{B}$ in fuzzy topological space (X, $\tilde{T}$), $\tilde{A} \subseteq \tilde{B}$ if and only if $\tilde{A}$ is not q – coincident with $\tilde{B}^c$ (complement of $\tilde{B}$) and denoted by $\tilde{A} q \tilde{B}^c$.

Definition 2 – 5

A fuzzy point $x_\alpha \in \overline{\tilde{A}}$ (the fuzzy closure of fuzzy set $\tilde{A}$ in an fuzzy topological space (X , $\tilde{T}$) if and only if each q – nbd of x_α is q – coincident with $\tilde{A}$.

Definition 2 – 6

A fuzzy set $\tilde{A}$ in fuzzy topological space (X , $\tilde{T}$) is called open (closed) fuzzy regularly if and only if $(\overline{\tilde{A}})^o = \tilde{A}$ ($\tilde{A}^o$ the fuzzy interior of fuzzy set $\overline{A}$ in an fuzzy topological space (X , $\tilde{T}$)) (respectively $(\overline{\tilde{A}^o}) = \tilde{A}$).

Definition 2 – 7

A fuzzy point x_α is called a fuzzy δ - cluster point of a fuzzy set $\tilde{A}$ in an fuzzy topological space (X , $\tilde{T}$) if and only if every open fuzzy regularly q-nbd of x_α is q – coincident with $\tilde{A}$.

Definition 2 – 8

The set of all fuzzy δ -cluster points of $\tilde{A}$ is called the fuzzy δ -cluster of $\tilde{A}$, to be denoted by $[\tilde{A}]_\delta$.

Remark 2 – 1

A fuzzy set $\tilde{A}$ is fuzzy δ -closed if and only if $\tilde{A} = [\tilde{A}]_\delta$ and complement of a fuzzy δ -closed is called fuzzy δ - open.

Definition 2 – 9

If X , Y are fuzzy topological spaces and $\tilde{A}, \tilde{B}$ are fuzzy sets of X and Y respectively , then the fuzzy set $\tilde{A} \times \tilde{B}$ of X×Y is defined as $\mu_{(\tilde{A}\times\tilde{B})(x,y)} = \min(\mu_{\tilde{A}(x)}, \mu_{\tilde{B}(y)})$ where $x \in X$ and $y \in Y$.

Throughout the paper , by (X , $\tilde{T}$), (Y , $\tilde{T}_1$) etc. or simply by X , Y, Z , etc. we shall mean fuzzy topological spaces.

III. Fuzzy Super Continuous Functions

Definition 3 – 1

A mapping $f : X \to Y$ from fuzzy topological X to an fuzzy topological space Y is said to be fuzzy super continuous at a fuzzy point x_α of X if and only if for every q – nbd Ũ of f(x_α), there is a q – nbd $\tilde{V}$ of x_α such that $f(\overline{\tilde{V}})^o \subseteq U$ (equivalently , we can take Ũ ,Ṽ to be open fuzzy set). f is said to be fuzzy super continuous on X if and only if it is fuzzy super continuous at each fuzzy point of X.

Definition 3 – 2

A function f : X ⟶ Y from an fuzzy topological space X to an fuzzy topological space Y is said to be fuzzy continuous at a fuzzy point x_α of X if and only if for every q – nbd Ũ of f(x_α) there is a q – nbd $\tilde{V}$ of x_α such that $f(\tilde{V}) \subseteq \tilde{U}$. f is called fuzzy continuous on X if and only if f is fuzzy continuous at each fuzzy point of X.

Remark 3 – 1

If a mapping f : (X, $\tilde{T}$) $\longrightarrow$ (Y, $\tilde{F}$) , from an fuzzy topological space (X, $\tilde{T}$) to an fuzzy topological space (Y, $\tilde{F}$) is fuzzy super continuous at a fuzzy point x_α of X, then f is fuzzy continuous at x_α, but the converse is false.

Theorem 3 – 1

For a mapping f : (X, $\tilde{T}$) $\longrightarrow$ (Y, $\tilde{F}$), from an fuzzy topological space (X, $\tilde{T}$) to an fuzzy topological space (Y, $\tilde{F}$), the following are equivalent:

(a) f is a fuzzy super continuous .

(b) $f([\tilde{A}]_\delta) \subseteq \overline{f(\tilde{A})}$, for every fuzzy set $\tilde{A}$ in X.

(c) $[f^{-1}(\tilde{A})]_\delta \subseteq f^{-1}(\overline{\tilde{A}})$, for every fuzzy set $\tilde{A}$ in Y.

(d) For every closed fuzzy set $\tilde{A}$ in Y, $f^{-1}(\tilde{A})$ is fuzzy δ-closed in X.

(e) For every open fuzzy set $\tilde{A}$ in Y, $f^{-1}(\tilde{A})$ is fuzzy δ-open in X.

(f) For every open fuzzy point x_α of X and for each open fuzzy q-nbd $\tilde{M}$ of f(x_α), there is a fuzzy δ- open q-nbd $\tilde{N}$ of x_α such that $f(\tilde{N}) \subseteq \tilde{M}$.

Proof. (a) $\rightarrow$(b) :

Let $x_\alpha \in [\tilde{A}]_\delta$ and Ũ be any open fuzzy q-nbd of $y_\alpha = f(x_\alpha)$, where y=f(x).

By (a) , there is open fuzzy q-nbd $\tilde{N}$ of x_α such that $f(\overline{\tilde{N}})^o \subseteq \tilde{U}$.

Now , if $x_\alpha \in [\tilde{A}]_\delta$ then $(\overline{\tilde{N}})^o q \tilde{A}$ so that

$f(\overline{\tilde{N}})^o q f(\tilde{A})$ and hence $\tilde{U} q f(\tilde{A})$.

Therefore $f(x_\alpha) \in \overline{(f(\tilde{A}))}$ and it follows that

$x_\alpha \in f^{-1}\overline{(f(\tilde{A}))}$.

Thus $[\tilde{A}]_\delta \subseteq f^{-1}\overline{(f(\tilde{A}))}$ and hence $f([\tilde{A}]_\delta) \subseteq \overline{f(\tilde{A})}$.

(b) $\rightarrow$ (c) :

By (b) , if $f([f^{-1}(\tilde{A})]_\delta) \subseteq \overline{f[f^{-1}(\tilde{A})]} \subseteq \overline{\tilde{A}}$

Then it follows that $[f^{-1}(\tilde{A})]_\delta \subseteq f^{-1}(\overline{\tilde{A}})$.

(c) $\rightarrow$ (d) : Immediate

(d) $\rightarrow$ (e) : Clear

(e) $\rightarrow$ (f) : $\tilde{N} = f^{-1}(\tilde{M})$ serves the purpose.

(f) $\rightarrow$ (a) : For any fuzzy point x_α and open fuzzy q-nbd $\tilde{V}$ of f(x_α) , there is a fuzzy δ - open q-nbd $\tilde{N}$ of x_α such that $f(\tilde{N}) \subseteq \tilde{V}$.

Then $1 - \tilde{N} = \tilde{G}$ (say) is fuzzy δ -closed in X and $x_\alpha \notin \tilde{G}$.

So the exist a open fuzzy regularly q-nbd $\tilde{M}$ of x_α such that $\tilde{M}$ is not q-coincident with $\tilde{G}$.

Now, $x_\alpha q\, \tilde{M} \to x_\alpha q\left(\overline{\tilde{M}}\right)^o \subseteq 1-\tilde{G} = \tilde{N}$

From which it follows that $f\left(\overline{\tilde{M}}\right)^o \subseteq f(\tilde{N}) \subseteq \tilde{V}$ and thus f is fuzzy super continuous.

Theorem 3 – 2

A function $f: X \to Y$ from fuzzy is a fuzzy super continuous if and only if for any fuzzy point x_α of X and for each q-nbd $\tilde{M}$ of $f(x_\alpha)$, there q q-nbd $\tilde{N}$ of x_α such that $f(\overline{\tilde{N}}^o)^o \subseteq \tilde{M}$.

Lemma 3 – 1

Let $f: X \to Y$ be a mapping from an fuzzy topological space X to an fuzzy topological space Y the following are equivalent:

(a) For any fuzzy point x_α of X and for any q-nbd $\tilde{M}$ of $f(x_\alpha)$, there is a q-nbd $\tilde{N}$ of x_α such that $f(\overline{\tilde{N}^o}) \subseteq \tilde{M}$

(b) For any fuzzy point x_α of X and for any q-nbd $\tilde{M}$ of $f(x_\alpha)$, there is a q-nbd $\tilde{N}$ of x_α such that $\overline{f(\overline{\tilde{N}}^o)} \subseteq \tilde{M}$.

Proof : it is immediate .

Corollary 3 – 1

If one of the conditions of Lemma 3 -1 holds then f is fuzzy super continuous.
That the converse of the above corollary is false is shown by the following example.

Example 3 – 1

Let X be a non-empty set and $a \in X$.

Consider $\tilde{T} = \{0, 1, \tilde{A}\}$,

Where $\mu_{\tilde{A}}(a) = 1/3$ and $\mu_{\tilde{A}}(x)=0$, for $x \neq a$ ($x \in X$), and consider the identity map

$f: (X, \tilde{T}) \to (X, \tilde{T})$.

We consider the fuzzy point a_α of X , where $\alpha = \frac{5}{6}$.

Then $\tilde{A}$ is open fuzzy q-nbd of $f(a_\alpha)$,

But $f(\overline{\tilde{U}^o}) \supset \tilde{A}$, for $\tilde{U} = \tilde{A}$ or 1.

Hence condition (a) of lemma 3 – 1 fails. But if clearly fuzzy super continuous.

Lemma 3 – 2

Let A , X , Y be fuzzy topological spaces and $f_1: A \to X$ and $f_2: A \to Y$ be any functions .

Let $f: A \to X \times Y$ be defined by $f(a) = (f_1(a), f_2(a))$, for $a \in A$ where $X \times Y$ is provided with the product fuzzy topology.

(a) If $\tilde{B}, \tilde{U}_1, \tilde{U}_2$ are fuzzy sets in A, X and Y respectively such that $f(\tilde{B}) \subseteq \tilde{U}_1 \times \tilde{U}_2$ then $f_1(\tilde{B}) \subseteq \tilde{U}_1$ and $f_2(\tilde{B}) \subseteq \tilde{U}_2$.

(b) If $\tilde{W}$ is open fuzzy q-nbd of $f(x_\alpha) = (f_1(x), f_2(x))_\alpha$ in $X\times Y$,

Then there exist open fuzzy q-nbd $\tilde{U}$ of $(f_1(x))_\alpha$ in X and open fuzzy q-nbd $\tilde{V}$ of $(f_2(x))_\alpha$ in Y such that $f(x_\alpha) q\, (\tilde{U}\times\tilde{V}) \subseteq \tilde{W}.$

(c) If $f_1(\tilde{A}_1)\times f_2(\tilde{A}_2) \subseteq \tilde{U}_1\times\tilde{U}_2$, where $\tilde{A}_1, \tilde{A}_2$ are fuzzy sets in A and $\tilde{U}_1, \tilde{U}_2$ are fuzzy sets of x and y respectively , then $f(\tilde{V}) \subseteq \tilde{U}_1\times\tilde{U}_2$ where $\tilde{V} = \tilde{A}_1 \cap \tilde{A}_2$.

Proof. (a) we prove that $f_1(\tilde{B}) \subseteq \tilde{U}_1$.

In fact , Let $x \in X$, then

$f(\mu_{\tilde{B}}(x)) = \sup_{z\in f_1^{-1}(x)} \mu_{\tilde{B}}(x)$

Now , $z \in f_1^{-1}(x) \to f_1(z) = x$.

Let $f_2(z) = y_1 \in Y.$ Then

$f(z) = (x\ \ , y_1) \to z \in f^{-1}[(x\ \ , y_1)] \to$ (1)

Now

$f(\mu_{\tilde{B}})[(x,y)] = \sup_{t\in f^{-1}[(x,y_1)]} \mu_{\tilde{B}}(t)$

Obviously

$\mu_{\tilde{B}}(z) \subseteq \sup_{t\in f^{-1}[(x,y_1)]} \mu_{\tilde{B}}(t)$

By (1).

But

$$\sup_{t\in f^{-1}[(x,y_1)]} \mu_{\tilde{B}}(t) \subseteq \mu(\tilde{U}_1\times\tilde{U}_2)[(x,y_1)] \to \mu_{\tilde{B}}(z) \subseteq \mu_{\tilde{U}_1\times\tilde{U}_2}[(x,y_1)]$$

$$\to \mu_{\tilde{B}}(z) \subseteq \mu_{\tilde{U}_1}(x) \qquad (2)$$

Since (2) is true for all $z \in f_1^{-1}(x)$, we have $f_1(\mu_{\tilde{B}})(x) \subseteq \mu_{\tilde{U}_1}(x)$ and hence $f_1(\tilde{B}) \subseteq \tilde{U}_1.$

Similarly , we can prove that $f_2(\tilde{B}) \subseteq \tilde{U}_2$.

(b) Since $\tilde{W}$ is open fuzzy set in $X\times Y$, we have $\tilde{W} = \cup(\tilde{U}_\beta \times \tilde{V}_\mu)$, where the $\tilde{U}_\beta$'s are open set in X and the $\tilde{V}_\mu$'s are open fuzzy set in Y.

Then obviously $\tilde{U}_\beta \times \tilde{V}_\mu \subseteq \tilde{W}$, for each β and each μ.

We claim that for some β and some μ.

$f(x_\alpha) q\ \tilde{U}_\beta\times\tilde{V}_\mu$

In not then for each β and each μ

$\alpha + \min(\mu_{\tilde{U}_\beta}(f_1(x)), \mu_{\tilde{V}_\mu}(f_2(x)) \le 1$

So that $\alpha + \mu_{(\tilde{U}_\beta\times\tilde{V}_\mu)}(f(x)) \le 1$, for each β and each μ, And thus $\alpha + \text{Sup}[\mu_{(\tilde{U}_\beta \times \tilde{V}_\mu)}(f(x)) \le 1$.

But then $\alpha + \mu_{\tilde{W}}(f(x)) \le 1$ and hence $f(x_\alpha) q \tilde{W}$, a contradiction .

Therefore $f(x_\alpha) q \tilde{U}_\beta \times \tilde{V}_\mu$, for some β and some μ ,

So that $\alpha + \min(\mu_{\tilde{U}_\beta}(f_1(x)), \mu_{\tilde{V}_\mu}(f_2(x)) > 1$.

Then $f_1(x_\alpha) q \tilde{U}_\beta$ and $f_2(x_\alpha) q \tilde{V}_\mu$ we have proved our desired result.

(c) Straight forward and left.

Theorem 3 – 3

Let A, X, Y be fuzzy topological spaces and $f_1 : A \to X,\ f_2 : A \to Y$ be any functions. Then $f : A \to X \times Y$, defined by
$f(x) = (f_1(x), f_2(x)),$ for all $x \in A$, is fuzzy super continuous iff f_1 and f_2 are fuzzy super continuous.

Proof.

Let x_α be a fuzzy point of A and $\tilde{U}_1, \tilde{U}_2$ be open fuzzy q-nbds of $f_1(x_\alpha)$ and $f_2(x_\alpha)$ in X and Y respectively.

Then $\tilde{U}_1 \times \tilde{U}_2$ is clearly open fuzzy q-nbd of $f(x_\alpha)$.

i.e., of $(f(x))_\alpha$.

Then by fuzzy super continuity of f , there is open fuzzy q-nbd $\tilde{V}$ of x_α in A such that $f(\overline{\tilde{V}})^o \subseteq \tilde{U}_1 \times \tilde{U}_2$.

By lemma 3 – 2 (a) , we then have

$f_1(\overline{\tilde{V}})^o \subseteq \tilde{U}_1$ and $f_2(\overline{\tilde{V}})^o \subseteq \tilde{U}_2$.

So that f_1 and f_2 are fuzzy super continuous conversely ,

Let x_α be any fuzzy point of A and $\tilde{W}$ be any open fuzzy q-nbd of f(x_α) in X×Y.

Then by lemma 3 – 2 (b) , there exist open fuzzy q-nbds $\tilde{U}_1$ of $f_1(x_\alpha)$ and $\tilde{U}_2$ of $f_2(x_\alpha)$ such that $f(x_\alpha) q \tilde{U}_1 \times \tilde{U}_2 \subseteq \tilde{W}$. Also since f_1 and f_2 are fuzzy super continuous , there exist open fuzzy q-nbd $\tilde{V}_1$ and $\tilde{V}_2$ of x_α in A such that $f_1(\overline{\tilde{V}}_1)^o \subseteq \tilde{U}_1$ and $f_2(\overline{\tilde{V}}_2)^o \subseteq \tilde{U}_2$, so that
$f_1(\overline{\tilde{V}}_1)^o \times f_2(\overline{\tilde{V}}_2)^o \subseteq \tilde{U}_1 \times \tilde{U}_2$

Now by lemma 3 – 2 (c) we have $f(\overline{\tilde{V}})^o \subseteq \tilde{U}_1 \times \tilde{U}_2$, where $\tilde{V} = \tilde{V}_1 \cap \tilde{V}_2$ and $\tilde{V}$ is obviously open fuzzy q-nbd .

Hence f is fuzzy super continuous function.

References

[1]. S.P. Arya and R. Gupta, on strongly continuous mapping, kyungpook Math. J. 14(2) (1974) 131 – 143 .

[2]. K.K. Azad, on fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity , J. Math. Anal. Appl. 82 (1) (1981) 14 – 32.

[3]. Casanovas J. Torrens, J., An axiomatic approach to fuzzy cardinalities of finite fuzzy sets , fuzzy sets and systems. 133 (2003), no.2, 193-209.

[4]. C.L. Chang, Fuzzy topological spaces , J. Math. Anal. Appl. 24 (1968) 182 – 190.

[5]. B. Hutton and I.L. Reilly, Separation axioms in fuzzy topological spaces, Fuzzy sets and systems 3 (1980) 127 – 141 .

[6]. N. Levine , strong continuity in topological spaces, Amer. Math monthly 67 (1960) 269.

[7]. S. Nada , on fuzzy topological spaces, Fuzzy sets and systems 19(1986) 193 – 197.
[8]. Pao – Ming Pu and Ying – Ming Liu, Fuzzy topology, In. Neighborhood structure of fuzzy point and Moore-smith convergence , J. Math. Anal. Appl. 76 (1980) 571 – 599.
[9]. C.K. Wong , Fuzzy topology ; product and quotient theorems. J. Math. Anal. Appl. 45 (1974).
[10]. C.K. Wong , Fuzzy points and local properties of fuzzy topology , J. Math. Anal. Appl. 46 (1974) 316 – 328.
[11]. L.A. Zadeh, Fuzzy sets , inform , and control 8 (1965) 338 – 353.