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On The Ergodic Behaviour of Fuzzy Markov Chains

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Abstract: *Stochastic stability of Markov chains has an almost complete theory and forms a foundation for several other general techniques. A fuzzy Markov system is proposed and describe both determined and random behavior of complex dynamic systems. In this paper we study the ergodic behavior of a fuzzy Markov chain, and Consequently their weak and strong ergodic behavior.*
Keywords: *Fuzzy Markov Chain, Fuzzy Transition Probability Matrix, Non-Stationary Markov chain.*

I. Introduction

A fuzzy Markov system is proposed to describe both determined and random functioning of Complex dynamic systems. Most fuzzy logic applications are intended for Control and analytic purpose[5,4]. Another group of application is system state prediction[3] conventional fuzzy systems cannot operate with random phenomena.

Control processes in real life plants consist of determined and random elements. Stochastic processes can be described using a Markov modeling approach[2]. However, this approach allows simulation of a limited number of system states depending on state quantification. Furthermore the transition probability matrix must have large size to achieve high accuracy of modeling. This disadvantage can be avoided using a combination of Markov modeling with fuzzy logic.

In order to extend the application area of both techniques a fuzzy Markov modeling approach was proposed[1].

Therefore fuzzy Markov systems could be used for smooth non-linear approximation of a multidimensional probability density function. In case, a Markov model represents a fuzzy inference system with the transition probability matrix stored within the rule base.

Stochastic processes with a dynamic system can often be assumed to be stationary and ergodic. In this case the Markov chain is homogeneous and its dynamics are described by the transition probability matrix P. In this paper we study the ergodic behavior of fuzzy Markov chains and consequently the concepts weak ergodicity and strong ergodicity of fuzzy Markov chains.

II. Fuzzy Markov Chain

In this paper we proposed the set of possible limiting distributions for finite state Markov chain with fuzzy transition probabilities by which we mean a non-stationary Markov chain defined by the stochastic process.

{X(t); t=0,1,2…} wtth transition probabilities $P_{ij}(t)=P\{X(t+1)=j|X(t)=i\}$ $1\leq i, j \leq n$

Which satisfy the condition $\alpha_{ij}\leq P_{ij}(t) \leq\beta_{ij}$ for each t=0,1,2,… where $0 \leq \alpha_{ij}\leq \beta_{ij}\leq 1$

Let $S=\{x;x=(x_1,\ldots x_n\},\sum_{i=1}^{n} x_i = 1 ; x \geq 0\}$ i.e. the set of all n-dimensional probability vectors. The norm of a vector $x\in R^n$ is defined by $\|x\| = \sum_{i=1}^{n} |x|$ and we topologize the closed subsets of the metric space (S, || ||) with the Hausdorff metric d defined by

$$\delta(A,B) = \max_{x\in A} \min_{y\in B} ||x - y||$$

$$d(A,B)=\mathrm{Max}[\delta(A,B),\delta(B,a)]$$

for any closed $A,B\subseteq S$. We also define

$x_i(t)=P_r\{X(t)=i\}$

f_i, denote the fuzzy states of a Markov chain without loss of generality let f_r denote the initial fuzzy state, f_s denote the terminal fuzzy state and f_j denote the inter mediate fuzzy state.

III. Ergodic Coefficients of Fuzzy Matrix

Definition:3.1 Let P be a fuzzy stochastic matrix the ergodic coeffecient of P denoted as α(P) is defined by

$$\alpha(P) = 1 - \sup_{fr,fs} \sum_{fj=1}^{\infty} [P_{frfj} - P_{fsfj}]^{+} \qquad (1.1)$$

Where $[P_{frfj} - P_{fsfj}]^{+} = \max(0, P_{frfj} - P_{fsfj})$.

Theorem: 3.1 Let P be a fuzzy stochastic matrix, then

$$\alpha(P) = \inf_{fr,fs} \sum_{fj=1}^{\infty} \min(P_{frfj}, P_{fsfj})$$

Proof: Let fr and fs be fixed,
Since $(P_{frfj} - P_{fsfj})^+ = [P_{frfj} - \min(P_{frfj}, P_{fsfj})]$ and since $\sum_{fj=1}^{\infty} P_{frfj} = 1$
We have

$$1 - \sum_{fj=1}^{\infty} [P_{frfj} - P_{fsfj}]^+ = 1 - \sum_{fj=1}^{\infty} [P_{frfj} - \min(P_{frfj}, P_{fsfj})]$$

$$= \sum_{fj=1}^{\infty} \min(P_{frfj}, P_{fsfj})$$

Taking the infimum of both sides over fr,fs we get

$$\inf_{fr,fs} \sum_{fj=1}^{\infty} \min(P_{frfj}, P_{fsfj}) = \inf_{fr,fs} [1 - \sum_{fj=1}^{\infty} (P_{frfj} - P_{fsfj})]^+$$

$$= 1 - \sup_{fr,fs} \sum_{fj=1}^{\infty} [P_{frfj} - P_{fsfj}]^+$$

It sometimes more convenient to use 1- α(P) instead of α(P) itself. In view of this we define

δ(P)=1- α(P)

and δ(P) the delta coefficient of P.
therefore $\delta(P) = 1 - \inf_{fr,fs} \sum_{fj=1}^{\infty} \min(P_{frfj}, P_{fsfj})$

Theorem: 3.2 If P and Q are fuzzy stochastic matrices the δ(QP)< δ(Q)δ(P).
Proof: In definition 3.1 we introduced the notation $a^+=\max(0,a)$. If we introduce a^- to denote max(0,-a) then we have $a=a^+-a^-$.
Employing this notation we see that for any two rows i and k of a fuzzy stochastic matrix Q we have

$$\sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^+ - \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^-$$

This is true since

$$\sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^+ - \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^-$$

$$= \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})$$

$$= 1 - 1$$

$$= 0$$

If we define
QP = R = (γ_{frfj})
Then δ(QP)= δ(R)

$$= \sup_{fr,fs} \sum_{fl=1}^{\infty} [\gamma_{frfl} - \gamma_{fsfl}]^+$$

For the moment fix fr and fs and consider

$$\sum_{fl=1}^{\infty} [\gamma_{frfl} - \gamma_{fsfl}]^+ = \sum_{fl=1}^{\infty} [\sum_{fj} q_{frfj} P_{fjfl} - q_{fsfj} P_{fjfl}]^+ \qquad (1.2)$$

Let E={l: $\sum_{fj} (q_{frfj} - q_{fsfj}) P_{fjfl} > 0$}
That is E denotes those columns l, for which the values$\gamma_{frfl} - \gamma_{fsfl}$ is positive using the set E, (1.2) can be written

$$\sum_{fl \in E} \sum_{fl=1}^{\infty} [q_{frfj} - q_{fsfj}] P_{fjfl} \qquad (1.3)$$

The order summation can be interchanged using Funinis theorem so (1.3) is equal to

$$\sum_{fl=1}^{\infty} [q_{frfj} - q_{fsfj}] \sum_{fl \in E} P_{fjfl} = \sum_{fj=1}^{\infty} [(q_{frfj} - q_{fsfj})^+ - (q_{frfj} - q_{fsfj})^-] \sum_{fl \in E} P_{fjfl}$$

$$= \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{+} \sum_{fl\in E} P_{fjfl} - (q_{frfj} - q_{fsfj})^{-} \sum_{fl\in E} P_{fjfl}$$

Now since all the terms in this difference are non-negative the difference is made larger if the first term is increased and the second decreased. That is in place of $\sum_{l\in E} P_{fjfl}$ we substitute $sup_{fj} \sum_{fl\in E} P_{fjfl}$ in the first term of the difference and
$inf_{l\in E} \sum_{l\in E} P_{fjfl}$ in the second term, using the first that

$$\sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{+} = \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{-}$$

We get

$$\sum_{fl=1}^{\infty} [\gamma_{frfl} - \gamma_{fsfl}]^{+} \leq \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{+} [sup_{fj} \sum_{l\in E} P_{fjfl} - inf_{fj} \sum_{l\in E} P_{fjfl}]$$

$$= \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{+} sup_{fj1fj2} \sum_{l\in E} (P_{fj1fl} - P_{fj2fl})$$

$$< \sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{+} sup_{fj1fj2} \sum_{l=1}^{\infty} (P_{fj1fl} - P_{fj2fl})^{+}$$

The last expression is simplified to

$$\sum_{fj=1}^{\infty} (q_{frfj} - q_{fsfj})^{+} \delta(p)$$

So taking the supremums of both sides over fr and fs we get

$$\delta(QP) < \delta(Q)\delta(P)$$

Theorem: 3.3 For all matrices A and B the following inequality holds $\| AB \| < \| A \| . \| B \|$

Proof: The case where $\| A \|$ or $\| B \|$ is either zero or infinite, is easily done. Therefore assume that $0 < \| A \| < \infty$ and $0 < \| B \| < \infty$. Note that the (fr,fj)th element of AB is given by

$$\sum_{fk=1}^{\infty} a_{frfs} - b_{fsfj}$$

Then

$$\| AB \| = \sup_{fr} \sum_{fj=1}^{\infty} \sum_{fs=1}^{\infty} a_{frfs} - b_{fsfj}$$

$$< \sup_{fr} \sum_{fj=1}^{\infty} \sum_{fs=1}^{\infty} a_{frfs}\, b_{fsfj}$$

By Funinis theorem the last expression is equal to

$$\sup_{fr} \sum_{fs=1}^{\infty} a_{frfs} \sum_{fs=1}^{\infty} b_{fsfj}$$

$$< \sup_{fr} \sum_{fs=1}^{\infty} a_{frfs} \sup_{fs} \sum_{fj=1}^{\infty} b_{fsfj}$$

$= \| A \| . \| B \|$

IV. Weak Ergodicity

In this section we give several theorems in which the ergodic coefficient can be used to determine whether a non-stationary Markov chain is weakly ergodic.

Definition: 4.1 A non-stationary Markov chain is called Weakly ergodic if for all m $\lim_{fs\to\infty} \sup_{f(0)g(0)} \| f^{(m,fs)} - g^{(m,fs)} \| = 0$

Where $f^{(0)}$ and $g^{(0)}$ are starting vectors.

Theorem: 4.1 A non-stationary fuzzy Markov chain is weakly Ergodic if and only if for all m

$$\delta\left(P^{(m,fs)}\right) \to 0 \; as \; fs \to \infty$$

Proof: Assume that for all m, $\delta\left(P^{(m,fs)}\right) \to 0$ as $fs \to \infty$.

Let $f^{(0)}$ and $g^{(0)}$ be any two starting vectors and let m and fs be fixed. Define a fuzzy stochastic matrix Q such that the first row is $f^{(0)}$and the remaining row are $g^{(0).}$

Consider the matrix $QP^{(m,fs)}$=R. The first row of the matrix R is $f^{(m,fs)}$ and the remaining rows are $g^{(m,fs)}$. Therefore since the value of δ(R) is determined by the rows of R we have

$$\delta\left(QP^{(m,fs)}\right) = \delta(R)$$
$$= \frac{1}{2} \sup_{frfj} \sum_{fl=1}^{\infty} \gamma_{frfl} - \gamma_{fjfl}$$
$$= \frac{1}{2} \sup_{frfj} \sum_{fl=1}^{\infty} f_l^{(m,fs)} - g_l^{(m,fs)}$$
$$= \frac{1}{2} \parallel f_l^{(m,fs)} - g_l^{(m,fs)} \parallel$$

Using theorem 3.2 and the fact that δ(Q)<1, we note that

$$\parallel f_l^{(m,fs)} - g_l^{(m,fs)} \parallel$$
$$= 2\delta\left(QP^{(m,fs)}\right)$$
$$\leq 2\delta(Q)\delta\left(P^{(m,fs)}\right)$$
$$< 2\delta\left(P^{(m,fs)}\right)$$

By assumption the right hand side goes to zero for each m as fs→∞. Further more it goes to zero independently if $f^{(0)}$ and $g^{(0)}$. So the chain is Weakly ergodic.

Conversely assume that for all m, $\sup_{f(0)g(0)} \parallel f^{(m,fs)} - g^{(m,fs)} \parallel \to 0 \; as \; k \to \infty$. Define $f^{(0)}$ to be a starting vector with a one in the ith position and zero elsewhere and define $g^{(0)}$ to be starting vector with a one in the jth position and zeros elsewhere.

Note that the vectors

$$f^{(0)}\, p^{(m,fs)} = f^{(m,fs)} \; and$$
$$g^{(0)}\, p^{(m,fs)} = g^{(m,fs)}$$

Are the ith and jth rows of $p^{(m,fs)}$respectively./ So

$$\sum_{fl=1}^{\infty} P_{frfl}{}^{(m,fs)} - P_{fjfl}{}^{(m,fs)}$$
$$= \parallel f^{(m,fs)} - g^{(m,fs)} \parallel$$
$$< \sup_{f(0)g(0)} \parallel f^{(m,fs)} - g^{(m,fs)} \parallel$$

Since the Inequality holds for all fr, fj it follows that

$$2\delta\left(P^{(m,fs)}\right) = \sup_{frgj} \sum_{fl=1}^{\infty} P_{frfl}{}^{(m,fs)} - P_{fjfl}{}^{(m,fs)}$$
$$< \sup_{f(0)g(0)} \parallel f^{(m,fs)} - g^{(m,fs)} \parallel$$

And the last term tends to zero for all m as fs→ ∞ by assumption.

Theorem: 4.2 Let {Xn} be a non-stationary fuzzy Markov chain with transition matrices $\{Pn\}_{n=1}^{\infty}$. The chain {Xn} is weakly ergodic if and only if there exists a subdivision of P_1,P_2…. In to blocks of matrices $[P_1P_2P_3\ldots.P_n]$ $[P_{n1+1}P_{n2+1}\ldots.P_{nj+1}]$ such that

$\sum_{fl=0}^{\infty} \alpha(P^{(nj,nj+1)}) = \infty \; where$ n_0=0.

Proof: The first part of the proof depends on the following result from analysis.

If $\{\in j\}_{j=1}^{\infty}$ is a sequence of numbers with 0<$\in j$<1 for all j then the product$\{\prod\}_{n=1}^{\infty}(1-\in_j)$ diverges to zero as n→∞ if and only if

$\sum_{j=m}^{\infty} \in_j = \infty$. If $\sum_{fl=0}^{\infty} \alpha(P^{(nfj,nfj+1)}) = \infty \; for \; all \; i.$

Using δ(P)=1- α(P) we see that

$$\prod_{fj=i}^{fl} \delta\left(P^{(nfj,nfj+1)}\right) = \prod_{fj=i}^{fl} \left[1 - \alpha\left(P^{(nfj,nfj+1)}\right)\right] \qquad (1)$$

As fl→∞.

Finally let m given and define fr=min{fj:n_{fj}>m} and for fs>m define l=max{fj:n_{fj}<k} and note that fl→∞ as fs→∞. Then using (1) and theorem we have

$$\delta(P^{(m,fs)}) \le \delta(P^{(m,nfr)}) \prod_{fj=i}^{fl-1} \delta\left(P^{(nfj,nfj+1)}\right) . \delta\left(P^{(nfj,nfj+1)}\right) \to 0 \quad as\ fs \to \infty$$

Conversely assume that the chain is Weakly ergodic that is for all m

$$\delta(P^{(m,fs)}) \to 0 \quad as\ k \to \infty$$

This implies that for all m

$$\alpha(P^{(m,fs)}) \to 1 \quad as\ k \to \infty$$

Hence for m=0=n_0 there exists n1 such that $\alpha(P^{(0,n1)}) > 1/2$. Likewise given n1 there exists n2>n1 such that $\alpha(P^{(n1,n2)}) > 1/2$

Proceeding this way we get

$$\sum_{fl=0}^{fs} \alpha(P^{(nfj,nfj+1)}) > \frac{fs+1}{2}$$

Which diverges as $fs \to \infty$

Hence we have constructed a partition of the original sequence of matrices p1,p2,…. In to blocks satisfying

$$\sum_{fl=0}^{\infty} \alpha(P^{(nfj,nfj+1)}) = \infty$$

V. Strong Ergodicity

In this section we present some theorems that give sufficient conditions for a chain to be strongly ergodic.

Theorem: 5.1 A non- stationary fuzzy Markov chain is strongly ergodic if and only if there is a sequence of constant fuzzy stochastic matrices {Q_m} and for each m , there is a sequence of constant stochastic matrices {Q_{mk}} such that

$$(i)\ \lim_{fs\to\infty} \| P^{(m,fs)} - Q_{mfs} \| = 0 \ \ and$$

$$(ii)\ \lim_{fs\to\infty} \| Q_{mfs} - Q_m \| = 0$$

Proof: Assume sequence of constant matricas {Qm} and {Qmfs} satisfying conditions (i) and (ii) exist. Since

$$\| P^{(m,fs)} - Q_m \| < \| P^{(m,fs)} - Q_{mfs} \| + \| Q_{mfs} - Q_m \|$$

It follows that for all m,

$$\lim_{fs\to\infty} \| P^{(m,fs)} - Q_{mfs} \| = 0$$

Clearly if Qm=0 for all m, then by theorem,

A non stationary fuzzy Markov chain with transition matrices {pn} is strongly ergodic if and only if there exists a constant matrix Q such that for each m

$$\lim_{fs\to\infty} \| P^{(m,fs)} - Q \| = 0$$

The chain will be strongly ergodic.

In other words, it suffices to show that Qm is the same constant matrix for all m. It is easy to show that PmQm=Qm

We also know from theorem that for any two matrices A and B,

$\| AB \| < \| A \| . \| B \|$ hence we get

$$\| Q_{m-1} - Q_m \| < \| Q_{m-1} - P^{(m,fs)} \| + \| P_m P^{(m,fs)} - P_m Q_m \| + \| P_m Q_m - Q_m \|$$
$$= \| Q_{m-1} - P^{(m-1,fs)} \| + \| P_m (P^{(m,fs)} - Q_m \|$$
$$< \| Q_{m-1} - P^{(m-1,fs)} \| + \| P_m \| \| (P^{(m,fs)} - Q_m \|$$
$$< \| Q_{m-1} - P^{(m-1,fs)} \| + \| (P^{(m,fs)} - Q_m \|$$

By letting $k \to \infty$ we get $\| Q_{m-1} - Q_m \| = 0$ which implies that $Q_{m-1} = Q_m$ for all m.

Conversely of the chain is strongly ergodic then by setting Qm=Qmfs=Q for all m and fs it follows that (i) and (ii) are true.

Definition: 5.1 Let a be the class of stochastic matrices P for which there exists at least one non-negative left eigen vector corresponding to the eigen value ψ such that

$$\| \psi \| = 1$$

Theorem: 5.2 Let {Pn} be a sequence of transition matrices corresponding to a non-stationary weakly ergodic Markov chain with Pn∊a for all n. If there exists a corresponding sequence of left eigen vectors ψn satisfying

$\sum_{fj=1}^{\infty} \| \psi_{fj} - \psi_{fj+1} \| < \infty$ (2)

Then the chain is strongly ergodic.

Proof: The condition imposed on the left eigon vectors is stronger than assuming $\{\psi n\}_{n=1}^{\infty}$ converges in norm to some vector ψ. Hence we can define $\psi = \lim_{n\to\infty} \psi_n$ and note that $\| \psi_n - \psi \| \to 0\ as\ n \to \infty$.

Since all of the ψ_n's have the property that their components are non-negative and add to one ψ will also have this property.

Define Q to be the constant stochastic matrix with each row equal to ψ.

In order to show {Pn} is strongly ergodic it is sufficient to show $\| (P^{(m,fs)} - Q_m \| \to 0\ as\ k \to \infty$ for all m.

For notational convenience let Qn denote the constant stochastic matrix with rows equal to ψ_n. Let m be fized using the triangle inequality and the fact that $P^{(m,fs)} = P^{(m,fl)}P^{(fl,fs)}$ we get

$$\| P^{(m,fs)} - Q \| < \| P^{(m,fs)} - Q_{fs} \| + \| Q_{fs} - Q \|$$

$$< \| P^{(m,fl)}P^{(fl,fs)} - Q_{fl+1}P^{(fl,fs)} \| + \| Q_{fl+1}P^{(fl,fs)} - Q_{fs} \| + \| Q_{fs} - Q \| \quad (3)$$

In order to prove that

$$\lim_{fs\to\infty} \| P^{(m,fs)} - Q_{mfs} \| = 0$$

We let ∈>0 be given and show that there exists k such that for all fs>k $\| P^{(m,fs)} - Q \| < \epsilon$ we do this by making each of the three terms on the right hand side of (3) less than $\epsilon/3$

We first consider the middle term of the right hand side of (3) and note that since $Q_{fl+1}P^{(fl,fs)} = Q_{fl+1}$ we have

$$Q_{fl+1}P^{(fl,fs)} = Q_{fl+1}P^{(fl+1,fs)}$$

$$= Q_{fl+1}P^{(fl+1,fs)} - Q_{fl+2}P^{(fl+1,fs)} + Q_{fl+2}P^{(fl+1,fs)}$$

$$= (Q_{fl+1} - Q_{fl+2})P^{(fl+1,fs)} + Q_{fl+2}P^{(fl+1,fs)}$$

Repeating this procedure on $Q_{fl+2}P^{(fl+1,fs)}$ we get

$$Q_{fl+1}P^{(fl,fs)} = (Q_{fl+1} - Q_{fl+2})P^{(fl+1,fs)} + (Q_{fl+2} - Q_{fl+3})P^{(fl+2,fs)} + Q_k$$

Hence using the triangle inequality theorem and the fact that $\delta(P^{(fj,fs)}) < 1$ we get

$$\| Q_{fl+1}P^{(fl,fs)} - Q_{fs} \| = \| \sum_{j=fl+1}^{fs-1} (Q_{fj} - Q_{fj+1})P^{(fl,fs)} \|$$

$$< \sum_{j=fl+1}^{fs-1} \| (Q_{fj} - Q_{fj+1})P^{(fj,fs)} \|$$

$$< \sum_{j=fl+1}^{fs-1} \| (Q_{fj} - Q_{fj+1}) \| \delta P^{(fj,fs)}$$

$$< \sum_{j=fl+1}^{fs-1} \| (Q_{fj} - Q_{fj+1}) \| \quad (4)$$

Since by construction Qfj has all its rows equal to ψ_{fj}, it follows that

$$\| (Q_{fj} - Q_{fj+1}) \| = \| (\psi_{fj} - \psi_{fj+1}) \|$$

Hence using assumption (2) we can choose fl*>m such that for all k>fl*

$$\| Q_{fl*+1}P^{(fl*,fs)} - Q_{fs} \| = \| \sum_{j=fl*+1}^{fs-1} (Q_{fj} - Q_{fj+1}) \|$$

$$= \| \sum_{j=fl*+1}^{fs-1} (\psi_{fj} - \psi_{fj+1}) \|$$

$<\epsilon/3$

With fl* fixed, next consider the first term of the right hard side of eigen vector. Since $P^{(m,fl*)}$ and Q_{fl*+1} are stochastic matrices it follows that

$$\| P^{(fl*,fs)} - Q_{fl*+1} \| < 2$$

So by the theorem

$$\| P^{(m,fl*)}P^{(fl*,k)} - Q_{fl*+1}P^{(fl*,fs)} \| \leq \| P^{(m,fl*)} - Q_{fl*+1} \| \delta P^{(fl*,fs)}$$

$$\leq 2\delta P^{(fl*,fs)}$$

Using the assumption that the chain is weakly ergodic we can find k1>fl* such that for all fs>k1

$$\delta P^{(fl*,fs)} <\in/6$$

For such values of fs

$$\| P^{(m,fl*)} P^{(fl*,k)} - Q_{fl*+1} P^{(fl*,fs)} \| < \in/3$$

For the third term on the right hand side of eigen vector we note that ψ_k converges in norm to ψ and so

$$\log_{fs\to\infty} \| Q_{fs} - Q \| = 0$$

Here there exist k2 such that for all fs>k2 we have

$$\| Q_{fs} - Q \| < \in/3$$

Therefore for
Fs>max(k1,k2) we have
$\| P^{(m,fs)} - Q \| < \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in.$

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