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# Doubly Geodetic Numberof a Graph

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***Abstract:*** *Geodetic number of a graph is one of the widely studied graph theoretic parameters concerning geodesic convexity in graphs. In this paper, we introduce a variant of this parameter namely, doubly geodetic number of a graph. For a connected graph $G$, a set $S$ of vertices of $G$ is called a doubly geodetic set of $G$ if each vertex in $V - S$ lies on atleast two distinct geodesics of vertices in $S$. The doubly geodetic number $\ddot{d}g(G)$ of $G$ is the minimum cardinality of a doubly geodetic set. Any doubly geodetic set of cardinality $\ddot{d}g(G)$ is called $\ddot{d}g$-set of $G$. In this paper, the doubly geodetic number of certain standard graphs is determined.*
***Keywords:*** *doubly geodetic number, doublygeodetic set, geodesic, geodetic number, geodetic set.*

## I. Introduction

Let $G = (V, E)$ be a connected graph with node set $V = V(G)$ and the edge set $E = E(G)$. If $e = uv \in E(G)$, then the vertices $u$ and $v$ are adjacent. For vertices $u$ and $v$ in $G$, the distance $d(u, v)$ is the length of a shortest $u-v$ path in $G$. An $u-v$ path of length $d(u, v)$ is called an $u - v$ geodesic. A vertex $w$ is said to lie on an $u - v$ geodesic $P$ if $w$ is a vertex of $P$ including the vertices $u$ and $v$. The eccentricity $e(u)$ of a vertex $u$ is defined by $e(u) = max\{d(u, v) : v \in V\}$. The minimum and the maximum eccentricity among vertices of $G$ is its radius $r$ and diameter $d$, respectively. For graph theoretic notation and terminology, we follow [1,2].

In [1] Harary et al introduced a graph theoretical parameter, the geodetic number of a graph and further studied it in [3,4,5,6]. The geodetic closure of the set $S \subset V(G)$ is $S^c = \{x \in V: (\exists u, v \in S), x$ is in some $u-v$ geodesic$\}$. The geodetic number of a graph $G$ is defined as $g(G) = min\{|S| : S \subset V and S^c = V\}$. An equivalent definition for geodetic number of a graph $G$ is given in[4] as follows, Let $I(u, v)$ be the set (interval) of all vertices lying on some $u - v$ geodesic of $G$, and for a nonempty subset $S$ of $V(G)$, $I(S) = \bigcup_{u,v\in S} I(u, v)$. The set $S$ of vertices of $G$ is called a geodetic set in $G$ if $I(S) = V$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of aminimum geodetic set in $G$ is called the geodetic number $g(G)$. In [6], it is shown that theproblem of determining the geodetic number of a graph is an NP-hard problem. The geodetic number of a graph is also referred as geodomination number [7]. Chartrand, Harary, Swart and Zhang were the first to study the geodetic concepts in relation to domination. Later, it was further studied by several others.[8,9].

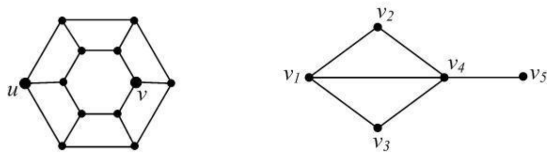
The edge geodetic number of graph was introduced in [10] and further studied in [11]. In [12], Santhakumaran et al introduced a variant, the double geodetic number of a graph. A set $S$ of vertices of $G$ is called a double geodetic set of $G$ if for each pair of vertices $x, y$ in $G$ there exist vertices $u, v$ in $S$ such that $x, y \in I(u, v)$.The double geodetic number $dg(G)$ of $G$ is minimum cardinality of a double geodetic set. Any geodetic set of cardinality $dg(G)$ is called $dg$-set of $G$. The geodetic concepts have many remarkable applications in communication network design and designing the route for a shuttle. The edge geodetic set has more real life applications than the geodetic sets. In particular, they are more advantageous in the case of regulating and routing the goods vehicles to transport the commodities to important places. The geodetic concepts are also applied to other areas like telephone switching centers, facility location, distributed computing, image and video editing, neural networks and data mining.

## II. Doubly Geodetic Number of a Graph

In this section, we formally define the doubly geodetic number of a graph.Let $G$ be a connected graph with at least two vertices. A vertex $x$ is said to be geodominated by the pair of vertices$\{u, v\}$ if lies $x$ lie on some $u - v$ geodesic. The geodetic interval $I[u, v]$ consists of $u, v$ together with all vertices geodominated by the pair$\{u, v\}$.

Let $|I[u, v]|$ denote the number of vertices in $I[u, v]$. If $|I[u, v]| = d(u, v) + 1$, then there exist a unique $u - v$ geodesic. If $|I[u, v]| \geq d(u, v) + 1$, then there exists more than one geodesics between $u$ and $v$. Let $g_p(u, v)$ be a $u - v$ geodesic and $I[g_p(u, v)]$ consists of $u, v$ together with all vertices in $g_p(u, v)$. Its clear that $|I[g_p(u, v)]| = d(u, v) + 1$.We say that two geodesics $g_p(a, b)$ and $g_q(u, v)$ are distinct if $I[g_p(a, b)] \neq I[g_q(u, v)]$.A set $S$ of vertices of $G$ is called a doubly geodetic set of $G$ if each vertex in $V - S$ lies on at least two distinct geodesics of vertices in $S$.The doubly geodetic number of $\ddot{d}g(G)$ isminimum cardinality of a doubly geodetic set. Any doubly geodetic set of cardinality $\ddot{d}g(G)$ is called $\ddot{d}g$-set of $G$. In other words, in the $I[S]$ of the doubly geodetic set$S$ of$G$ the vertices of $V - S$ should occur at least twice.

**Example 2.1.** For the graph$G_1$in Fig.1(a), it is clear that$S_1 = \{u, v\}$is a$\ddot{d}g$-set of$G_1$. Thus $\ddot{d}g(G_1) = 2 = g(G_1)$. The double geodetic number and geodetic number of a graphcan be different. For the graph $G_2$in Fig.1(b), it is clear that no 2-element or no 3-element subset of $G_2$ is a doubly geodetic set of $G_2$. $S_2 = \{v_1, v_2, v_3, v_5\}$ is a doubly geodetic set, so $\ddot{d}g(G_2) = 4$. But$g(G_2) = 3$.




(a) (b)

**Fig 1:** (a) $\ddot{d}g = g$, (b) $dg \neq g$

A vertex $v$ in a graph $G$ is an extreme vertex if the subgraph induced by its neighborhood is complete. Every extreme vertex of a graph is an end-vertex of every geodesic containing it. A set $S$ of vertices of a connected graph $G$ is called a cutset of $G$ if the graph $G - S$ is not connected. In particular, a vertex $v \in V(G)$ is a cut-vertex of $G$ if $G - v$ is disconnected. The connectivity $\kappa(G)$ of a connected non-complete graph $G$ is the minimum cardinality of a cutset of $G$. The following theorems will be used in the sequel.

**Theorem 2.2.** [7] Every geodetic set of a graph$G$contains its extreme vertices. In particular,if the set of extreme vertices $S$ of $G$is a geodetic set of$G$, then $S$ is the unique minimum geodetic set of $G$.

**Theorem 2.3.** [7] Let$G$be a connected graph with a cut vertex$v$. Then every geodetic setof $G$ contains at least one vertex from each component of $G - v$.

## III. Main Results

**Theorem 3.1.** For any graph $G$, $2 \leq g(G) \leq \ddot{d}g(G) \leq n$.
**Proof:** A geodetic set needs at least two vertices and therefore$g(G) \geq 2$. It is clear that every doubly geodetic set is also a geodetic set and so $g(G) \leq \ddot{d}g(G)$. Since the set of all vertices of $G$ is a doubly geodetic set of $G$,$\ddot{d}g(G) \leq n$.

**Remark 3.2.** The bounds in theorem 3.1 are sharp. For the complete graph$K_n (n \geq 2)$, wehave $\ddot{d}g(K_n) = n$ and for the graph $G$, in Fig.1(a),$\ddot{d}g = 2$. The graphs with double geodetic number 2 are investigated in the sequel.

**Theorem 3.3.** Every$\ddot{d}g$-set of a graph contains its extreme vertices.
**Proof:** Since every doubly geodetic set is a geodetic set, the result follows from theorem 2.2.

**Corollary 3.4.** For a graph $G$ of order $n$ with $k$extreme vertices, $max\{2, k\} \leq \ddot{d}g(G) \leq n$.
**Proof:** This follows from theorems 3.1 and 3.3.

**Theorem 3.5.** Let$G$be a connected graph with a cut vertex$v$. Then each doubly geodeticset contains at least one vertex from each component of $G - v$.
**Proof:** This follows from theorem 2.3 and the fact that every doubly geodetic set is a geodetic set.

**Theorem3.6.** Let $T$ be a tree with $n$ vertices and $l$ leaves, then $\ddot{d}g(T) = g(T) = l$, where $l \geq 3$.
**Proof:** Let $S$ be the set of all end-vertices of $T$. By theorem 3.3,$\ddot{d}g(T) \geq S$. On the other hand, for an internal vertex $v$ of $T$, there exist end-vertices $x, y$ of $T$such that $v$ lies on the unique $x\!-y$geodesic in $T$. Thus, an internal vertex $v$of $T$, will lie on exactly $\binom{l}{2}$ distinct geodesics of vertices in $S$. Therefore, $d = |S|$. Since every geodetic set $S$of $T$must contain $S$by theorem 3.3, $S$is the unique minimum doubly geodetic set.

**Corollary3.7.** For integers$k, n$, such that $3 \leq k \leq n$there exist a connected graph$G$with$g(G) = \ddot{d}g(G) = k$.
**Proof:** For $k = n$, let $G = K_n$. Then, $g(G) = \ddot{d}g(G) = n = k$. Also, for each pair of integers$k, n$with $3 \leq k < n$, there exists a tree of order$n$with$k$end vertices. Hence the result followsfrom theorem 3.6.
**Theorem 3.8.** For any two positive integers$a, b$with$a \geq b + 1$ and$b > 2$ there exists a connected graph with

$|V(G)| = a$, $\ddot{d}g(G) = b$.
**Proof:** Let$P: u_0, u_1, \dots, u_{(a-b)}$be a path. Consider the graph$G$constructed from$P$byjoining $b-1$ new vertices to $u_0$. The graph $G$ is a tree of order $a$, with $b$ leaves. Then by theorem 3.6, $\ddot{d}g(G) = b$.

**Theorem 3.9.** If$G = K_n$or$K_n - e$, *i.e.* a graph obtained from$K_n$by removing an edge$e$, then $\ddot{d}g(G) = n$.
**Proof:** By theorem 3.1, $\ddot{d}g(G) \geq g(G)$. Also for $G = K_n$, we have, $\ddot{d}g(G) = g(G) = n$. It remains to show that, for $G = K_n - e$, $\ddot{d}g(G) = n$. Let $e = (x, y)$, where$x, y \in V(G)$ then $G$can be redrawn as in Fig.2. Each vertices $x$and $y$are adjacent to $n-2$ vertices of clique $K_{n-2}$ in $G$. Since $x$ and $y$ are extreme vertices, they both belong to $\ddot{d}g$ −set. Also every vertex in the clique $K_{n-2}$ will lie on only one geodesic. Thus$\ddot{d}g(G) = n$.

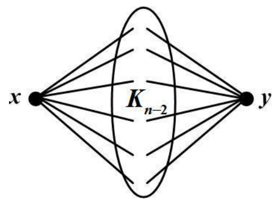


**Fig.**2: $K_n - e$, where $e = xy$

**Theorem3. 10.** If $G = K_n - \{e_1, e_2\}$, then $\ddot{d}g(G) = \begin{cases} 3, & if\, e_1, e_2\, are adjacent \\ 4, & if\, e_1, e_2\, are not adjacent \end{cases}$
**Proof:** Let $G = K_n - \{e_1, e_2\}$. We have the following two cases:
Case(i): When$e_1$ and $e_2$ are adjacent. Let $e_1 = xy$ and $e_2 = xz$ for some $x, y, z \in V(G)$, then $G = K_n - \{e_1, e_2\}$, can be redrawn as in Fig.3(a). Since $x, y, z$ are extreme vertices, they all belong to $\ddot{d}g$-set. Also, every vertex$V(G) - \{x, y, z\}$ lies on each of the $x - y$ and $x - z$ geodesics. Thus, $\ddot{d}g(G) = 3$.
Case(ii): When $e_1$ and $e_2$ are not adjacent. Let $e_1 = uv$ and $e_2 = xy$ for some $u, v, x, y \in V(G)$, then $G = K_n - \{e_1, e_2\}$, can be redrawn as in Fig.3(b). Since $u, v, x, y$ are extreme vertices, they all belong to $\ddot{d}g$-set. Also, every vertex $V(G) - \{u, v, x, y\}$ lie on each of the $u - v$ and $x - y$ geodesics. Thus, $\ddot{d}g(G) = 4$.

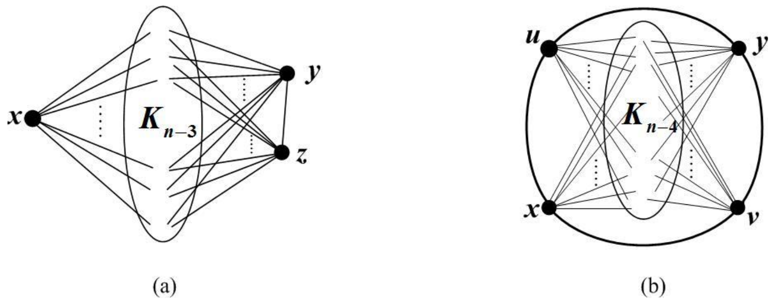



**Fig 3:** (a) $K_n - \{e_1, e_2\}$, where $e_1 = xy$ and $e_2 = xz$,
(b) $K_n - \{e_1, e_2\}$, where $e_1 = uv$ and $e_2 = xz$.

**Theorem3. 11.** If $P_n$be a path of order$n \geq 3$, then $\ddot{d}g(P_n) = 3$.
**Proof:** Let$P_n: u_1, u_2, \dots, u_n$be the path. The two end-vertices $u_1, u_n$ belong to $\ddot{d}g$-set say $S$. All the internal vertices lie exactly on only one $u_1 - u_n$geodesic. Inclusion of any one internal vertex to set $S$, will make all vertices of $V(P_n) - S$ doubly geodominated. Thus, $\ddot{d}g(P_n) = 3$

**Theorem3.12.** Let $C_n$be a cycle of order$n \geq 4$, then $\ddot{d}g(G) = \begin{cases} 4, & n is even \\ 5, & n is odd \end{cases}$
**Proof:** Let$C_n: u_1, u_2, \dots, u_n, u_1$be the cycle.
Case (i): When $n$ is even. Consider the induced subgraph $[A]$ on the vertices $u_1, u_2, \dots, u_{(n/2)+1}$. Clearly $[A] \sim P_{(n/2)+1}$, therefore by theorem 3.11 at least 3 vertices are required to doubly geodominate $V[A]$. Similarly for the induced subgraph $[B]$ on the vertices $u_{(n/2)+1}, u_{(n/2)+2}, \dots, u_1$, at least 3 vertices are required to doubly geodominate $V[B]$. Since $[A]$ and $[B]$ have two common vertices, at least $3 + 3 - 2$ i.e. 4 vertices are required to doubly geodominate $V(G)$. Thus $\ddot{d}g(G) \leq 4$. Let $S = \{u_1, u_2, u_{(n/2)+1}, u_{(n/2)+2}\}$, clearly $S$ is a $\ddot{d}g(G)$-set. Thus $\ddot{d}g(C_n) = 4$.
Case (ii): When $n$ is odd. Consider the induced subgraph $[A]$ on the vertices $u_1, u_2, \dots, u_{(n/2)+1}$. $[A] \sim P_{[n/2]+1}$,

therefore by theorem 3.11 at least 3 vertices are required to doubly geodominate $V[A]$. Similarly for the induced subgraph $[B]$ on the vertices $u_{[n/2]+1}, u_{[n/2]+2}, \dots, u_1$ at least 3 vertices are required to doubly geodominate $V\,[B]$. Since $[A]$ and $[B]$ have one common vertex, atleast 3+3-1 i.e. 5 vertices are required to doubly geodominate $V(G)$. Thus $\ddot{d}g(C_n) \leq 5$. Let $S = \{u_1, u_2, u_{[n/2]+1}, u_{[n/2]+2}, u_n\}$, clearly$S$is a $\ddot{d}g$-set. Thus$\ddot{d}g(C_n) = 5$.

**Theorem 3.13.** For integers$p_1 \leq p_2 \leq \cdots \leq p_k,\ p_i \geq 3,\ 1 \leq i \leq k$, let $G = K_{p_1, p_2, \dots, p_k}$ be a complete $k$-partite graph. Then $\ddot{d}g(G) = \min\{p_1, 6\}$

**Proof:** Let$U = \{u_1, u_2, \dots, u_{p_1}\}$and$W = \{w_1, w_2, \dots, w_{p_2}\}$be the two partite setsof $G$ of least cardinality, where $p_1 \leq p_2$. First, consider $3 \leq p_1 \leq 5$, clearly $S = U$ is a minimum doubly geodetic set of $G$. Now, let $p_1 \geq 6$and let $S = \{u_1, u_2, u_3, w_1, w_2, w_3\}$. Since, every vertex of $U - S$ lies on $w_1 - w_2, w_2 - w_3$and$w_1 - w_3$geodesics. Also, every vertex of$W - S$lies on$u_1 - u_2, u_2 - u_3$and $u_1 - u_3$ geodesics. And clearly every vertex of $V(G) - (U \cup W)$ lies on atleast two geodesics of vertices in $S$. It follows that$\ddot{d}g(G) \leq 6$.

It remains to prove that,$\ddot{d}g(G) \geq 6$. Let$X$be any 5-element subset of $V$. Let $A, B, C, D, E$ be any five partite set of $G$. If$X \subset A$ i.e. $|X \cap A| = 5$, then vertices of $A \backslash X$ does not lie on any geodesics of vertices in $X$. If $|X \cap A| = 4$ and $|X \cap B| = 4$then vertices of $A \backslash X$ does not lie on any geodesics of vertices in $X$.If $|X \cap A| = 3$ and $|X \cap B| = 2$then vertices of $A \backslash X$ lie on exactly one geodesicof vertices in $X$.If $|X \cap A| = 3$ and $|X \cap B| = |X \cap C| = 1$then vertices of $A \backslash X$ does not lie on any geodesics of vertices in $X$.If $|X \cap A| = |X \cap B| = 2$ and $|X \cap C| = 1$then vertices of $A \backslash X$ lie on exactly one geodesicof vertices in $X$.If $|X \cap A| = 2$ and $|X \cap B| = |X \cap C| = |X \cap D| = 1$then vertices of $A \backslash X$ does not lie on any geodesics of vertices in $X$.If $|X \cap A| = |X \cap B| = |X \cap C| = |X \cap D| = |X \cap E| = 1$then vertices of $A \backslash X$ does not lie on any geodesics of vertices in $X$.It follows that$\ddot{d}g(G) \geq 6$.

**Corollary 3.14.**For complete bipartite $K_{r,s}$and complete tripartite$K_{r,s,t}$graphs,$\ddot{d}g(K_{r,s}) = \min\{r, s, 6\}$ and$\ddot{d}g(K_{r,s,t}) = \min\{r, s, t, 6\}$, where integers $r, s, t \geq 3$.

**Proof:** The result followsfrom theorem 3.13.

**Theorem 3.15.** For the grid$G_{r,s}$,$\ddot{d}g(G_{r,s}) = 4$, where integers$r, s \geq 2$.

**Proof:** Let$G = G_{r,s}, r, s \geq 2$ and let $S$ be a $\ddot{d}g$-set of $G$. Suppose $S$ is a 3-element subset of $V(G)$. Then by theorem 3.12, there exist a vertex in the outer boundary, that is the outer cycle $C_{2(r+s-2)}$,such that it lies on at most one geodesic of vertices of$S$.See Fig.4(a).Therefore,$\ddot{d}g(G) \geq 4$. It can be easily verified that,$S = \{a, b, c, d\}$, where$a, b, c, d$arethe corner vertices is a doubly geodetic set of $G$.See Fig.4(b). Thus, it follows that$\ddot{d}g(G) = 4$.

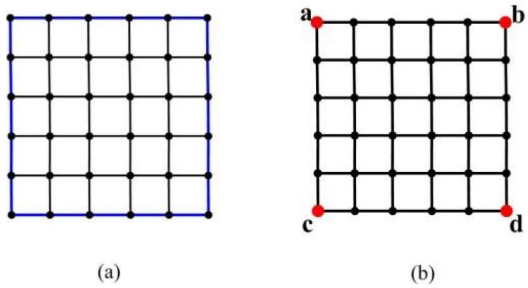



**Fig.4: (a)** The blue line denotes the outer cycle $C_{20}$ of a $6 \times 6$ grid,
(b) $\ddot{d}g$-set of a $6 \times 6$ grid.

**Theorem 3.16.** Let $G$be a graph satisfying the following two conditions: (i) $g(G) = 2$. (ii) Let$\{u, v\}$ be the geodetic set of $G$. For every $u - v$ geodesic say $(u, x_1, x_2, \dots, x_{D-1}, v)$ there exist another distinct $u - v$ geodesic say $(u, y_1, y_2, \dots, y_{D-1}, v)$ such that $x_i \neq y_i, x_j = y_j$ and $x_k \neq y_k$ for $1 \leq i < j < k \leq D - 1$. Then,$\ddot{d}g(G) = 2$.

**Proof:** Let$P_x(u - v)$ be the$u - v$geodesic$(u, x_1, x_2, \dots, x_{D-1}, v)$and$P_y(u - v)$be the$u - v$geodesic$(u, y_1, y_2, \dots, y_{D-1}, v)$.By (ii)wehave,$(u, x_1, x_2, \dots, x_i, x_j, x_k, \dots, x_{D-1}, v)$and$(u, y_1, y_2, \dots, y_i, y_j, y_k, \dots, y_{D-1}, v)$are also geodesics between $u$ and $v$. Thus all the vertices of$P_x(u - v)$and$P_y(u - v)$ lie on atleast two distinct geodesics. Since the above argument holdsgood for every $u - v$ geodesics we have, $\ddot{d}g(G) = 2$.

**Theorem 3.17.** Let$G$be a graph satisfying the following two conditions: (i)$g(G) = 2$. (ii) Let$\{u, v\}$ be the geodetic set of $G$. For every $u - v$ geodesic say $(u, x_1, x_2, \ldots, x_{D-1}, v)$ there exist another distinct $u - v$ geodesic say $(u, y_1, y_2, \ldots, y_{D-1}, v)$ such that$(x_i, y_{i+1})$ and $(x_{i+1}, y_i) \in E(G)$ for some $i$.Then,$\ddot{d}g(G) = 2$.

**Proof:** Let$P_x(u - v)$ be the$u - v$geodesic $(u, x_1, x_2, \ldots, x_{D-1}, v)$and $P_y(u - v)$be the$u - v$geodesic$(u, y_1, y_2, \ldots, y_{D-1}, v)$.By (ii) we have,$(u, x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_{D-1}, v)$and$(u, y_1, y_2, \ldots, y_i, y_{i+1}, \ldots, y_{D-1}, v)$are also geodesics between $u$ and $v$. Thus all the vertices of$P_x(u - v)$and$P_y(u - v)$ lie on atleast two distinct geodesics. Since the above argument holdsgood for every $u - v$ geodesics we have, $\ddot{d}g(G) = 2$.

## IV. Conclusion

In this paper, we haveformally defined the doubly geodetic number of a graph and studied its properties. Furthermore, we have obtained the doubly geodetic number of trees, cycle, complete graph, complete $k$-partite graph and grid.The computational complexity for this problem is under study.

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