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Weak Width of Subgroups

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I. Introduction

A subgroup H of G is malnormal in G if for any g 2 G such that g =2 H the intersection H \ gℵ1Hg is trivial. Most subgroups are neither normal nor malnormal, so the study of the intersection pattern of conjugates of a subgroup is an interesting problem. It is closely connected to the study of the behavior of di_erent lifts of subspaces of topological spaces in covering spaces. Malnormality of a subgroup has been generalized in di_erent ways. One of them, namely the height, introduced in [3], has been used by Agol in [1] and [2] in his proof of Thurston's conjecture that 3-manifolds are virtual bundles. In this paper we introduce yet another generalization of malnormality. It is a new invariant of the conjugacy class of a subgroup H of G, which we call the weak width of a subgroup. Like malnormality, the weak width measures only the cardinality of the intersections of H with its conjugates in G. In section 4 we prove that quasiconvex subgroups of negatively curved groups have _nite weak width, which might simplify Agol's proof. In section 2 we review the de_nitions and the basic properties of the width and the height of a subgroup. In section 3 we give examples showing that height, width, and weak width are di_erent invariants of a subgroup.

Remark 1. *Note that if $g_i \in Hg_jH$, hence $g_i = h_1g_jh_2$ with h_1 and h_2 in H, then $H \cap g_i^{-1}Hg_i = H \cap (h_1g_jh_2)^{-1}H(h_1g_jh_2) = H \cap (h_2^{-1}g_j^{-1}Hg_jh_2) = h_2^{-1}(H \cap g_0^{-1}Hg_0)h_2$. So the cardinality of the set $H \cap g_i^{-1}Hg_i$ is equal to the cardinality of the set $H \cap g_j^{-1}Hg_j$.*

Remark 1 motivates the following definitions.

Definition 1. *Let H be a subgroup of a group G. We say that the elements $\{g_i | 1 \leq i \leq n\}$ of G are strongly H-essentially distinct if $Hg_iH \neq Hg_jH$ for $i \neq j$. Conjugates $g_i^{-1}Hg_i$ of H by strongly H-essentially distinct elements are called strongly essentially distinct conjugates.*

Definition 2. *We say that the weak width of an infinite subgroup H of G in G, denoted $WeakWidth(H, G)$, is n if there exists a collection of n strongly essentially distinct conjugates $\{H, g_1^{-1}Hg_1, \cdots, g_{n-1}^{-1}Hg_{n-1}\}$ of H in G such that the intersection $H \cap g_i^{-1}Hg_i$ is infinite for all $1 \leq i \leq n-1$ and n is maximal possible. We define the weak width of a finite subgroup of G to be 0.*

Note that if $WeakWidth(H, G) = n$, then in any set of $n+1$ strongly essentially distinct conjugates $\{H, g_1^{-1}Hg_1, \cdots, g_n^{-1}Hg_n\}$ of H in G there exists an element $g_i^{-1}Hg_i$ which has finite intersection with H.

II. Height and width

The following definitions were introduced in [3] and [4].

Definition 3. *Let H be a subgroup of a group G. We say that the elements $\{g_i | 1 \leq i \leq n\}$ of G are H-essentially distinct if $Hg_i \neq Hg_j$ for $i \neq j$. Conjugates $g_i^{-1}Hg_i$ of H by H-essentially distinct elements are called essentially distinct conjugates.*

If g_i and g_j are not H-essentially distinct, then $g_i^{-1}Hg_i = g_j^{-1}Hg_j$, hence it is interesting to investigate the intersections of the conjugates of H only if they are H-essentially distinct. However, essentially distinct conjugates need not be distinct. For example, let $G =< a_1, a_2 | a_1a_2 = a_2a_1 >$ be a free abelian group of rank 2 and let $H =< a_1 >$ be a subgroup of G. The conjugates $a_2^{-1}Ha_2$ and H are essentially distinct, but $a_2^{-1}Ha_2 = H$.

Definition 4. *We say that the height of an infinite subgroup H of G in G, denoted by $Height(H,G)$, is n if there exists a collection of n essentially distinct conjugates of H in G such that the intersection of all the elements of the collection is infinite and n is maximal possible. We define the height of a finite subgroup of G to be 0.*

Note that if $Height(H,G)=n$ then the intersection of any set of $n+1$ essentially distinct conjugates of H in G is finite. It was shown in [3] that subgroups of negatively curved groups have finite height in the ambient group.

Definition 5. *We say that the width of an infinite subgroup H of G in G, denoted by $Width(H,G)$, is n if there exists a collection of n essentially distinct conjugates of H in G such that the intersection of any two elements of the collection is infinite and n is maximal possible. We define the width of a finite subgroup of G to be 0.*

Note that if $Width(H,G)=n$ then in any set of $n+1$ essentially distinct conjugates of H in G there exist two elements with finite intersection. It was shown in [4] and, later, in [5] that quasiconvex subgroups of negatively curved groups have finite width in the ambient group.
It follows from the above definitions that $Width(H,G)$ and $Height(H,G)$ are invariants of the conjugacy class of H in G.

Note also that $Height(H,G) \leq Width($1, however, it is not clear if there is any relationship between $WeakWidth(H,$ (d $Width(H,G)$.

Infinite normal subgroups of infinite ind(ve infinite height, width, and weak width in the ambient group. More generaf an infinite subgroup has infinite index in its normalizer, then the subgrou; infinite height, width, and weak width in the ambient group.

If G is torsion-free and H is infinite, tH is malnormal in G if and only if $Height(H,G) = Width(H,G) = WeakWid,G) = 1$.

III. Examples

The following examples demonstrate that $WeakWidth(H,G)$, $Width(H,G)$, and $Height(H,G)$ are distinct invariants of the conjugacy class of H in G.

Let X be a set and let $X^* = \{x, x^{-1} | x \in X\}$, where for $x \in X$ we define $(x^{-1})^{-1} = x$. Denote the equality of two words in X^* by "$\equiv$".

Example 1. *Let F be a free group of rank 4 generated by the elements x_1, x_2, x_3, x_4, let $G = < F, t | t^4 = 1, t^{-1}x_i t = x_{(i+1)mod4} | 1 \leq i \leq 4 >$, and let $H_1 = < x_1, x_2 >$. We claim that $WeakWidth(H_1, G) = 3$, but $Height(H_1, G) = Width(H_1, G) = 2$.*

In order to prove the claim we will list all essentially distinct and all strongly essentially distinct conjugates of H_1 in G which have non-trivial intersection with H_1.

Let $H_i = < x_i, x_{(i+1)mod4} | 1 \leq i \leq 4 >= t^{(-i+1)} H_1 t^{(i-1)}$ be conjugates of H_1 in G. As $t^i \notin F$ for $i \not\equiv 0 \pmod 4$, the conjugates $\{H_i | 1 \leq i \leq 4\}$ are strongly essentially distinct. Note that $H_2 \cap H_1 = < x_2 >, H_4 \cap H_1 = < x_1 >, H_3 \cap H_1 = < 1 >$, and $H_2 \cap H_4 = < 1 >$. Hence $WeakWidth(H_1, G) \geq 3$, $Height(H_1, G) \geq 2$, and $Width(H_1, G) \geq 2$.

In order to determine how other conjugates of H_1 *intersect, we consider* $g \in G$ *such that the intersection* $g^{-1}H_1g \cap H_1$ *is non-trivial. As we are interested only in essentially distinct conjugates of* H_1*, we can assume that* g *is a shortest element in the coset* H_1g*.*

As t *normalizes* F*, we have* $g = wt^k, 0 \leq k \leq 3$*, with* w *a reduced word in* F*. If* w *is trivial, then* $g^{-1}H_1g = t^{-k}H_1t^k = H_{1+k}$*, and the intersection pattern of the subgroups* $\{H_i | 1 \leq i \leq 4\}$ *is described above.*

If w *is non-trivial, let* $v \in H_1$ *be a non-trivial reduced word such that* $g^{-1}vg = (t^{-k}w^{-1})v(wt^k) \in H_1$*. Then* $w^{-1}vw \in t^kH_1t^{-k} = t^{-(4-k)}H_1t^{4-k} = H_{(1-k)mod4}$*. As* w *and* v *are reduced words in a free group* F*, there exist decompositions* $w \equiv w_1w_2$ *and* $v \equiv w_1v_0w_1^{-1}$ *(where* $\equiv$ *denotes equality of words) with*

$w^{-1}vw = (w_2^{-1}w_1^{-1})(w_1v_0w_1^{-1})(w_1w_2) = w_2^{-1}v_0w_2$*, where* $w_2^{-1}v_0w_2$ *is a reduced word in* $H_{(1-k)mod4}$*. Then* $v_0 \in H_{(1-k)mod4}$ *and* $w_2 \in H_{(1-k)mod4}$*. As* $v \in H_1$*, it follows that* $w_1 \in H_1$ *and* $v_0 \in H_1$*. However, as* $g = wt^k = w_1w_2t^k$ *is shortest in the coset* H_1g*,* w_1 *should be trivial. Hence* $w = w_2 \in H_{(1-k)mod4}$*. As a non-trivial word* v_0 *belongs to* $H_1 \cap H_{(1-k)mod4}$*, it follows that* $(1-k) \pmod 4$ *is equal to either* $1, 2$ *or* 4*. Hence if* $(1-k) \pmod 4 \equiv 3$*, so* $k = 2$*, then for any* $r \in F$ *the intersection* $(rt^2)^{-1}H_1(rt^2) \cap H_1$ *is trivial.*

If $(1-k) \pmod 4 \equiv 1$ *then* $w = w_2 \in H_1$*, contradicting again the fact that* g *is shortest in the coset* H_1g*. Hence either* $(1-k) \pmod 4 \equiv 2$ *and* $k = 3$*, or* $(1-k) \pmod 4 \equiv 4$ *and* $k = 1$*.*

If $k = 3$*, then* $g = wt^3$ *with* $w \in H_2$*. Note that the essentially distinct elements of the infinite collection of the conjugates* $\{(wt^3)^{-1}H_1(wt^3) | w \in H_2\}$ *intersect each other trivially. Indeed, consider* $w_0 \in H_2$ *and* $w \in H_2$ *such that the intersection* $(t^{-3}w^{-1})H_1(wt^3) \cap (t^{-3}w_0{}^{-1})H_1(w_0t^3)$ *is non-trivial. Then the intersection* $H_1 \cap (w_0t^3)(t^{-3}w^{-1})H_1(wt^3)(t^{-3}w_0{}^{-1})$ *is non-trivial. As* H_1 *is malnormal in* F*, it follows that* $w_0w^{-1} = (w_0t^3)(t^{-3}w^{-1}) \in H_1$*, so the conjugates* $(t^{-3}w^{-1})H_1(wt^3)$ *and* $(t^{-3}w_0{}^{-1})H_1(w_0t^3)$ *are not essentially distinct. Therefore the family of the conjugates* $\{(wt^3)^{-1}H_1(wt^3) | w \in H_2\}$ *does not contribute to* $Width(H_1, G)$*.*

Similarly, if $k = 1$*, hence* $g = ut$ *with* $u \in H_4$*, the essentially distinct elements of the infinite collections of the conjugates* $\{(ut)^{-1}H_1(ut) | u \in H_4\}$ *intersect each other trivially.*

Also for $w \in H_2$ *and* $u \in H_4$ *the intersection* $(t^{-3}w^{-1})H_1(wt^3) \cap (t^{-1}u^{-1})H_1(ut)$ *is trivial. Indeed, the cardinality of that intersection is equal to the cardinality of the intersection* $(ut)(t^{-3}w^{-1})H_1(wt^3)(t^{-1}u^{-1}) \cap H_1$*. However,* $(wt^3)(t^{-1}u^{-1}) = wt^2u^{-1} = (w(t^2u^{-1}t^{-2})t^2 = rt^2$ *with* $r \in F$*, and we have mentioned above that for all* $r \in F$ *the intersection* $(rt^2)^{-1}H_1(rt^2) \cap H_1$ *is trivial. So the infinite family of conjugates* $\{(ut)^{-1}H_1(ut) | u \in H_4\}$ *does not contribute to* $Width(H_1, G)$*, therefore* $Height(H_1, G) = Width(H_1, G) = 2$*.*

Note that for any $w \in H_2$*,* $wt^3 = t^3(t^{-3}wt^3) \in t^3H_1 \subseteq H_1t^3H_1$*, hence all the elements* $\{(wt^3) | w \in H_2\}$ *are strongly* H_1*-equivalent to* t^3*, so the conjugates of* H_1 *by those elements do not contribute to the weak width of* H_1*. Similarly, all the elements* $\{(ut)^{-1} | u \in H_4\}$ *are strongly* H_1*-equivalent to* t*, so the conjugates of* H_1 *by those elements do not contribute to the weak width of* H_1 *either. Therefore,* $WeakWidth(H, G) = 3$*.*

Example 2. *Let G be as in Example 1, and let $L_1 =< x_1, x_2, x_3 >$. We claim that $WeakWidth(L_1, G) = Width(L_1, G) = 4$, but $Height(L_1, G) = 3$.*

Let $L_i =< x_i, x_{(i+1)mod4}, x_{(i+2)mod4} | 1 \leq i \leq 4 >= t^{(-i+1)} L_1 t^{(i-1)}$ be conjugates of L_1 in G. As $t^i \notin F$ for $i \not\equiv 0 \pmod 4$, the conjugates $\{L_i | 1 \leq i \leq 4\}$ are strongly essentially distinct. By observation, the elements of the set $\{L_i | 1 \leq i \leq 4\}$ have infinite pairwise intersections, hence $WeakWidth(L_1, G) \geq 4$ and $Width(L_1, G) \geq 4$. Also the intersection $\bigcap_{i=1}^{3} L_i$ is infinite, so $Height(L_1, G) \geq 3$. Note also that the intersection $\bigcap_{i=1}^{4} L_i$ is trivial.

Using the same argument as in Example 1 we can show that there are only three families of L_1-essentially distinct elements in G such that the conjugates of L_1 by these elements intersect L_1 non-trivially. They are $\{(wt^3) | w \in L_2\}, \{ut | u \in L_4\}$, and $\{st^2 | s \in L_3\}$. Just as in Example 1, the malnormality of L_1 in F implies that the essentially distinct conjugates in each family intersect each other trivially, hence $Width(L_1, G) = 4$. Also as in Example 1 these elements are strongly L_1-essentially equivalent to t^3, t, and t^2, respectively, so $WeakWidth(L_1, G) = 4$.

Suppose $Height(H, G) \geq 4$. Then there are 3 essentially distinct conjugates M_2, M_3, and M_4 of L_1 such that the intersection $L_1 \cap (\bigcap_{i=2}^{4} M_i)$ is infinite. The preceding paragraph implies that the M_i's must come one from each of the families of conjugates of L_1 described above, i.e. M_2, M_3 and M_4 are conjugates of L_1 by wt^3, ut, and st^2 respectively, with $w \in L_2, u \in L_4$, and $s \in L_3$. Let h_1, h_2, h_3, and h_4 in L_1 be such that $h_4 = t^{-3}w^{-1}h_1wt^3 = t^{-2}s^{-1}h_2st^2 = t^{-1}u^{-1}h_3ut$. Note that $t^{-3}wt^3 \in L_1, t^{-3}h_1t^3 \in L_4, t^{-2}st^2 \in L_1, t^{-2}h_2t^2 \in L_3, t^{-1}ut \in L_1$, and $t^{-1}h_3t \in L_2$. Then $t^{-3}w^{-1}h_1wt^3 = r_1^{-1}q_1r_1$ with $r_1 \in L_1$ and $q_1 \in L_4$, $t^{-2}s^{-1}h_2st^2 = r_2^{-1}q_2r_2$ with $r_2 \in L_1$ and $q_2 \in L_3$, and $t^{-1}u^{-1}h_3ut = r_3^{-1}q_3r_3$ with $r_3 \in L_1$ and $q_3 \in L_2$. As $r_1^{-1}q_1r_1 = r_2^{-1}q_2r_2 = r_3^{-1}q_3r_3$, it follows that $q_2 = l_1^{-1}q_1l_1 = l_2^{-1}q_3l_2$ with l_1 and l_2 in L_1. We can assume that all the words l_1, l_2, q_1, q_2, and q_3 are reduced. Then, as in Example 1, there exist decompositions $l_1 \equiv p_1p_2$ and $q_1 \equiv p_1q_1'p_1^{-1}$ such that $q_2 = (p_1p_2)^{-1}(p_1q_1'p_1^{-1})(p_1p_2) = p_2^{-1}q_1'p_2$, and $p_2^{-1}q_1'p_2$ is a reduced word in F.

As $r_1^{-1}q_1r_1 = r_2^{-1}q_2r_2 = r_3^{-1}q_3r_3 = h_4 \in L_1$, it follows that $q_1 \in L_1 \cap L_4 =< x_1, x_2 >, q_2 \in L_1 \cap L_3 =< x_1, x_3 >$, and $q_3 \in L_1 \cap L_2 =< x_2, x_3 >$. As $q_1 \in< x_1, x_2 >$ and $q_2 \in< x_1, x_3 >$, it follows that $q_1' = x_1^n$ for $n \in \mathbf{N}$.

Simirlarly, there exist decompositions $l_2 \equiv c_1c_2$ and $q_3 \equiv c_1q_3'c_1^{-1}$ such that $q_2 = (c_1c_2)^{-1}(c_1q_3'c_1^{-1})(c_1c_2) = c_2^{-1}q_3'c_2$, and $c_2^{-1}q_3'c_2$ is a reduced word in F. As $q_3 \in< x_2, x_3 >$ and $q_2 \in< x_1, x_3 >$, it follows that $q_3' = x_3^m$ for $m \in \mathbf{N}$. Then a conjugate of $q_1' = x_1^n$ is equal to a conjugate of $q_3' = x_3^m$ in a free group F. This can happen only if q_1' and q_3' are trivial, hence q_2 is trivial. Therefore, the intersection of L_1 with all three families of conjugates is trivial, so $Height(L_1, G) = 3$.

IV. Quasiconvex subgroups of negatively curved groups have finite weak width

We will use the following notation.

Let G be a group generated by the set X^*. As usual, we identify the word in X^* with the corresponding element in G. Let $Cayley(G)$ be the Cayley graph of G with respect to the generating set X^*. The set of vertices of $Cayley(G)$ is G, the set of edges of $Cayley(G)$ is $G \times X^*$, and the edge (g, x) joins the vertex g to gx.

Definition 6. *The label of the path $p = (g, x_1)(gx_1, x_2) \cdots (gx_1x_2 \cdots x_{n-1}, x_n)$ in $Cayley(G)$ is the word $Lab(p) \equiv x_1 \cdots x_n$. The length of the path p, denoted by $|p|$, is the number of edges forming it. The inverse of a path p is denoted by $\bar{p}$,*

Remark 2. *Let H be a K-quasiconvex subgroup of G, let η be a geodesic in $Cayley(G)$ with $Lab(\eta) \in H$, and let $\eta'\eta''$ be any decomposition of η. There exists a path c with $|c| \le K$ which begins at the terminal vertex of η' such that $Lab(\eta' c) \in H$ and $Lab(\bar{c}\eta'') \in H$. Indeed, if η begins (and , hence, ends) at an element of H such c exists by the definition of K-quasiconvexity. In the general case, we can find such c using translation in $Cayley(G)$.*

The following result was essentially proven in [4]. We include a streamlined version of the proof.

Theorem 1. *If H is a quasiconvex subgroup of a negatively curved group G, then $WeakWidth(H, G)$ is finite.*

Proof. As G is finitely generated, there exists a finite number N of elements in G of length not greater than $2K + 2\delta$, hence there exist at most N strongly H-essentially distinct elements $\{g_i \in G\}$ such that the shortest representative of the double coset Hg_iH is not longer than $2K + 2\delta$. Then Lemma 1 implies that the only strongly H-essentially distinct conjugates of H which might have infinite intersection with H are the conjugates of H by the elements in the set $\{g_i | 1 \le i \le N\}$. Therefore $WeakWidth(H, G) \le N$.

Lemma 1. *Let H be a K-quasiconvex subgroup of a δ-negatively curved group G and g be an element in G. If every element of the double coset HgH is longer than $2K + 2\delta$, then the intersection $H \cap g^{-1}Hg$ is finite.*

Proof. Remark 1 implies that it is sufficient to prove Lemma 1 for a shortest representative g_0 of the double coset HgH.

We will show that all the elements in the intersection $H \cap g_0^{-1}Hg_0$ are shorter than $2K + 8\delta + 2$, so that the intersection is finite, as required.

Consider $h \in H \cap g_0^{-1}Hg_0$. Let $h_0 \in H$ be such that $h = g_0^{-1}h_0g_0$. Let p_1, p_{h_0}, p_2 and p_h be geodesics in $Cayley(G)$ such that $p_1p_{h_0}p_2\bar{p}_h$ is a closed path, p_1 (hence also p_h) begins at 1, $Lab(p_1) = g_0^{-1}, Lab(p_{h_0}) = h_0, Lab(p_2) = g_0$, and $Lab(p_h) = h = g_0^{-1}h_0g_0$.

Let v be a middle vertex of p_h and let q be the initial subpath of p_h ending at v. As $Lab(p_h) = h \in H$, Remark 2 implies that there exists a path s with $|s| \le K$ which begins at v such that $Lab(qs) \in H$. Let t be a shortest path which begins at v and ends at some vertex w of p_{h_0} and let q' be the initial subpath of p_{h_0} terminating at w. As $Lab(p_{h_0}) = h_0 \in H$, Remark 2 implies that there exists a path s' with $|s'| \le K$ which begins at w such that $Lab(q's') \in H$. Then $Lab(\bar{p}_1) = g_0 = Lab(q's')Lab(\bar{s}'\bar{t}s)Lab(\bar{s}\bar{q})$. As $Lab(q's') \in H$ and $Lab(\bar{s}\bar{q}) = Lab^{-1}(qs) \in H$, it follows that $Lab(s'\bar{t}s) \in Hg_0H$.

As $HgH = Hg_0H$, the assumption of Lemma 1 that any element of the double coset HgH is longer than $2K + 2\delta$ implies that $|s'\bar{t}s| > 2K + 2\delta$. But then $|t| > 2K + 2\delta - |s| - |s'| > 2\delta$, hence the distance from v to p_{h_0} is greater than 2δ.

As G is δ-negatively curved, a side p_h of the geodesic 4-gon $p_1p_{h_0}p_2\bar{p}_h$ belongs to the 2δ-neighborhood of the union of the other three sides, so the above discussion implies that v belongs to the 2δ-neighborhood of $p_1 \cup p_2$. Assume that there exists a path y of length less than 2δ which begins at a vertex u of p_1 and ends at v. Consider the decomposition $p_1 = p_1' p_1''$, where p_1' ends at u. As $g_0 = Lab(\bar{p}_1) = Lab(\bar{p}_1'' ys) Lab(\bar{s}\bar{q})$ and $Lab(\bar{s}\bar{q}) \in H$, it follows that $Lab(\bar{p}_1'' ys) \in g_0 H$. As g_0 is a shortest representative of $g_0 H$, it follows that $|g_0| = |p_1'| + |p_1''| \leq |\bar{p}_1'' ys| = |s| + |y| + |p_1''|$. Hence $|p_1'| \leq |s| + |y| \leq K + 2\delta$, so $|q| \leq |p_1'| + |y| \leq K + 4\delta$. However, as v is a middle point of p_h, $|h| = |p_h| \leq 2|q| + 1 < 2K + 8\delta + 2$.

Similarly, if v belongs to the 2δ-neighborhood of p_2, it follows that $|h| < 2K + 8\delta + 2$, proving Lemma 1.

V. Question

Is there a simple relation between the width and the weak width?

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